p -HYPONORMAL OPERATORS ARE SUBSCALAR

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ABSTRACT. We prove that if $R, S \in B(\mathbf{X})$, R, S are injective, then RS is subscalar if and only if SR is subscalar. As corollaries, it is shown that p -hyponormal operators ($0 < p \leq 1$) and log-hyponormal operators are subscalar; also w-hyponormal operators T with $\text{Ker}T \subset \text{Ker}T^*$ and their generalized Aluthge transformations $T(r, 1-r)$ ($0 < r < 1$) are subscalar.

1. INTRODUCTION AND NOTATIONS

Let $\mathbf{X}$ denote a Banach space. $T \in B(\mathbf{X})$ is said to be generalized scalar ([1]) if there exists a continuous algebra homomorphism

$$\Phi: \varepsilon(\mathbb{C}) \longrightarrow B(\mathbf{X})$$

with $\Phi(1) = I$ and $\Phi(z) = T$. Here $\varepsilon(\mathbb{C})$ denotes the algebra of all infinitely differentiable functions on $\mathbb{C}$ with the topology defined by the uniform convergence of such functions and their derivatives ([2]). An operator similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces is called subscalar. Subscalar operators are subdecomposable ([1]).

Let $\mathbf{H}$ denote an infinite-dimensional separable Hilbert space, $T \in B(\mathbf{H})$. T is said to be p -hyponormal ($p > 0$) if $(T^*T)^p \geq (TT^*)^p$, i.e., $|T|^{2p} \geq |T^*|^{2p}$. A p -hyponormal operator is called hyponormal if $p = 1$ and semi-hyponormal if $p = \frac{1}{2}$ ([3], [4]). The Löwner-Heinz inequality implies that if T is q -hyponormal, then it is p -hyponormal for any $0 < p \leq q$. An invertible operator T is said to be log-hyponormal ([5]) if $\log(T^*T) \geq \log(TT^*)$.

For $T \in B(\mathbf{H})$, let $T = U|T|$ be the polar decomposition of T , and for $s, t \geq 0$, $T(s, t) = |T|^s U |T|^t$ is called the generalized Aluthge transformation of T . $T(\frac{1}{2}, \frac{1}{2}) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is denoted by $\hat{T}$. It is known that if T is p -hyponormal ($p > 0$), then $\hat{T}$ is semihyponormal and $(\hat{\hat{T}}) = \hat{T}(\frac{1}{2}, \frac{1}{2})$ (denoted by $\tilde{T}$) is hyponormal. If T is log-hyponormal, then $\hat{T}$ is semihyponormal. T is said to be w-hyponormal ([6]) if $|\hat{T}| \geq |T| \geq |\hat{T}^*|$, so $\hat{T}$ is semihyponormal if T is w-hyponormal. It is shown in [6] that p -hyponormal ($p > 0$) and log-hyponormal operators are w-hyponormal.

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M. Putinar showed ([7]) that hyponormal operators are subscalar; Y. Chu showed ([8]) that semihyponormal operators are also subscalar. Are all p -hyponormal ($p > 0$) operators subscalar? Some authors have discussed this problem and obtained certain partial answers. We are going to derive a result: if $R, S \in B(\mathbf{X})$, R, S injective, then RS is subscalar if and only if SR is subscalar. Applying this result to generalized Aluthge transformation of operators, we get an affirmative and stronger answer to the above question: all p -hyponormal operators and log-hyponormal operators are subscalar; w -hyponormal operators T with $\text{Ker}T \subset \text{Ker}T^*$ and their generalized Aluthge transformation $T(r, 1-r)$ ($0 < r < 1$) are also subscalar.

2. MAIN RESULTS

Let $\mathbf{X}$ denote a Banach space. $R, S \in B(\mathbf{X})$, operators RS and SR have many common properties, including spectral properties, subdecomposability, nontriviality of invariant and hyperinvariant subspaces, etc. (see [9], [10]). It is proved in [9] that if R, S are injective, then RS is subdecomposable if and only if SR is subdecomposable, whence one can derive that p -hyponormal operators ($p > 0$) are subdecomposable (see also [11]). We are now going to present a similar result for subscalarity.

Theorem A. *If $R, S \in B(\mathbf{X})$, where R and S are injective, then RS is subscalar if and only if SR is subscalar.*

Some lemmas are to be considered first. $\varepsilon(U, \mathbf{X})$ denotes the Fréchet space of all $\mathbf{X}$ -valued C^∞ -functions, i.e., infinitely continuously differentiable functions on U ([1]). T is said to have property $(\beta)_\varepsilon$ (denoted by $T \in (\beta)_\varepsilon$) if for each open set U in $\mathbb{C}$ the operator

$$T_z : \varepsilon(U, \mathbf{X}) \longrightarrow \varepsilon(U, \mathbf{X}), \quad f \longmapsto (T - z)f$$

is a topological monomorphism, i.e., $T_z f_n \rightarrow 0$ implies $f_n \rightarrow 0$ for $f_n \in \varepsilon(U, \mathbf{X})$.

Lemma 1. *For $T \in B(\mathbf{X})$, the following statements are equivalent:*

- (1) T is subscalar;
- (2) $T \in (\beta)_\varepsilon$, i.e., for each open subset U of $\mathbb{C}$, $f_n \in \varepsilon(U, \mathbf{X})$, $(T - z)f_n \rightarrow 0$ ($n \rightarrow \infty$) implies $f_n \rightarrow 0$ ($n \rightarrow \infty$);
- (3) $T_z = T - z$ acts one-to-one and with closed range on $\varepsilon(U, \mathbf{X})$ for each open subset U of $\mathbb{C}$.

Proof. (1) $\iff$ (2). See corollary 4.6 of [1].

(2) $\implies$ (3). Suppose $f \in \varepsilon(U, \mathbf{X})$, $T_z f = 0$. Take $f_n = f$ ($n = 1, 2, \dots$); then $T_z f_n \rightarrow 0$ ($n \rightarrow \infty$), $f = \lim_{n \rightarrow \infty} f_n = 0$. T_z is one-to-one on $\varepsilon(U, \mathbf{X})$.

Suppose that $f_n, g \in \varepsilon(U, \mathbf{X})$, $\lim_{n \rightarrow \infty} T_z f_n = g$. Then $T_z(f_n - f_m) \rightarrow 0$ ($n, m \rightarrow \infty$). This implies that $f_n - f_m \rightarrow 0$ ($n, m \rightarrow \infty$) and $f_n \rightarrow f \in \varepsilon(U, \mathbf{X})$ ($n \rightarrow \infty$) since the Fréchet space $\varepsilon(U, \mathbf{X})$ is complete. It follows that $g = \lim_{n \rightarrow \infty} T_z f_n = T_z f$. Therefore the range of T_z is closed.

(3) $\implies$ (2). Suppose that $f_n \in \varepsilon(U, \mathbf{X})$, $n = 1, 2, \dots$, $T_z f_n \rightarrow 0$ ($n \rightarrow \infty$). Let $E = \{T_z f : f \in \varepsilon(U, \mathbf{X})\}$; E is a closed subspace of $\varepsilon(U, \mathbf{X})$ and is therefore a Fréchet space, too. $S = T_z : \varepsilon(U, \mathbf{X}) \rightarrow E$ is one-to-one and subjective. In view that S is a continuous linear mapping from $\varepsilon(U, \mathbf{X}) \rightarrow E$, the open mapping theorem implies that S^{-1} is continuous and hence that $f_n = S^{-1}T_z f_n \rightarrow 0$ ($n \rightarrow \infty$). $\square$

Lemma 2. Let $U \subset \mathbb{C}, U$ be open, $0 \in U$; $g(z) \in \varepsilon(U, \mathbf{X}), \varphi(z) \in \varepsilon(U \setminus \{0\}, \mathbf{X})$, $Q \in B(\mathbf{X})$ being injective, $g(z) = (Q - z)\varphi(z), z \in U \setminus \{0\}$. $\{z_n\}$ and $\{t_n\}$ are two sequences in $\mathbb{C}$ converging to 0. Let (1) $\Phi(z) = \varphi(z)$ or (2) $\Phi(z) = \frac{\partial^k \varphi(z)}{\partial u_1 \cdots \partial u_k}$ ($u_i = x$ or $y, i = 1, 2, \dots, k$), φ and its derivatives of order $n < k$ are continuous at $z = 0$. If $\lim_{n \rightarrow \infty} \Phi(z_n)$ and $\lim_{n \rightarrow \infty} \Phi(t_n)$ both exist, then $\lim_{n \rightarrow \infty} \Phi(z_n) = \lim_{n \rightarrow \infty} \Phi(t_n)$.

Proof. (1) $\Phi(z) = \varphi(z)$. By hypothesis,

$$g(z_n) = (Q - z_n)\varphi(z_n), g(0) = \lim_{n \rightarrow \infty} g(z_n) = Q \lim_{n \rightarrow \infty} \varphi(z_n).$$

Similarly $g(0) = Q \lim_{n \rightarrow \infty} \varphi(t_n)$. The injectivity of Q then implies that

$$\lim_{n \rightarrow \infty} \Phi(z_n) = \lim_{n \rightarrow \infty} \Phi(t_n).$$

(2) $\Phi(z) = \frac{\partial^k \varphi(z)}{\partial u_1 \cdots \partial u_k}$ It is clear by calculation and induction that

$$G(z) = \frac{\partial^k g(z)}{\partial u_1 \cdots \partial u_k} = (Q - z)\Phi(z) - \sum_{i=1}^m a_i \psi_i(z), \quad z \in U \setminus \{0\}.$$

Here $\psi_i = \varphi$ or derivatives of φ of order $n < k$, a_i are constants, $1 \leq i \leq m$. Since $g \in \varepsilon(U, \mathbf{X})$, hence

$$G(0) = \lim_{n \rightarrow \infty} G(z_n) = Q \lim_{n \rightarrow \infty} \Phi(z_n) - \sum_{i=1}^m a_i \lim_{n \rightarrow \infty} \psi_i(z_n),$$

$$G(0) = \lim_{n \rightarrow \infty} G(t_n) = Q \lim_{n \rightarrow \infty} \Phi(t_n) - \sum_{i=1}^m a_i \lim_{n \rightarrow \infty} \psi_i(t_n).$$

The continuity of ψ_i at $z = 0$ implies that $\lim_{n \rightarrow \infty} \psi_i(z_n) = \lim_{n \rightarrow \infty} \psi_i(t_n)$, $1 \leq i \leq m$. Consequently we have $Q \lim_{n \rightarrow \infty} \Phi(z_n) = Q \lim_{n \rightarrow \infty} \Phi(t_n)$ and $\lim_{n \rightarrow \infty} \Phi(z_n) = \lim_{n \rightarrow \infty} \Phi(t_n)$ by the injectivity of Q . $\square$

Lemma 3. Let $0 \in U \subset \mathbb{C}, U$ be open, $h(z), g(z) \in \varepsilon(U, \mathbf{X})$, $h(0) = 0, Q \in B(\mathbf{X})$ be injective. If $g(z) = (Q - z)\frac{h(z)}{z}$, $z \in U \setminus \{0\}$, then we can define

$$\varphi(z) = \begin{cases} h(z)/z, & z \in U \setminus \{0\}, \\ y_0, & z = 0 \end{cases}$$

($y_0 \in \mathbf{X}$), such that $\varphi(z) \in \varepsilon(U, \mathbf{X})$.

Proof. It follows from the Taylor formula that

$$(1) \quad h(z) = xh'_x(0) + yh'_y(0) + R(z), \quad z \in U.$$

Here $R(z) = o(|z|)$, i.e., $\frac{R(z)}{|z|} \rightarrow 0$ ($z \rightarrow 0$). Hence

$$\varphi(z) = \frac{h(z)}{z} = \frac{xh'_x(0) + yh'_y(0)}{x + iy} + \frac{R(z)}{z}, \quad z \in U \setminus \{0\}.$$

By (1), $\lim_{x \rightarrow 0} \varphi(x) = \lim_{x \rightarrow 0} \frac{h(x)}{x} = h'_x(0)$ and $\lim_{y \rightarrow 0} \varphi(iy) = \lim_{y \rightarrow 0} \frac{h(iy)}{iy} = \frac{h'_y(0)}{i}$ both exist. Lemma 2 implies that $h'_x(0) = \frac{h'_y(0)}{i}$, whence

$$\begin{aligned} \varphi(z) &= \frac{h'_x(0)(x + iy)}{x + iy} + \frac{R(z)}{z} \\ &= h'_x(0) + \frac{R(z)}{z}, \quad z \in U \setminus \{0\}. \end{aligned}$$

Define $y_0 = h'_x(0)$; then we have

$$(C_0) \quad \varphi(z) = \begin{cases} h'_x(0) + \frac{R(z)}{z}, & z \in U \setminus \{0\}, \\ h'_x(0), & z = 0, \end{cases}$$

$R(z) \in \varepsilon(U, \mathbf{X})$, $R(z) = o(|z|)$ and the continuity of $\varphi(z)$ in U follows at once.

Suppose that Φ is a k th order derivative of φ (including $\varphi^{(0)} = \varphi$) and that φ and all derivatives of φ of order $n \leq k$ are continuous at $z = 0$. Φ satisfies the condition

$$(C_k) \quad \Phi(z) = \begin{cases} a + \frac{G(z)}{z^{k+1}}, & z \in U \setminus \{0\}, \\ a, & z = 0, \end{cases}$$

$G(z) \in \varepsilon(U, \mathbf{X})$, $G(z) = o(|z|^{k+1})$, $a \in \mathbf{X}$ being a constant vector.

We are going to show that Φ'_x and Φ'_y exist and are continuous at $z = 0$ and satisfy condition (C_{k+1}) similar to (C_k) .

Applying the Taylor formula, we have

$$(2) \quad \begin{aligned} G(z) &= G(0) + G_1(z) + \cdots + G_i(z) + \cdots + G_{k+2}(z) + R_1(z), \\ G_i(z) &= \frac{1}{i!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^i G(0), \quad 1 \leq i \leq k+2, \quad R_1(z) = o(|z|^{k+2}). \end{aligned}$$

The hypothesis $G(z) = o(|z|^{k+1})$ implies that

$$G(0) \equiv G_1(z) \equiv \cdots \equiv G_{k+1}(z) \equiv 0$$

and hence that

$$(3) \quad \frac{\partial^n G}{\partial^m x \partial^{n-m} y}(0) = 0, \quad 1 \leq n \leq k+1, \quad 0 \leq m \leq n.$$

It follows from (2) and (3) that

$$(4) \quad G(z) = \frac{G_x^{(k+2)}(0)}{(k+2)!} x^{k+2} + a_1 x^{k+1} y + \cdots + a_{k+2} y^{k+2} + R_1(z).$$

Here $a_1, a_2, \dots, a_{k+2}$ are constant vectors.

Similarly we have (noticing that $G'_x(z) \in \varepsilon(U, \mathbf{X})$, too), by (3),

$$(5) \quad \begin{aligned} G'_x(z) &= \frac{1}{(k+1)!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^{k+1} G'_x(0) + R_2(z) \\ &= \frac{G_x^{(k+2)}(0)}{(k+1)!} x^{k+1} + b_1 x^k y + \cdots + b_{k+1} y^{k+1} + R_2(z). \end{aligned}$$

Here $b_1, b_2, \dots, b_{k+1}$ are constant vectors, $R_2(z) = o(|z|^{k+1})$.

From (4) and (5), we obtain

$$\Phi'_x(0) = \lim_{x \rightarrow 0} \frac{\Phi(x) - \Phi(0)}{x} = \lim_{x \rightarrow 0} \frac{G(x)}{x^{k+2}} = \frac{G_x^{(k+2)}(0)}{(k+2)!} = a_0,$$

say, and

$$\begin{aligned} \Phi'_x(z) &= \frac{zG'(z) - (k+1)G(z)}{z^{k+2}} \\ &= \frac{a_0 x^{k+2} + \alpha_1 x^{k+1} y + \cdots + \alpha_{k+2} y^{k+2}}{(x + iy)^{k+2}} + \frac{F(z)}{z^{k+2}} \quad (z \in U \setminus \{0\}). \end{aligned}$$

Here $\alpha_1, \alpha_2, \dots, \alpha_{k+2}$ are constant vectors, $F(z) = zR_2(z) - (k+1)R_1(z) = o(|z|^{k+2})$.

Substituting $x = r \cos \theta, y = r \sin \theta$ into the first term of the right-hand side, we obtain

$$\Phi'_x(z) = f(\theta) + \frac{F(z)}{z^{k+2}}, \quad z \in U \setminus \{0\}.$$

$\lim_{r \rightarrow 0} \Phi'_x(z) = \lim_{r \rightarrow 0} \Phi'_x[r(\cos \theta + i \sin \theta)] = f(\theta)$ exists for every θ , $0 \leq \theta \leq 2\pi$, therefore it follows from Lemma 2 that

$$f(\theta) = f(0) = a_0 = \Phi'_x(0).$$

$\Phi'_x(z)$ satisfies the following condition:

$$(C_{k+1}) \quad \Phi'_x(z) = \begin{cases} a_0 + \frac{F(z)}{z}, & z \in U \setminus \{0\}, \\ a_0, & z = 0, \end{cases}$$

$F(z) \in \varepsilon(U, \mathbf{X})$, $F(z) = o(|z|^{k+2})$, whence Φ'_x is continuous at $z = 0$.

Similarly we have that Φ'_y exists and is continuous at $z = 0$ and satisfies

$$(C_{k+1}) \quad \Phi'_y(z) = \begin{cases} b_0 + \frac{E(z)}{z^{k+2}}, & z \in U \setminus \{0\}, \\ b_0, & z = 0, \end{cases}$$

$E(z) \in \varepsilon(U, \mathbf{X})$, $E(z) = o(|z|^{k+2})$, here b_0 being a constant vector.

We have proved that $\varphi^{(0)} = \varphi$ is continuous at $z = 0$ and satisfies condition (C_0) . It follows by mathematical induction that φ has continuous derivatives of any order at $z = 0$, i.e., $\varphi \in \varepsilon(U, \mathbf{X})$. $\square$

Proof of Theorem A. By Lemma 1, it is sufficient to show that if R and S are injective and $RS - z$ is one-to-one and has closed range, then $SR - z$ is one-to-one and has closed range, too.

Let U be any open subset of $\mathbb{C}$, $F(z) \in \varepsilon(U, \mathbf{X})$, $(SR - z)F(z) = 0$. It follows that

$$(RS - z)RF(z) = R(SR - z)F(z) = 0 \quad (z \in U).$$

Since $RS - z$ is one-to-one, hence $RF(z) = 0$ and the injectivity of R implies $F(z) = 0$ ($z \in U$) so that $SR - z$ is one-to-one on $\varepsilon(U, \mathbf{X})$.

Suppose that $g \in \varepsilon(U, \mathbf{X})$, $f_n \in \varepsilon(U, \mathbf{X})$ ($n = 1, 2, \dots$) and $(SR - z)f_n(z) \rightarrow g(z)$ on $\varepsilon(U, \mathbf{X})$. Then $(RS - z)Rf_n(z) = R(SR - z)f_n(z) \rightarrow Rg(z) \in \varepsilon(U, \mathbf{X})$. Since $RS - z$ has closed range, there exists $f(z) \in \varepsilon(U, \mathbf{X})$ such that $Rg(z) = (RS - z)f(z)$ and hence

$$(6) \quad RSf(z) = Rg(z) + zf(z).$$

Let $h(z) = Sf(z) - g(z)$. Then $h(z) \in \varepsilon(U, \mathbf{X})$ and

$$\begin{aligned} Rh(z) &= R(Sf(z) - g(z)) = zf(z) \\ (SR - z)h(z) &= S[zf(z)] - zSf(z) + zg(z) = zg(z) \end{aligned}$$

$$(7) \quad (SR - z)\frac{h(z)}{z} = g(z), \quad z \in U \setminus \{0\}.$$

If $0 \notin U$, then $\frac{h(z)}{z} \in \varepsilon(U, \mathbf{X})$, $g(z)$ belongs to the range of $SR - z$ on $\varepsilon(U, \mathbf{X})$, that is, $SR - z$ has closed range on $\varepsilon(U, \mathbf{X})$.

If $0 \in U$, then by (6), $Rg(0) = RSf(0)$, $g(0) = Sf(0)$, hence $h(0) = Sf(0) - g(0) = 0$. By Lemma 3, taking $Q = SR$, we can define

$$\varphi(z) = \begin{cases} \frac{h(z)}{z}, & z \in U \setminus \{0\}, \\ y_0, & z = 0, \end{cases}$$

such that $\varphi(z) \in \varepsilon(U, \mathbf{X})$. By (7) and the continuity of φ at $z = 0$, we have

$$(SR - z)\varphi(z) = g(z), \quad z \in U,$$

and hence that $SR - z$ has closed range on $\varepsilon(U, \mathbf{X})$. The conclusion that SR is subscalar now follows. $\square$

Now let us turn to the applications of Theorem A. Let $\mathbf{H}$ denote a separable Hilbert space in the sequel.

Corollary 1. *Let $T \in B(\mathbf{H})$, $\text{Ker}T \subset \text{Ker}T^*$, $s \geq 0$, $0 < r < t$. Then $T(s, t)$ is subscalar if and only if $T(s+r, t-r)$ is subscalar; especially $T = T(0, 1)$ is subscalar if and only if $T(r, 1-r)$ ($0 < r < 1$) is subscalar.*

Proof. Since $\text{Ker}T \subset \text{Ker}T^*$, $\text{Ker}T$ reduces T and $T = \theta \oplus T_1$; here $\theta = T|_{\text{Ker}T}$, $T_1 = T|_{(\text{Ker}T)^\perp}$, $\text{Ker}T_1 = \{0\}$. It is clear that $T(\alpha, \beta) = \theta \oplus T_1(\alpha, \beta)$ ($\alpha, \beta > 0$). Suppose that $T_1 = U_1|T_1|$ is the polar decomposition of T_1 , $R = |T_1|^r$, $S = |T_1|^s U_1 |T_1|^{t-r}$; then $\text{Ker}R = \text{Ker}S = \{0\}$, $RS = T_1(s+r, t-r)$, $SR = T_1(s, t)$. By Theorem A, it concludes that $T_1(s, t)$ is subscalar if and only if $T_1(s+r, t-r)$ is subscalar and hence that $T(s, t)$ is subscalar if and only if $T(s+r, t-r)$ is subscalar. $\square$

Corollary 2. *Let $T \in B(\mathbf{H})$ be a p -hyponormal ($0 < p \leq 1$) operator or a log-hyponormal operator. Then T is subscalar.*

Proof. Suppose T is a p -hyponormal ($0 < p \leq 1$) operator. Then we have $\text{Ker}T \subset \text{Ker}T^*$ ([12], Lemma 1), $\hat{T} = \hat{T}(\frac{1}{2}, \frac{1}{2})$ is hyponormal and therefore subscalar ([7]). By Corollary 1, this implies that $\hat{T} = T(\frac{1}{2}, \frac{1}{2})$ and $T = T(0, 1)$ are subscalar.

Suppose T is a log-hyponormal operator. Then T is invertible, $\text{Ker}T = \text{Ker}T^* = \{0\}$. Since $\hat{T} = T(\frac{1}{2}, \frac{1}{2})$ is $\frac{1}{2}$ -hyponormal and therefore subscalar, T is also subscalar by Corollary 1. $\square$

Corollary 3. *Let $T \in B(\mathbf{H})$ be w -hyponormal and $\text{Ker}T \subset \text{Ker}T^*$. Then T and their generalized Aluthge transformation $T(r, 1-r)$ ($0 < r < 1$) are subscalar.*

Proof. By definition, $|\hat{T}| \geq |\hat{T}^*|$. This implies that $\hat{T} = T(\frac{1}{2}, \frac{1}{2})$ is $\frac{1}{2}$ -hyponormal and subscalar. Since $\text{Ker}T \subset \text{Ker}T^*$, it follows from Corollary 1 that T and $T(r, 1-r)$ ($0 < r < 1$) are subscalar. $\square$

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