L^p ESTIMATES FOR A CLASS OF OSCILLATORY INTEGRALS

G. SAMPSON

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ABSTRACT. We obtain a simpler proof of Theorem 3.1 of *The complete (L^p, L^p) mapping properties of some oscillatory integrals in several dimensions*, by G. Sampson and P. Szeptycki (Canad. Math. J. **53** (5) (2001), 1031–1056).

1. INTRODUCTION

The purpose of this article is to obtain simpler arguments to the $L^p(\mathbb{R}_+^2)$ -estimates of some of the operators that appear in [SS]. To be more precise, we found a simpler proof of Theorem 3.1 of [SS] (also see Theorem 1.2 of [SS]). For these purposes it suffices to consider, with $a_1, a_2, b_1, b_2 \geq 1$, the operator given by

$$(1.1) \quad Tf(x) = \sum_{m,n \geq 1} K_{mn}f(x), \text{ where } K_{mn}f(x) = \int_{\mathbb{R}_+^2} k_{mn}(x, y)f(y)dy, x \in \mathbb{R}_+^2,$$

with the kernel $k_{mn}(x, y) = \psi_m(x)\psi_n(y)e^{ix^a \cdot y^b} \varphi(x, y)\beta(x - y)$, $x^a \cdot y^b = x_1^{a_1}y_1^{b_1} + x_2^{a_2}y_2^{b_2}$, $\psi_n(x) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$ ($n = (n_1, n_2)$), $\psi_{n_l}(x_l) = \psi(2^{-n_l}x_l)$, $l = 1, 2$, where $0 \leq \psi \in C^\infty(\mathbb{R})$, $\text{supp } \psi(t) \subset [\frac{1}{4}, 1]$ is so chosen that $\sum_{r=1}^\infty \psi(2^{-r}t) = 1$ for $t > 1$. Thus here $n_1, n_2 \geq 1$. Also we suppose that $0 \leq \beta(x) \leq 1$, $\beta(x) = 1$ if $|x| \geq 2$, and $\beta(x) = 0$ if $|x| \leq 1$ and $\beta(x) \in C^\infty(\mathbb{R}^2)$.

We suppose that $\varphi(x, y)$ in (1.1) satisfies

$$(1.2) \quad |\partial_x^\alpha \partial_y^\beta \varphi(x, y)| \leq C_{\alpha\beta} |x - y|^{-|\alpha| - |\beta|}, \text{ for all } \alpha, \beta.$$

Note the φ 's that satisfy (1.2) are bounded and so it follows that our L^p result will also hold if in (1.1) we replace $\beta(x - y)$ by 1.

We improve Theorem 3.1 of [SS], which we showed in the case $\varphi(x, y) = |x - y|^{i\tau}$, τ real, but here, in Theorem 1.1, we obtain this result for any $\varphi(x, y)$ that satisfies (1.2).

Theorem 1.1. *Assume $\varphi(x, y)$ satisfies (1.2). If $1 \leq b_l \leq a_l$, $l = 1, 2$, $\frac{b_1}{a_1} = \frac{b_2}{a_2}$, then*

$$(1.3) \quad \|Tf\|_p \leq C\|f\|_p, \text{ for } p = \frac{a_1 + b_1}{a_1}.$$

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For real numbers s , we define $\bar{s} = (s, s)$, and also for any $a, b \in \mathbb{R}^2$ we say that $a \geq b$ if $a_1 \geq b_1$ and $a_2 \geq b_2$. In this paper, we always assume that $a, b \geq \bar{1}$. For $y = (y_1, y_2) \in \mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$, we set $y^b = y_1^{b_1} y_2^{b_2}$, while for a given measurable function f , we set $f(y^{\frac{1}{b}}) = f(y_1^{\frac{1}{b_1}}, y_2^{\frac{1}{b_2}})$. Also for $1 \leq p < \infty$, we take $\frac{1}{p} + \frac{1}{p'} = 1$. Finally, we denote by C , indexed if needed, positive constants depending only on the kernel k , and C in different places may stand for different constants.

2. THE MAIN RESULT

We begin with the operators given by

$$(2.1) \quad S_{mn}f(x) = \int_{\mathbb{R}_+^2} k_{mn}(x^{\frac{1}{a}}, y^{\frac{1}{b}}) f(y) dy$$

where $k_{mn}(x, y)$ is defined in (1.1).

We begin by obtaining L^2 -estimates of the operators in (2.1); we apply the Cotlar and Stein theorem that appears on p. 278 of [St].

Lemma 2.1. *Let $a, b \geq \bar{1}$, $a_l \cdot b_l > 1$, $l = 1, 2$ and $\varphi(x, y)$ satisfy (1.2). Then for some $\delta > 0$,*

$$(2.2) \quad \|S_{mn}f\|_2 \leq C d_{mn} \|f\|_2,$$

with $d_{mn} = \frac{1}{2^{\delta(n_1+m_1+n_2+m_2)}}$, $n_1, m_1, n_2, m_2 \geq 1$.

Proof. We argue as on pp. 282-285 of [St] (applied to the operator S_{mn}) and we consider the kernel

$$(2.3) \quad s_{\vec{i}\vec{j}}(\xi, \eta; m, n) = \int a_{\vec{i}}(\eta, x) a_{\vec{j}}(\xi, x) e^{2\pi i x \cdot (\eta - \xi)} dx;$$

suppose $n_1 = \max(n_1, n_2, m_1, m_2) = m^*$ with $\epsilon = (\epsilon_1, 0)$, $\epsilon_1 > 0$,

$$a_{\vec{j}}(\xi, x) = \psi_n(x^{\frac{1}{b}}) \psi_m(\xi^{\frac{1}{a}}) \varphi(\xi^{\frac{1}{a}}, x^{\frac{1}{b}}) \beta(x^{\frac{1}{b}} - \xi^{\frac{1}{a}}) \phi(x^{\bar{1}-\epsilon} - j') \phi(\xi^{\bar{1}-\epsilon} - j),$$

$\vec{j} = (j, j') \in \mathbb{Z}^4$. Because of the support restriction imposed by $\psi_n(x^{\frac{1}{b}}) \psi_m(y^{\frac{1}{a}})$ to our operator S_{mn} , we get that $2^{(m-2)a(\bar{1}-\epsilon)} \leq j \leq 2^{(m+4)a(\bar{1}-\epsilon)}$, $2^{(n-2)b(\bar{1}-\epsilon)} \leq j' \leq 2^{(n+4)b(\bar{1}-\epsilon)}$.

Case 1. $|\xi_1 - \eta_1| \geq 2^{-\frac{m^* \epsilon_1}{2}}$.

$$(2.4) \quad (i\partial_{x_1})^{2N} e^{i2\pi x \cdot (\eta - \xi)} = ((2\pi)^2 |\eta_1 - \xi_1|^2)^N e^{i2\pi x \cdot (\eta - \xi)};$$

therefore from (2.3) and (2.4)

$$(2.5) \quad s_{\vec{i}\vec{j}} = \int \frac{(i\partial_{x_1})^{2N} [a_{\vec{i}}(\eta, x) a_{\vec{j}}(\xi, x)]}{((2\pi)^2 |\eta_1 - \xi_1|^2)^N} e^{i2\pi x \cdot (\eta - \xi)} dx.$$

First suppose that $b_1 > 1$ and take $\epsilon_1 = 1 - \frac{1}{b_1}$. It follows that

$$(2.6) \quad |(\partial_{x_1})^{2N} [a_{\vec{i}}(\eta, x) a_{\vec{j}}(\xi, x)]| \leq C_N x_1^{-2N\epsilon_1} \psi_n(x^{\frac{1}{b}}) \leq C_N 2^{m^*(-2Nb_1\epsilon_1)}.$$

Next suppose that $b_1 = 1$ and take $\epsilon_1 = 1 - \frac{1}{a_1}$. Also assume that $|n_1 - m_1| \geq 3$; therefore $\beta(x^{\frac{1}{b}} - y^{\frac{1}{a}}) = 1$. For the differentiated term in the integrand of (2.5) we notice that

- (a) $|(\partial_{x_1})^{2N} \psi_n(x^{\frac{1}{b}})| \leq \frac{C_N}{2^{2Nm^*}},$
- (b) $|(\partial_{x_1})^{2N} \varphi(\xi^{\frac{1}{a}}, x^{\frac{1}{b}})| \leq C_N [\xi_1^{\frac{1}{a_1}} - x_1]^{-2N},$ and
- (c) $|(\partial_{x_1})^{2N} \phi(x^{\bar{1}-\epsilon} - j')| \leq C_N |x_1|^{-2N\epsilon_1}.$

Since $n_1 \geq m_1 + 3$, we get from (a), (b), (c) that for $b_1 = 1$

$$(2.7) \quad |(\partial_{x_1})^{2N} [a_{\vec{i}}(\eta, x) a_{\vec{j}}(\xi, x)]| \leq \frac{C_N}{2^{m^*(2N\epsilon_1)}}.$$

Thus, if $|\eta_1 - \xi_1| \geq 2^{-\epsilon_1 \frac{m^*}{2}}$, it follows from (2.5), (2.6), and (2.7) that

$$(2.8) \quad |s_{\vec{i}\vec{j}}| \leq C_N \phi(\xi^{\bar{1}-\epsilon} - j) \phi(\eta^{\bar{1}-\epsilon} - i) \frac{2^{N\epsilon_1 m^*}}{2^{m^*(2N\epsilon_1)}}.$$

If instead $n_1 \leq m_1 + 3$ and $b_1 = 1$, then $m^* - 3 \leq m_1 \leq m^*$ and we note that $\|S_{mn}\|_{2,2} = \|S_{mn}^*\|_{2,2}$. We apply our method to the operator S_{mn}^* ; here $a_1 > 1$ plays the role of b_1 and so we argue as we did in (2.6).

Case 2. $|\xi_1 - \eta_1| \leq 2^{-\epsilon_1 \frac{m^*}{2}}$.

Using (2.3) with the standard arguments we get

$$|s_{\vec{i}\vec{j}}| \leq \frac{\chi(|\xi_1 - \eta_1| \leq 2^{-\epsilon_1 \frac{m^*}{2}}) \psi_m(\xi^{\frac{1}{a}}) \psi_m(\eta^{\frac{1}{a}}) \phi(\xi^{\bar{1}-\epsilon} - j) \phi(\eta^{\bar{1}-\epsilon} - i)}{(1 + |\eta - \xi|^2)^{N_1}}$$

in case that $i' - j' \in Q_1$, and we get that

$$(2.9) \quad \int |s_{\vec{i}\vec{j}}| \leq C_N \frac{1}{2^{\epsilon_1 \frac{m^*}{2}} (1 + |\vec{i} - \vec{j}|^2)^{N_1}}.$$

Therefore from (2.8) and (2.9) and the restriction on j, j' we get that

$$\sum_{\vec{j}} (\gamma_1 + \gamma_2)(\vec{j}) \leq C_N \frac{2^{\frac{1}{2}(n_1 b_1 + m_1 a_1) \epsilon_1} 2^{\frac{1}{2} m^* (a_1(1-\epsilon_1) + a_2 + b_1(1-\epsilon_1) + b_2)}}{2^{m^* \frac{N\epsilon_1}{2}}} + C_N \frac{1}{2^{\frac{\epsilon_1}{4} m^*}},$$

and select N so that $N\epsilon_1 = 2(b_1 + b_2 + a_1 + a_2) + 8$, and therefore

$$(2.10) \quad \|S_{mn}\|_{2,2} \leq \frac{C}{2^{4\delta m^*}} \leq \frac{C}{2^{\delta(m_1 + n_1 + m_2 + n_2)}},$$

which completes our proof. $\square$

Next we set

$$(2.11) \quad U_{mn}g(x) = \int_{\mathbb{R}_+^2} k_{mn}(x^{\frac{b}{a}}, y) g(y) dy$$

and we study the operator

$$(2.12) \quad Ug(x) = \sum_{m,n \geq \bar{1}} U_{mn}g(x).$$

Lemma 2.2. *With the hypothesis as in Lemma 2.1 we get*

$$(2.13) \quad \int_{\mathbb{R}_+^2} |U_{mn}f(x)|^2 dx \leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |f(y)|^2 dy,$$

where d_{mn} is defined below (2.2).

Proof. Now since $b \geq \bar{1}$ and $S_{mn}g(x)$ is supported in $x \geq \frac{\bar{1}}{2}$, we get that

$$\begin{aligned} \int_{\mathbb{R}_+^2} x^{\frac{1}{b}-\bar{1}} |S_{mn}g(x)|^2 dx &\leq \int_{\mathbb{R}_+^2} |S_{mn}g(x)|^2 dx \\ &\leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |\psi_n(y^{\frac{1}{b}}) g(y)|^2 dy, \end{aligned}$$

and this last estimate follows from Lemma 2.1. We write out the left side of this estimate to get

$$\begin{aligned} \int_{\mathbb{R}_+^2} x^{\frac{1}{b}-\bar{1}} |\psi_m(x^{\frac{1}{a}})| \int_{\mathbb{R}_+^2} \psi_n(y^{\frac{1}{b}}) e^{ix \cdot y} \varphi(x^{\frac{1}{a}}, y^{\frac{1}{b}}) \beta(x^{\frac{1}{a}} - y^{\frac{1}{b}}) g(y) dy|^2 dx \\ \leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |\psi_n(y^{\frac{1}{b}}) g(y)|^2 dy, \end{aligned}$$

and set $x = v^b, y = u^b$. We then get that

$$\begin{aligned} \int_{\mathbb{R}_+^2} |\psi_m(x^{\frac{b}{a}})| \int_{\mathbb{R}_+^2} \psi_n(y) e^{ix^b \cdot y^b} \varphi(x^{\frac{b}{a}}, y) \beta(x^{\frac{b}{a}} - y) g(y^b) y^{b-\bar{1}} dy|^2 dx \\ \leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |\psi_n(y^{\frac{1}{b}}) g(y)|^2 dy. \end{aligned}$$

Next set $f(y) = g(y^b) y^{b-\bar{1}}$; therefore it follows that

$$\begin{aligned} \int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+^2} k_{mn}(x^{\frac{b}{a}}, y) f(y) dy \right|^2 dx \\ \leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |\psi_n(y^{\frac{1}{b}}) f(y^{\frac{1}{b}}) y^{\frac{1}{b}-\bar{1}}|^2 dy. \end{aligned}$$

Setting $v = y^{\frac{1}{b}}, dy = v^{b-\bar{1}} dv$, we get that

$$\begin{aligned} \leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |\psi_n(v) f(v) v^{\bar{1}-b}|^2 v^{b-\bar{1}} dv \\ \leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |\psi_n(y) f(y)|^2 y^{\bar{1}-b} dy, \end{aligned}$$

and since here $b \geq \bar{1}$ and $y \geq \frac{\bar{1}}{2}$, we obtain our result (2.13). $\square$

Next we form the operators

$$(2.14) \quad J_{mn}f(x) = x_1 x_2 U_{mn}f(x) \text{ with the measure } d\nu(x) = \frac{dx_1 dx_2}{x_1^2 x_2^2}.$$

Lemma 2.3. *With the hypothesis as in Lemma 2.1, for the operators defined in (2.14) we get that*

$$(2.15) \quad \|J_{mn}f\|_{p,\nu}^p = \int_{\mathbb{R}_+^2} x^{\bar{p}-\bar{2}} |U_{mn}f(x)|^p dx \leq C d_{mn}^{2(p-1)} \|f\|_p^p,$$

for $1 < p \leq 2$.

Proof. We first show that

$$(2.16) \quad \nu(\{x : |J_{mn}f(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_1.$$

Without any loss of generality, we can suppose that $f(y) \geq 0$ and $\|f\|_1 > 0$. Also set $E_\lambda = \{x : |J_{mn}f(x)| > \lambda\}$ and so obtaining (2.16) we need to estimate $\nu(E_\lambda)$.

Set for $l = 1$ and 2 , $I_{m_l} = [(\frac{2^{m_l}}{4})^{\frac{a_l}{b_l}}, (2 \cdot 2^{m_l})^{\frac{a_l}{b_l}}]$, and notice that $J_{mn}f(x)$ is supported in $I_{m_1} \times I_{m_2}$. Also, $\lambda < |x_1 x_2 U_{mn}f(x)| \leq \psi_m(x^{\frac{b}{a}}) \|f\|_1 x_1 x_2$. Thus, $x_2 > \frac{\lambda}{\|f\|_1 x_1}$ and $x_2 \in I_{m_2}$. Hence if $\frac{\lambda}{\|f\|_1 x_1} > (2 \cdot 2^{m_2})^{\frac{a_2}{b_2}}$, then the set $E_\lambda = \emptyset$ and

so $\nu(E_\lambda) = 0$. We need only concern ourselves with the cases where $\frac{\lambda}{\|f\|_1 x_1} \leq (2 \cdot 2^{m_2})^{\frac{a_2}{b_2}}$.

Then we get

$$(2.17) \quad \begin{cases} \text{(a) } \int_{\frac{\lambda}{\|f\|_1 x_1}}^{(2 \cdot 2^{m_2})^{\frac{a_2}{b_2}}} \frac{1}{x_2^2} dx_2 \leq \frac{\|f\|_1 x_1}{\lambda}, \text{ if } \frac{\lambda}{\|f\|_1 x_1} \in I_{m_2}, \\ \text{and} \\ \text{(b) } \int_{I_{m_2}} \frac{1}{x_2^2} dx_2 \leq \frac{C}{2^{m_2} \frac{a_2}{b_2}}, \text{ if } \frac{\lambda}{\|f\|_1 x_1} < (2 \cdot 2^{m_2})^{\frac{a_2}{b_2}}. \end{cases}$$

Therefore we get from (2.17) that

$$\begin{aligned} \nu(E_\lambda) &= \int_{I_{m_1}} \left(\int_{\frac{\lambda}{\|f\|_1 x_1}}^{(2 \cdot 2^{m_2})^{\frac{a_2}{b_2}}} \frac{1}{x_2^2} dx_2 \right) \frac{dx_1}{x_1^2} \\ &\quad + \int_{\frac{\lambda}{\|f\|_1} \left(\frac{4}{2^{m_2}} \right)^{\frac{a_2}{b_2}}}^{(2 \cdot 2^{m_1})^{\frac{a_1}{b_1}}} \left(\int_{I_{m_2}} \frac{1}{x_2^2} dx_2 \right) \frac{dx_1}{x_1^2} = I + II, \\ I &\leq \frac{\|f\|_1}{\lambda} \int_{I_{m_1}} \frac{1}{x_1} dx_1 \leq C \frac{\|f\|_1}{\lambda}, \text{ and} \\ II &\leq C \left(\frac{4}{2^{m_2}} \right)^{\frac{a_2}{b_2}} (2^{m_2})^{\frac{a_2}{b_2}} \frac{\|f\|_1}{\lambda} \leq C \frac{\|f\|_1}{\lambda}. \end{aligned}$$

Thus we obtain our estimate for (2.16).

Next it follows from Lemma 2.2 that

$$(2.18) \quad \|J_{mn}f\|_{2,\nu}^2 = \int_{\mathbb{R}_+^2} |U_{mn}f(x)|^2 dx \leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |f(y)|^2 dy.$$

Now we shall prove that (2.15) follows from (2.16) and (2.18).

To complete this proof, we take

$$f_1^\lambda(y) = \begin{cases} f(y), & \text{if } 0 \leq f(y) \leq \lambda\alpha, \\ 0, & \text{elsewhere,} \end{cases}$$

and $f(y) = f_1^\lambda(y) + f_2^\lambda(y)$, and we take $\alpha = \frac{1}{d_{mn}^2}$, d_{mn} defined below (2.2).

Thus (2.16) and (2.18) imply that

$$(2.19) \quad \begin{cases} \text{(a) } \nu(\{|J_{mn}f_1^\lambda(x)| > \lambda\}) \leq \frac{C d_{mn}^2}{\lambda^2} \|f_1^\lambda\|_2^2, \\ \text{and} \\ \text{(b) } \nu(\{|J_{mn}f_2^\lambda(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f_2^\lambda\|_1. \end{cases}$$

Now for $1 < p < 2$ we get that

$$\begin{aligned} \|J_{mn}f\|_{p,\nu}^p &= 2^p p \int_0^\infty \lambda^{p-1} \nu(\{|J_{mn}f(x)| > 2\lambda\}) d\lambda \\ &\leq C_p \int_0^\infty \lambda^{p-1} [\nu(\{|J_{mn}f_1^\lambda(x)| > \lambda\}) + \nu(\{|J_{mn}f_2^\lambda(x)| > \lambda\})] d\lambda \\ &= I + II. \end{aligned}$$

Thus (2.19) implies that

$$\begin{aligned}
I + II &\leq C_p \left(\int_0^\infty \lambda^{p-1} \left[\frac{d_{mn}^2}{\lambda^2} \left(\int_{\mathbb{R}_+^2} |f_1^\lambda(y)|^2 dy \right) + \frac{1}{\lambda} \left(\int_{\mathbb{R}_+^2} |f_2^\lambda(y)| dy \right) \right] d\lambda \right) \\
&= C_p \left(\int_{\mathbb{R}_+^2} [d_{mn}^2 \left(\int_0^\infty \lambda^{p-3} |f_1^\lambda(y)|^2 d\lambda \right) + \left(\int_0^\infty \lambda^{p-2} |f_2^\lambda(y)| d\lambda \right)] dy \right) \\
&\leq C_p \left(\int_{\mathbb{R}_+^2} [d_{mn}^2 \left(\int_{\{y: 0 < f(y) \leq \lambda\alpha\}} \lambda^{p-3} |f(y)|^2 d\lambda \right) \right. \\
&\quad \left. + \left(\int_{\{y: f(y) > \lambda\alpha\}} \lambda^{p-2} |f(y)| d\lambda \right)] dy \right) \\
&\leq C_p \left(\int_{\mathbb{R}_+^2} d_{mn}^2 |f(y)|^2 \left(\int_{\frac{f(y)}{\alpha}}^\infty \lambda^{p-3} d\lambda \right) dy + \int_{\mathbb{R}_+^2} |f(y)| \left(\int_0^{\frac{f(y)}{\alpha}} \lambda^{p-2} d\lambda \right) dy \right) \\
&\leq C_p \left(\frac{d_{mn}^2}{2-p} \int_{\mathbb{R}_+^2} \frac{|f(y)|^p}{\alpha^{p-2}} dy + \frac{1}{p-1} \int_{\mathbb{R}_+^2} \frac{|f(y)|^p}{\alpha^{p-1}} dy \right) \\
&\leq C_p (d_{mn}^2 d_{mn}^{2(p-2)} + d_{mn}^{2(p-1)}) \|f\|_p^p \leq C_p d_{mn}^{2(p-1)} \|f\|_p^p,
\end{aligned}$$

which completes our proof (note $\alpha = \frac{1}{d_{mn}^2}$). $\square$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We notice that in case $a_l = b_l \geq 1, l = 1$ or 2 , our result follows from Proposition 4.3 of [SS]. Now to do the remaining cases. It follows from (2.12) and Lemma 2.3 that

$$\begin{aligned}
(2.20) \quad &\left(\int_{\mathbb{R}_+^2} x^{\bar{p}-2} |Uf(x)|^p dx \right)^{\frac{1}{p}} \leq \sum_{m,n \geq 1} \left(\int_{\mathbb{R}_+^2} x^{\bar{p}-2} |U_{mn}f(x)|^p dx \right)^{\frac{1}{p}} \\
&\leq C \|f\|_p \sum_{m,n \geq 1} d_{mn}^{\frac{2}{p}} \leq C \|f\|_p,
\end{aligned}$$

with d_{mn} defined below (2.2).

Since $Tf(x) = Uf(x^{\frac{b}{a}})$, then

$$\|Tf\|_p = \left(\int_{\mathbb{R}_+^2} x^{\frac{b}{a}-\bar{1}} |Uf(x)|^p dx \right)^{\frac{1}{p}},$$

and if we take $\bar{p} - \bar{2} = \frac{b}{a} - \bar{1}$ or $p = \frac{b_1}{a_1} + 1 = \frac{b_2}{a_2} + 1$, then we get the desired result. $\square$

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849

E-mail address: sampsgm@math.auburn.edu