PSEUDOCOMPACT WHYBURN SPACES NEED NOT BE FRÉCHET

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ABSTRACT. We prove in ZFC that there exists a Tychonoff pseudocompact scattered AP-space of uncountable tightness. We give some sufficient and necessary conditions for a $\mathcal{P}$ -space to be AP as well as a characterization of AP-property in linearly ordered topological spaces.

1. INTRODUCTION

If X is a topological space and $F \subset X$, let us say that F is *almost closed* if $\overline{F} \setminus F$ is a one-point set. If $\overline{F} \setminus F = \{x\}$, we denote it by $F \rightarrow x$. It is natural to say that the topology of a space X is *determined by almost closed subspaces* if, for any non-closed $A \subset X$ and any $x \in \overline{A} \setminus A$, there is an almost closed $F \subset A$ such that $F \rightarrow x$. By analogy with the concepts of Fréchet–Urysohn and sequential spaces, we say that a topological space X is *weakly determined by almost closed subspaces* if, for any non-closed $A \subset X$, there is an almost closed $B \subset A$ such that $B \rightarrow x \notin A$.

The spaces determined by almost closed subspaces were first introduced by Whyburn [Wh] who baptized them *accessibility spaces* and studied the properties of pseudo-open maps onto accessibility spaces. Later Okromeshko [Ok] obtained more results in the same direction. Twenty-three years later this concept appeared in the paper of Pultr and Tozzi [PT] in the context of categorical topology. They called the relevant class “AP-spaces”, AP standing for “Approximation-by-Points”. When Bella [Be1] and Simon [Si] studied topological properties of AP-spaces being unaware of the papers [Wh] and [Ok], they used the terminology of [PT]. The spaces weakly determined by almost closed subsets were introduced in [Si] where they were called WAP-spaces, the first letter standing for “weakly”. In spite of being uninformative, the names AP and WAP became entrenched and were further used in the papers [Be2], [BY] and [TY].

The situation changed recently when Arhangel’skii communicated to the authors of this paper and other specialists in the field, that the concept of an AP-space was

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first introduced by Whyburn in [Wh]. That is why the authors of this paper have decided (after consultation with other specialists) to give credit to Whyburn and call AP-spaces *Whyburn spaces*. Of course, WAP-spaces automatically acquire the title “weakly Whyburn spaces”. As a transitional measure, we use the old terms in the Abstract and Introduction but the main text contains only new terminology.

The first to show that some AP-spaces are Fréchet was Bella [Be1] who proved that any scattered compact AP-space is Fréchet–Urysohn. In [BY] it was proved that “scattered” can be omitted in this result. Tkachuk and Yaschenko established that every countably compact AP-space is Fréchet–Urysohn and asked ([TY], Problem 4.7) whether every pseudocompact AP-space must also be Fréchet–Urysohn. The main result of this paper is an answer to Problem 4.7: we construct a ZFC example of a scattered pseudocompact zero-dimensional Tychonoff space which is AP but has uncountable tightness (and even uncountable α -tightness). We also study AP-property in P -spaces and give a characterization of the AP-property in linearly ordered spaces.

Our terminology is standard and follows [En]. *All spaces are assumed to be Hausdorff.* Given two infinite sets A and B we say that $A \subset^* B$ if $A \setminus B$ is finite. Analogously, if $f, g \in \omega^\omega$, then $f <^* g$ if there is $k \in \omega$ such that $f(n) < g(n)$ for all $n \geq k$. A set $F \subset \omega^\omega$ is *bounded* if there is a $g \in \omega^\omega$ such that $f <^* g$ for every $f \in F$. The symbol $\mathfrak{b}$ denotes the minimal cardinality of unbounded subsets of ω^ω . A family $D \subset \omega^\omega$ is *dominating* if, for any $f \in \omega^\omega$ there is $g \in D$ such that $f <^* g$. The symbol $\mathfrak{d}$ stands for the minimal cardinality of dominating subsets of ω^ω . It is clear that $\mathfrak{b} \leq \mathfrak{d}$ and the equality $\mathfrak{b} = \mathfrak{d}$ follows from CH as well as from MA [vD], so the statement $\mathfrak{b} = \mathfrak{d}$ is compatible with the usual axioms of ZFC.

We say that X is a P -space if any G_δ -subset of X is open. *The ω -modification* of a space (X, τ) is a space with the underlying set X whose topology is generated by G_δ -subsets of (X, τ) . A set $A \subset X$ is *Whyburn closed* if $\overline{F} \subset A$ for any almost closed $F \subset A$. If we want to emphasize that a set F is almost closed in a space X , we say that F is *X -almost closed* or *almost closed in X* .

2. THE WHYBURN PROPERTY IN LINDELÖF AND PSEUDOCOMPACT SPACES

We give examples of pseudocompact Whyburn spaces of uncountable tightness giving a complete answer to Problem 4.7 from [TY]. We also consider the Whyburn property in P -spaces and give its characterization in linearly ordered spaces.

2.1. Theorem. *If the equality $\mathfrak{b} = \mathfrak{d}$ holds, then there exists a Tychonoff scattered separable pseudocompact Whyburn space of uncountable tightness.*

Proof. Take a dominating family $\Phi = \{\varphi_\alpha : \alpha < \mathfrak{b}\}$ in $(\omega^\omega, <^*)$. Since $\mathfrak{b} = \mathfrak{d}$, we can suppose that Φ is well-ordered by $<^*$ so that $\alpha \rightarrow \varphi_\alpha$ is an order isomorphism and, in particular, $\varphi_\alpha <^* \varphi_{\alpha+1}$ for each $\alpha < \mathfrak{b}$. For the set $C = \omega \times \omega$, let $C_n = \{n\} \times \omega$ for each $n \in \omega$. Given $\varphi_\alpha \in \Phi$, we define the sets $D_\alpha = \{(m, n) \in C : n \leq \varphi_\alpha(m)\}$ and $U_\alpha = C \setminus D_\alpha$. Call a set $S \subset C$ a *selector* if $|S \cap C_n| \leq 1$ for any $n \in \omega$. By transfinite induction on $\alpha < \mathfrak{b}$ we define an almost disjoint family $\mathcal{A}$ on the set C with the following properties:

- (i) $\mathcal{A} = \bigcup \{\mathcal{A}_\alpha : \alpha < \mathfrak{b}\}$ and $\mathcal{A}$ consists of selectors on C ;
- (ii) the family $\mathcal{A} \cup \{C_n : n \in \omega\}$ is maximal almost disjoint;
- (iii) $A \subset D_\alpha$ and $A \subset^* U_\beta$ whenever $\beta < \alpha < \mathfrak{b}$ and $A \in \mathcal{A}_\alpha$.

For $\alpha = 0$, it is possible to choose a maximal almost disjoint family $\mathcal{A}_0$ of selectors contained in D_0 . Now assume that we have defined the families $\mathcal{A}_\beta$ for all $\beta < \alpha$, where $\alpha < \mathfrak{b}$. If $\alpha = \delta + 1$, then, since the set $D_\alpha \cap U_\delta$ is infinite, we can choose a maximal almost disjoint family $\mathcal{A}_\alpha$ of selectors contained in $D_\alpha \cap U_\delta$.

If $\alpha = \lim \alpha$, then choose any family $\mathcal{A}_\alpha$ of selectors such that

- (iv) every element of $\mathcal{A}_\alpha$ is a subset of D_α ;
- (v) $\mathcal{A}_\alpha$ is almost disjoint and the set $A \cap D_\beta$ is finite for any $\beta < \alpha$ and $A \in \mathcal{A}_\alpha$;
- (vi) $\mathcal{A}_\alpha$ is maximal with respect to the properties (iv) and (v).

We claim that the family $\mathcal{A} = \bigcup \{\mathcal{A}_\alpha : \alpha < \mathfrak{b}\}$ satisfies (i), (ii) and (iii). Let us check first that $\mathcal{A} \cup \{C_n : n \in \omega\}$ is almost disjoint. If $A, B \in \mathcal{A}$ are distinct sets, then $A \in \mathcal{A}_\alpha$ and $B \in \mathcal{A}_\beta$; assume without loss of generality that $\alpha \leq \beta$. If $\alpha < \beta$, then $A \subset D_\alpha$ and $B \cap D_\alpha$ is finite so the set $A \cap B$ is finite. If $\alpha = \beta$, then $A \cap B$ is finite because $\mathcal{A}_\alpha$ is almost disjoint. Since $\mathcal{A}$ consists of selectors, the family $\mathcal{A} \cup \{C_n : n \in \omega\}$ is also almost disjoint.

To see that $\mathcal{A}' = \mathcal{A} \cup \{C_n : n \in \omega\}$ is maximal, take any infinite $D \subset C$ such that $D \cap A$ is finite for any $A \in \mathcal{A}'$. Since $D \cap C_n$ is finite for each n , the set D contains an infinite selector D' ; we assume, without loss of generality, that $D = D'$, i.e., D is a selector. Since the family Φ dominates, we have $D \subset^* D_\gamma$ for some $\gamma < \mathfrak{b}$ and hence the set $D \cap D_\gamma$ is infinite. Denote by α the minimal element of the set $\{\delta < \mathfrak{b} : D \cap D_\delta \text{ is infinite}\}$. If $\alpha = \eta + 1$, then some infinite $E \subset D$ lies in $D_\alpha \cap U_\eta$. Since E is a selector and $\mathcal{A}_\alpha$ is a maximal family of selectors on $D_\alpha \cap U_\eta$, the set E has an infinite intersection with some $A \in \mathcal{A}_\alpha$ which is a contradiction. If α is a limit ordinal, then $H \subset D_\alpha$ for some infinite $H \subset D$. By definition of α the set $H \cap D_\beta$ is finite for each $\beta < \alpha$ and therefore H is a selector on D_α which satisfies properties (iv) and (v) for the elements of $\mathcal{A}_\alpha$. Since $\mathcal{A}_\alpha$ is a maximal family of selectors with (iv) and (v) the set $H \cap A$ is infinite for some $A \in \mathcal{A}_\alpha$ which gives the final contradiction.

For each $A \in \mathcal{A}$ we take any point $p_A \notin C$ and a point $p \notin C \cup \{p_A : A \in \mathcal{A}\}$; then $X = \{p\} \cup \{p_A : A \in \mathcal{A}\} \cup C$ is the underlying set of a space we need to prove our theorem. All points of C are isolated in X and the neighbourhood base of any p_A is the family $\{\{p_A\} \cup (A \setminus F) : F \text{ is a finite subset of } A\}$. Finally, a base at p is formed by the sets $O_\alpha = \{p\} \cup U_\alpha \cup \{p_A : A \in \bigcup_{\beta > \alpha} \mathcal{A}_\beta\}$, where $\alpha < \mathfrak{b}$.

It is immediate that X is a scattered zero-dimensional Tychonoff space. Observe that X is separable because C is dense in X . Note also that, for any $A \subset C$, we have $p \in \overline{A}$ if and only if $A \cap C_n$ is infinite for some $n \in \omega$. Now it is easy to show that X is pseudocompact. It suffices to verify that any infinite $D \subset C$ has an accumulation point in X . Since the family $\mathcal{A}' = \mathcal{A} \cup \{C_n : n \in \omega\}$ is maximal, the set $A \cap D$ is infinite for some $A \in \mathcal{A}'$. If $A = C_n$ for some n , then $p \in \overline{D} \setminus D$. If not, then $A \in \mathcal{A}$ and hence p_A is the respective accumulation point for D which proves that X is pseudocompact.

To see that X is Whyburn take any $D \subset X$ and any $x \in \overline{D} \setminus D$. Clearly, $x \in \{p\} \cup \{p_A : A \in \mathcal{A}\}$. If $x = p_A$ for some $A \in \mathcal{A}$, then $D \cap A$ is a sequence in D which converges to x . If $x = p$ and $x \in \overline{D} \cap \overline{C}$, then $S = D \cap C_n$ is infinite for some n and hence S is a sequence in D which converges to x . If $x \notin \overline{D} \cap \overline{C}$, then $x \in \overline{D} \cap \overline{P}$ where $P = \{p_A : A \in \mathcal{A}\}$. As a consequence $B = D \cap P$ is an almost closed subset of D with $B \rightarrow x$ so the Whyburn property of X is established. Observe finally that X has uncountable tightness because $p \in \overline{P}$ but $p \notin \overline{D}$ for any countable $D \subset P$. $\square$

To prove the next theorem, we need an auxiliary construction to be carried out for an arbitrary countably infinite set D which will be identified with the discrete space whose underlying set is D . The letter K stands for the Cantor set; the mappings $\pi_D : D \times K \rightarrow D$ and $\pi_K : D \times K \rightarrow K$ are the respective natural projections. If $A \subset D \times K$, then $A(d)$ denotes the set $\pi_K(A \cap (\{d\} \times K))$ for any $d \in D$. A set $A \subset D \times K$ is called *admissible* if it has the following properties:

- (i) the set $i(A) = \pi_D(A)$ is infinite;
- (ii) the set $A(d)$ is clopen in K for every $d \in i(A)$ and the family $\{A(d) : d \in i(A)\}$ is disjoint.

A family $\mathcal{A}$ of admissible subsets of $D \times K$ is called *essentially disjoint* if, for any distinct $A, B \in \mathcal{A}$, the set $\{d \in D : A(d) \cap B(d) \neq \emptyset\}$ is finite. Denote by $\mathcal{A}(D)$ any maximal essentially disjoint family of admissible subsets of $D \times K$. Assign a point p_A to any set $A \in \mathcal{A}(D)$.

We can now define our auxiliary space $P(D) = (D \times K) \cup \{p_A : A \in \mathcal{A}(D)\}$. The local bases at the points of $D \times K$ are given by their usual local bases in $D \times K$ (remember that D is endowed with the discrete topology). Given an $A \in \mathcal{A}(D)$, a local base at the point p_A is the family $\{\{p_A\} \cup (A \setminus (F \times K)) : F \subset D \text{ is finite}\}$.

2.2. Lemma. *The space $P(D)$ is Tychonoff, Fréchet–Urysohn, pseudocompact and zero-dimensional; $D \times K$ is an open subspace of $P(D)$ and the set $\{p_A : A \in \mathcal{A}(D)\}$ is closed and discrete.*

Proof. The space $P(D)$ is Hausdorff because $\mathcal{A}(D)$ is essentially disjoint; it is zero-dimensional because $U \cap (D \times K)$ is clopen in $D \times K$ for any basic neighbourhood U of any point p_A . The proof of all other properties except for pseudocompactness is easy so we will only check that $P(D)$ is pseudocompact. It suffices to show that, for any infinite family $\mathcal{U}$ of non-empty clopen subsets of $D \times K$, there is a point $x \in P(D)$ such that every neighbourhood of x meets infinitely many elements of $\mathcal{U}$.

If the family $\mathcal{U}_d = \{U \in \mathcal{U} : U(d) \neq \emptyset\}$ is infinite for some $d \in D$, then such a point exists because the space K is compact. If every $\mathcal{U}_d$ is finite, then the set $\{d \in D : U(d) \neq \emptyset \text{ for some } U \in \mathcal{U}\}$ is infinite and hence we can find an admissible set $B \subset D \times K$ and an injection $d \rightarrow U_d$ of the set $i(B)$ into $\mathcal{U}$ such that $B(d) \subset U_d(d)$ for each $d \in i(B)$. The family $\mathcal{A}(D)$ being maximal essentially disjoint, we can find $A \in \mathcal{A}(D)$ such that $A(d) \cap B(d) \neq \emptyset$ for infinitely many d 's. To finish our proof observe that each neighbourhood of the point p_A intersects infinitely many elements of $\mathcal{U}$. $\square$

2.3. Theorem. *There is (in ZFC) a Tychonoff pseudocompact (non-separable) scattered Whyburn space of uncountable $\mathfrak{o}$ -tightness.*

Proof. Denote by L the subset of $\mathfrak{c}^+$ which consists of limit ordinals of countable cofinality. Considering L with its order topology it is natural to say that a countably infinite set $S \subset L$ is a *sequence converging to* $\alpha \in L$ if $S \subset \alpha$, $\sup S = \alpha$ and $S \cap \beta$ is finite for any $\beta < \alpha$.

If $\alpha \in L$ and $\alpha = \beta + \omega$ for some $\beta < \mathfrak{c}^+$ (i.e., if α is an isolated point of L) let $\mathcal{D}_\alpha = \{[\beta, \alpha)\}$, where $[\beta, \alpha) = \{\gamma : \beta \leq \gamma < \alpha\}$. If α is not isolated in L , define $\mathcal{D}_\alpha$ to be any maximal almost disjoint family of sequences converging to α . It is a routine to prove that $\mathcal{D} = \bigcup \{\mathcal{D}_\alpha : \alpha \in L\}$ is a maximal almost disjoint family of countable subsets of $\mathfrak{c}^+$.

Apply Lemma 2.2 to fix the family $\mathcal{A}(D)$, the points $\{p_A : A \in \mathcal{A}(D)\}$ and the space $P(D)$ for every $D \in \mathcal{D}$. Choose a new point p and define $X = \{p\} \cup \{p_A :$

$A \in \mathcal{A}(D)$, $D \in \mathcal{D}$ to be the underlying set of the space we want to construct. Given a point $x = (\alpha, z) \in \mathfrak{c}^+ \times K$, take a clopen local base $\{W_n : n \in \omega\}$ of z in the space K and let $U_n = \{\alpha\} \times W_n$ for each $n \in \omega$. We declare the family $\mathcal{B}_x = \{U_n : n \in \omega\}$ to be a local base of the point x in X . Observe that the local bases thus defined generate the product topology on $\mathfrak{c}^+ \times K$ if $\mathfrak{c}^+$ is considered with the discrete topology.

If $A \in \mathcal{A}(D)$ for some $D \in \mathcal{D}$, then the local base $\mathcal{B}_x$ at the point $x = p_A$ is the same as in the space $P(D)$ (recall that $P(D) = D \times K \subset \mathfrak{c}^+ \times K$ is a subset of X). Finally, declare the local base of X at the point p to be the family $\mathcal{B}_p = \{V_\alpha : \alpha \in L\}$ where

$$V_\alpha = \{p\} \cup (\{\beta : \alpha < \beta < \mathfrak{c}^+\} \times K) \cup \{p_A : A \in \mathcal{A}(D), D \in \bigcup\{\mathcal{D}_\beta : \beta > \alpha\}\}$$

for each $\alpha \in L$.

Let us prove that X has all the required properties. It is immediate that X is a T_1 -space, so to prove that X is Tychonoff it suffices to show that it is zero-dimensional. We will establish a stronger fact, namely that all elements of the local bases defined above are clopen in X .

If $x = (\alpha, z) \in \mathfrak{c}^+ \times K$ and $U \in \mathcal{B}_x$, then $W = (\mathfrak{c}^+ \times K) \setminus U$ is open in $\mathfrak{c}^+ \times K$ and therefore W is a neighbourhood of any $y \in (\mathfrak{c}^+ \times K) \setminus U$. Given an $A \in \mathcal{A}(D)$ for some $D \in \mathcal{D}$, it is immediate from the definition of the local base at $y = p_A$ that $\{p_A\} \cup ((\mathfrak{c}^+ \times K) \setminus U)$ is an open neighbourhood of y which does not meet U . Now, if $y = p$, then $V_\beta \cap U = \emptyset$ for any $\beta \in L$, $\beta > \alpha$ and this finishes the proof that U is a clopen set.

Now we will show that, if $x = p_A$ for some $A \in \mathcal{A}(D)$, $D \in \mathcal{D}$, then every $U \in \mathcal{B}_x$ is a clopen set. Observe first that $W = (\mathfrak{c}^+ \times K) \setminus U$ is a clopen set in $\mathfrak{c}^+ \times K$ so every point from W is not in the closure of U . If $y = p_B$ is distinct from x , then either $B \in \mathcal{A}(D)$ or $i(A) \cap i(B)$ is finite because the family $\mathcal{D}$ is almost disjoint. Now observe that p_B belongs to the closure of a set $T \subset \mathfrak{c}^+ \times K$ if and only if T intersects infinitely many sets of the family $\{B(d) : d \in i(B)\}$. If $B \in \mathcal{A}(D)$ this is impossible for $T = A$ because the family $\mathcal{A}(D)$ is essentially disjoint and if $B \in \mathcal{A}(D')$ for some $D' \neq D$, then $i(B) \cap i(A)$ is finite being contained in the finite set $D \cap D'$. Finally, to see that p is not in the closure of U observe that $V_\alpha \cap U = \emptyset$ for any $\alpha \in L$ with $\alpha > \sup(D)$.

To establish that V_α is clopen for any $\alpha \in L$ observe that $V_\alpha \cap (\mathfrak{c}^+ \times K)$ is clopen in $\mathfrak{c}^+ \times K$ so no point of $(\mathfrak{c}^+ \times K) \setminus V_\alpha$ can be in the closure of V_α . If $p_A \notin V_\alpha$, then $A \in \mathcal{A}(D)$ for some $D \in \mathcal{D}_\beta$ with $\beta \leq \alpha$. By definition of $\mathcal{D}_\beta$ we have $\sup(D) \leq \alpha$ and therefore $\{p_A\} \cup (D \times K)$ is a neighbourhood of p_A which misses V_α . Thus we have finally proved that X is a Tychonoff zero-dimensional space.

It is easy to see that $\mathcal{U} = \{\{\alpha\} \times K : \alpha < \mathfrak{c}^+\}$ is a family of clopen subsets of X such that $p \in \bigcup \mathcal{U}$ but $p \notin \bigcup \mathcal{U}'$ for any $\mathcal{U}' \subset \mathcal{U}$ with $|\mathcal{U}'| \leq \mathfrak{c}$. This shows that X has uncountable o -tightness.

To establish that X is pseudocompact take any discrete family $\{U_n : n \in \omega\}$ of non-empty open subsets of X . Since $\mathfrak{c}^+ \times K$ is open and dense in X we can assume that U_n is a clopen subset of $\mathfrak{c}^+ \times K$ for each $n \in \omega$. Since each $K_\alpha = \{\alpha\} \times K$ is compact, only finitely many U_n 's can intersect each K_α . Thus, choosing smaller clopen sets and passing to an appropriate infinite subfamily, we can construct a discrete family $\{V_n : n \in \omega\}$ of non-empty clopen subsets of the space X such that, for all $m, n \in \omega$, we have $|i(V_n)| = 1$, $i(V_n) \cap i(V_m) = \emptyset$ if $n \neq m$ and

the family $\{\pi_K(V_n) : n \in \omega\}$ is disjoint. By maximality of $\mathcal{D}$ there exists a set $D \in \mathcal{D}$ such that the set $D' = D \cap (\bigcup\{i(V_n) : n \in \omega\})$ is infinite. It is easy to see that the set $B = \bigcup\{V_n : i(V_n) \subset D'\}$ is admissible. By maximality of the family $\mathcal{A}(D)$ there exists $A \in \mathcal{A}(D)$ such that $A(\alpha) \cap B(\alpha) \neq \emptyset$ for infinitely many α 's. As a consequence, p_A is an accumulation point of the family $\{V_n : n \in \omega\}$, a contradiction. Thus X is pseudocompact.

Finally we show that X is a Whyburn space. It is evident that all points of X except for p are Fréchet–Urysohn points so we must only check the Whyburn property at p . Let T be the set $\{p_A : A \in \mathcal{A}(D), D \in \mathcal{D}\}$. Suppose that $p \in \overline{A} \setminus A$ for some $A \subset X$. If $p \in \overline{A \cap T}$, then $B = A \cap T$ is an almost closed subset of A with $B \rightarrow p$. If not, then $p \in \overline{A \cap (\mathfrak{c}^+ \times K)}$ and therefore the cardinality of the set $A' = A \cap (\mathfrak{c}^+ \times K)$ is equal to $\mathfrak{c}^+$. Since there are only $\mathfrak{c}$ points in K , there exists a $t \in K$ such that the set $B = \{\alpha < \mathfrak{c}^+ : (\alpha, t) \in A'\}$ has cardinality $\mathfrak{c}^+$. It is easy to check that $F = B \times \{t\}$ is an almost closed subset of A with $F \rightarrow p$. $\square$

2.4. Remark. Any countably compact regular space of character $\leq \omega_1$ is semiradial under $\text{MA} + \neg\text{CH}$ [Be2, Theorem 5']. Since every semiradial space is weakly Whyburn [Be1, Proposition 1], we conclude that, *under $\text{MA} + \neg\text{CH}$, every regular countably compact space of character $\leq \omega_1$ is weakly Whyburn.* In particular, the space $\{0, 1\}^{\omega_1}$ is weakly Whyburn if $\text{MA} + \neg\text{CH}$ holds. On the other hand, if CH is fulfilled, then the space $\beta\omega$ embeds in $\{0, 1\}^{\omega_1}$ as a closed subspace and it is known (see [TY]) that $\beta\omega$ is not weakly Whyburn. This shows that *it is independent of ZFC whether $\{0, 1\}^{\omega_1}$ is weakly Whyburn.*

2.5. Theorem. *Suppose that X is a Whyburn Lindelöf P -space. Then $t(X) \leq \omega_1$.*

Proof. Take any $A \subset X$ and any $x \in \overline{A}$. Fix an almost closed $F \subset A$ with $F \rightarrow x$. Let γ be a maximal disjoint family of open subsets of F such that $x \notin \overline{V}$ for any $V \in \gamma$. The family γ is uncountable because $\overline{F}$ is a P -space. Since $x \in \overline{\bigcup \gamma}$, there is an almost closed set $G \subset \bigcup \gamma$ such that $G \rightarrow x$; clearly G is also uncountable. For any $V \in \gamma$ such that $V \cap G \neq \emptyset$ choose a point $x_V \in V \cap G$. The set $Q = \{x_V : V \in \gamma \text{ and } V \cap G \neq \emptyset\}$ is discrete and again uncountable; it is easy to see that every subset of Q is closed in G . Therefore $Q \setminus U$ is closed and discrete in X for any open neighbourhood U of the point x . Since X is Lindelöf, the set $Q \setminus U$ is countable. As a consequence, if we choose ω_1 points from Q we have constructed a set $B \subset A$ with $|B| \leq \omega_1$ and $x \in \overline{B}$. $\square$

2.6. Example. There exists a weakly Whyburn regular Lindelöf P -space of arbitrarily big tightness. Thus, not every regular Lindelöf P -space is Whyburn.

Proof. Take any cardinal $\kappa \geq \omega$ and denote by Y the ordinal $\kappa^+ + 1$ with the order topology. The ω -modification X of the space Y is a Lindelöf P -space [Us] and has tightness κ^+ . Since the topology of X is stronger than the order topology of Y , the space X is scattered and hence it is weakly Whyburn [TY]. If $\kappa = \omega_1$, then X is not Whyburn because its tightness is equal to ω_2 while any Whyburn Lindelöf P -space has tightness $\leq \omega_1$ by Theorem 2.5. $\square$

2.7. Proposition. *Every P -space X of character $\leq \omega_1$ is Whyburn.*

Proof. Take any $A \subset X$ and any $x \in \overline{A} \setminus A$. Fix a local base $\mathcal{B} = \{U_\alpha : \alpha < \omega_1\}$ at the point x such that $U_\beta \subset U_\alpha$ whenever $\alpha < \beta$. Such a choice is possible because X is a P -space. Choose a point $x_\alpha \in U_\alpha \cap A$ for each $\alpha < \omega_1$. The set

$F = \{x_\alpha : \alpha < \omega_1\} \subset A$ is almost closed because each countable subset of X is closed and $F \rightarrow x$ because $\mathcal{B}$ is a local base at x . $\square$

2.8. Corollary. *Every Lindelöf P -space of pseudocharacter $\leq \omega_1$ is Whyburn. In particular, if X is a Lindelöf P -space and $|X| \leq \omega_1$, then X is a Whyburn space.*

Proof. Recall that all spaces under consideration are assumed to be Hausdorff. A standard proof shows that every Hausdorff Lindelöf P -space is regular. It is also routine to prove that if a regular Lindelöf P -space has pseudocharacter $\leq \omega_1$, then its character is $\leq \omega_1$. Now apply Proposition 2.7. $\square$

Let L be a linearly ordered topological space. We denote by L^* the Dedekind completion of the space L and by “ $<$ ” the order on both L and L^* . To avoid confusion as to where the intervals are taken, we denote these indicating the relevant set as a subindex. Thus, given a point $x \in L$, we have the intervals $(\leftarrow, x]_L = \{y \in L : y \leq x\}$, $(\leftarrow, x]_{L^*} = \{y \in L^* : y \leq x\}$, $[x, \rightarrow)_L = \{y \in L : x \leq y\}$ and $[x, \rightarrow)_{L^*} = \{y \in L^* : x \leq y\}$. The intervals of other types (e.g., $(x, y)_L$, $(x, \rightarrow)_{L^*}$, etc.) are defined analogously.

2.9. Definition. Call a linearly ordered space L *left incomplete at a point $x \in L$* if there is an L^* -almost closed set $F \subset (L^* \setminus L) \cap (\leftarrow, x)_{L^*}$ such that $F \rightarrow x$. Analogously, the space L is *right incomplete at a point $x \in L$* if there is an L^* -almost closed set $F \subset (L^* \setminus L) \cap (x, \rightarrow)_{L^*}$ such that $F \rightarrow x$. A linearly ordered space L is *club incomplete* if, for any $x \in L$ with $\chi(x, (\leftarrow, x]_L) > \omega$ ($\chi(x, [x, \rightarrow)_L) > \omega$), the space L is left (right) incomplete at the point x .

2.10. Theorem. *A linearly ordered topological space L is Whyburn if and only if it is club incomplete.*

Proof. Suppose that L is club incomplete, $A \subset L$ and $x \in \overline{A} \setminus A$. We can assume without loss of generality that $A \subset (\leftarrow, x)_L$ or $A \subset (x, \rightarrow)_L$. Since these cases are identical, we assume that $A \subset (\leftarrow, x)_L$. If $\chi(x, (\leftarrow, x]_L) = \omega$, then there is nothing to prove so suppose that $\chi(x, (\leftarrow, x]_L) = \kappa > \omega$ and fix an L^* -almost closed set $F \subset (L^* \setminus L) \cap (\leftarrow, x)_{L^*}$ such that $F \rightarrow x$. Using the fact that κ is a regular cardinal it is easy to construct points $\{x_\alpha, y_\alpha, z_\alpha : \alpha < \kappa\}$ with the following properties:

- (i) $x_\alpha \in A$ and $y_\alpha, z_\alpha \in F$ for each $\alpha < \kappa$;
- (ii) $y_\alpha < x_\alpha < z_\alpha$ for every $\alpha < \kappa$;
- (iii) if $\alpha < \beta < \kappa$, then $z_\alpha < y_\beta$;
- (iv) the set $B = \{x_\alpha : \alpha < \kappa\}$ is cofinal in $(\leftarrow, x)_L$.

We claim that $B \subset A$ is an L -almost closed set with $B \rightarrow x$. The fact that $x \in \overline{B}$ is an immediate consequence of (iv), so let us establish that B is almost closed. It is easy to see that every accumulation point $t < x$ of the set B is the supremum of an infinite $B' \subset B$. However, (ii) and (iii) imply that there is an infinite $F' \subset F$ such that $\sup(F') = t$. The set $F \cap (\leftarrow, t]_{L^*}$ being closed, we have $t \in F \subset L^* \setminus L$ and therefore $t \notin L$. As a consequence, no point of $L \setminus (B \cup \{x\})$ is in the L -closure of B so the set B is almost closed and we proved that L is a Whyburn space.

Now, if L is a Whyburn space, suppose, for example, that $\chi(x, (\leftarrow, x]_L) > \omega$. Then $x \in (\leftarrow, x)_L$ and therefore there is a discrete $D \subset (\leftarrow, x)_L$ with $x \in \overline{D}$ [DTTW, Theorem 2.10]. Since L is Whyburn, there exists an almost closed $B \subset D$ with $B \rightarrow x$. The set $F = (\text{cl}_{L^*}(B) \setminus B) \cap (\leftarrow, x)_{L^*}$ is an L^* -almost closed subset of $(L^* \setminus L) \cap (\leftarrow, x)_{L^*}$ and $F \rightarrow x$. Hence L is club incomplete. $\square$

Recall that X is a Lindelöf Σ -space if X is a continuous image of a space Y which can be perfectly mapped onto a second countable space.

2.11. Proposition. *Let L be a linearly ordered Lindelöf Σ -space. Then L is Whyburn if and only if it is first countable.*

Proof. Since tightness coincides with character in linearly ordered spaces (see [En, 3.12.4(d)]), it suffices to prove that if L has the Whyburn property, then $t(L) \leq \omega$. Suppose, then that $A \subset L$ and $x \in \overline{A}$. We can assume without loss of generality that $A \subset (\leftarrow, x)$ or $A \subset (x, \rightarrow)$; since these cases are identical we suppose that $A \subset (\leftarrow, x)$. There exists a discrete $D \subset A$ with $x \in \overline{D}$ [DTTW, Theorem 2.10]. Since L is Whyburn, there exists an almost closed $B \subset D$ with $B \rightarrow x$. Observe that, for any open neighbourhood U of the point x , the set $B \setminus U$ is closed and discrete in L ; since L is Lindelöf, $B \setminus U$ has to be countable. Furthermore $B \cup \{x\}$ is a Lindelöf Σ -space being closed in L . If $x \in \overline{C}$ for some countable $C \subset B$, then we are done. If not, then $B \cup \{x\}$ is homeomorphic to one-point Lindelöfication of an uncountable discrete space and hence it cannot be a Lindelöf Σ -space (see e.g. [Ar, Proposition IV.6.15]). This contradiction shows that $t(L) \leq \omega$ and therefore L is first countable. $\square$

The following example shows that Proposition 2.11 is not true if only Lindelöf property of L is assumed.

2.12. Example. There exists a Lindelöf linearly ordered Whyburn space of uncountable tightness.

Proof. Let $M = (\omega_1 \times [0, 1)) \cup \{\omega_1\}$. The order is lexicographic and ω_1 is the largest element of M . It is well-known that M is compact (and hence complete) and ω_1 is the unique point of M of uncountable character. If $L = M \setminus (\omega_1 \times \{0\})$, then $M = L^*$ and $F = \omega_1 \times \{0\}$ is an L^* -almost closed subset of $(L^* \setminus L) \cap (\leftarrow, \omega_1)_M$ with $F \rightarrow \omega_1$. Apply Theorem 2.10 to conclude that L is a Whyburn space. Since the subspace $L \setminus U$ is second countable for each open $U \ni \omega_1$, the space L is Lindelöf. Finally, if $A = \omega_1 \times \{1/2\}$, then $\omega_1 \in \overline{A}$ but no countable subset of A contains ω_1 in its closure. $\square$

3. OPEN PROBLEMS

The topics of this paper are still far from being exhausted. To illustrate this we present a list of open problems; their solutions will certainly require methods different from those developed in this paper.

3.1. Problem. Suppose that G is a pseudocompact Whyburn group. Must G be Fréchet–Urysohn? What if G is separable?

3.2. Problem. Let G be a (weakly) Whyburn group. Suppose that H is a topological group and $f : G \rightarrow H$ is an open continuous surjective group homomorphism. Is H necessarily a (weakly) Whyburn space?

3.3. Problem. Is it true that, under $\text{MA} + \neg\text{CH}$, any Hausdorff countably compact space of character $\leq \omega_1$ is weakly Whyburn? What if we omit “countably compact”?

3.4. Problem. Is it true that, under $\text{MA} + \neg\text{CH}$, any (Tychonoff) space of weight $\leq \omega_1$ is weakly Whyburn?

3.5. Problem. Is it true that every (regular Lindelöf) P -space is weakly Whyburn?

3.6. Problem. Is it true that every (regular) P -space of cardinality $\leq \omega_1$ is a (weakly) Whyburn space?

3.7. Problem. For which compact spaces X is the ω -modification of X weakly Whyburn? For example, is the ω -modification of $\beta\omega \setminus \omega$ a weakly Whyburn space?

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