FUNCTORIAL EQUATIONS FOR LEXICOGRAPHIC PRODUCTS

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ABSTRACT. We generalize the main result of an earlier paper by the authors (*Exponentiation in power series fields*, Proc. Amer. Math. Soc. **125** (1997), 3177–3183) concerning the convex embeddings of a chain Γ in a lexicographic power Δ^Γ . For a fixed non-empty chain Δ , we derive necessary and sufficient conditions for the existence of non-empty solutions Γ to each of the lexicographic functorial equations

$$(\Delta^\Gamma)^{\leq 0} \simeq \Gamma, \quad (\Delta^\Gamma) \simeq \Gamma \quad \text{and} \quad (\Delta^\Gamma)^{< 0} \simeq \Gamma.$$

1. INTRODUCTION

Let us recall the definition of lexicographic products of ordered sets. Let Γ and Δ_γ , $\gamma \in \Gamma$, be non-empty totally ordered sets. For every $\gamma \in \Gamma$, we fix a distinguished element $0_\gamma \in \Delta_\gamma$. The **support** of a family $a = (\delta_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} \Delta_\gamma$ is the set of all $\gamma \in \Gamma$ for which $\delta_\gamma \neq 0_\gamma$. We denote it by $\text{supp}(a)$. As a set, we define $\mathbf{H}_{\gamma \in \Gamma}(\Delta_\gamma, 0_\gamma)$ to be the set of all families $(\delta_\gamma)_{\gamma \in \Gamma}$ with well-ordered support (with respect to fixed distinguished elements 0_γ). To relax the notation, we shall write $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$ instead of $\mathbf{H}_{\gamma \in \Gamma}(\Delta_\gamma, 0_\gamma)$ once the distinguished elements 0_γ have been fixed. Then the **lexicographic order** on $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$ is defined as follows. Given $a = (\delta_\gamma)_{\gamma \in \Gamma}$ and $b = (\delta'_\gamma)_{\gamma \in \Gamma} \in \mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$, observe that $\text{supp}(a) \cup \text{supp}(b)$ is well ordered. Let γ_0 be the least of all elements $\gamma \in \text{supp}(a) \cup \text{supp}(b)$ for which $\delta_\gamma \neq \delta'_\gamma$. We set $a < b :\Leftrightarrow \delta_{\gamma_0} < \delta'_{\gamma_0}$. Then $(\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma, <)$ is a totally ordered set, the **lexicographic product** (or **Hahn product**) of the ordered sets Δ_γ . We shall always denote by 0 the sequence with empty support in $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$.

Note that if all Δ_γ are totally ordered abelian groups, then we can take the distinguished elements 0_γ to be the neutral elements of the groups Δ_γ . Defining addition on $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$ componentwise, we obtain a totally ordered abelian group $(\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma, +, 0 <)$.

Lexicographic exponentiation of chains. If $\Delta = \Delta_\gamma$ for every $\gamma \in \Gamma$, we fix a distinguished element in Δ (the same distinguished element for every $\gamma \in \Gamma$), and denote it by 0_Δ . In this case we denote $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$ by Δ^Γ , and call it the

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lexicographic power Δ^Γ (with respect to 0_Δ). In other words, Δ^Γ is the set

$$\{s; s : \Gamma \rightarrow \Delta \text{ such that } \text{supp}(s) \text{ is well ordered in } \Gamma\},$$

ordered lexicographically.

This exponentiation of chains has its own arithmetic. In this paper we study some of its aspects (cf. also [K] and [H-K-M]). Note that if Γ and Δ are infinite ordinals, then lexicographic exponentiation does *not* coincide with ordinal exponentiation (cf. [H]).

Lexicographic powers appear naturally in many contexts. For example, $\mathbb{N}^\mathbb{N}$ is the order type of the nonnegative reals, and $\mathbb{Z}^\mathbb{N}$ that of the irrationals (cf. [R]). Also, 2^Γ is (isomorphic to) the chain of all well-ordered subsets of Γ , ordered by inclusion. The chain $2^\mathbb{N}$ has been studied in [H].

However, the main motivating example for us was that of generalized power series fields. If k is a real closed field and G a totally ordered divisible abelian group, then the field $k((G))$ of power series with exponents in G and coefficients in k is again real closed. The unique order of $k((G))$ is precisely the chain k^G . It was while studying such fields that our interest in the present problems arose. In [K-K-S], we considered the problem of defining an exponential function on $K = k((G))$, that is, an isomorphism f of ordered groups $f : (K, +, 0, <) \rightarrow (K^{>0}, \cdot, 1, <)$. We showed that the existence of f would imply that of a **convex embedding** (that is, an embedding with convex image) of the chain $G^{<0}$ into the chain $k^{G^{<0}}$. On the other hand, we proved:

Theorem 1. *Let Γ and Δ be non-empty totally ordered sets without greatest element, and fix an element $0_\Delta \in \Delta$. Suppose that Γ' is a cofinal subset of Γ and that $\iota : \Gamma' \rightarrow \Delta^\Gamma$ is an order preserving embedding. Then the image $\iota\Gamma'$ is not convex in Δ^Γ .*

Now for any ordered field k , the chain k has no last element. Similarly, $G^{<0}$ has no last element if G is nontrivial and divisible. So, using Theorem 1 one establishes that no exponentiation is possible on generalized power series fields.

If we omit the conditions on Γ and Δ in Theorem 1, the situation changes drastically. In this paper, we study conditions on the chains Γ and Δ under which a convex embedding of Γ in Δ^Γ exists. In particular, we seek non-empty solutions Γ to the functorial equations:

$$(\Delta^\Gamma)^{\leq 0} \simeq \Gamma, \quad (\Delta^\Gamma) \simeq \Gamma, \quad \text{and} \quad (\Delta^\Gamma)^{<0} \simeq \Gamma$$

(if T is any totally ordered set and $0 \in T$ is any element, we denote by $T^{\leq 0}$ the initial segment (including 0), and by $T^{<0}$ the strict initial segment (excluding 0) determined by 0 in T). None of the three equations hold if both Δ and Γ have no last element (for the first, this is trivial, and for the second and third it follows from Theorem 1). In Section 2 we start by proving a strong generalization of Theorem 1 (cf. Theorem 2). In Section 3, for each of the three functorial equations, we give simple characterizations of those chains Δ for which non-empty solutions Γ exist. In Section 4 we study simultaneous solutions to all three equations.

2. NONEXISTENCE OF CONVEX EMBEDDINGS

In this section, we shall prove that Theorem 1 remains true in the case where Δ is arbitrary, but 0_Δ is not the last element of Δ . This will follow from the following more general result.

Theorem 2. *Let Γ and Δ_γ , $\gamma \in \Gamma$, be non-empty totally ordered sets. For every $\gamma \in \Gamma$, fix an element 0_γ which is not the last element in Δ_γ . Suppose that Γ has no last element and that Γ' is a cofinal subset of Γ . Then there is no convex embedding*

$$\iota : \Gamma' \rightarrow \mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma.$$

Proof. For every $\gamma \in \Gamma'$, we choose an element $1_\gamma \in \Delta_\gamma$ such that $1_\gamma > 0_\gamma$. Take $d = (d_\gamma)_{\gamma \in \Gamma}$. If S is a well-ordered subset of Γ' such that $d_\gamma = 0_\gamma$ for all $\gamma \in S$, then we set

$$d \oplus S := (d'_\gamma)_{\gamma \in \Gamma} \quad \text{with} \quad d'_\gamma = \begin{cases} d_\gamma & \text{for } \gamma \notin S, \\ 1_\gamma & \text{for } \gamma \in S. \end{cases}$$

Observe that the support of $d \oplus S$ is contained in $\text{supp}(d) \cup S$ and thus, it is again well ordered. Note also that

$$(1) \quad S' \subsetneq S \Rightarrow d \oplus S' < d \oplus S.$$

Indeed, let γ_0 be the least element in $S \setminus S'$. Then $(d \oplus S')_{\gamma_0} = 0_{\gamma_0} < 1_{\gamma_0} = (d \oplus S)_{\gamma_0}$. On the other hand, if $\gamma \in \Gamma$ and $\gamma < \gamma_0$, then $(d \oplus S')_\gamma = (d \oplus S)_\gamma$: if $\gamma \in S$, then $\gamma \in S'$ (by minimality of γ_0) and $(d \oplus S')_\gamma = 1_\gamma = (d \oplus S)_\gamma$; if $\gamma \notin S$, then $\gamma \notin S'$ and $(d \oplus S')_\gamma = d_\gamma = (d \oplus S)_\gamma$.

Now suppose that $\iota : \Gamma' \rightarrow \mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$ is an order preserving embedding such that the image $\iota\Gamma'$ is convex in $\mathbf{H}_{\gamma \in \Gamma} \Delta_\gamma$. We wish to deduce a contradiction. The idea of the proof is the following. Let ON denote the class of ordinal numbers. We shall define an infinite $\text{ON} \times \mathbb{N}$ matrix with coefficients in Γ' , such that each column $(\gamma_\nu^{(n)})_{\nu \in \text{ON}}$ is a strictly increasing sequence in Γ' . Since Γ' is a set, every column of this matrix will provide a contradiction at the end of the construction (cf. the figure).

$$\begin{pmatrix} \gamma_0^{(1)} & \dots & \boxed{\gamma_0^{(n)}} & \boxed{\gamma_0^{(n+1)}} & \dots \\ \vdots & & \vdots & \vdots & \\ \vdots & & \vdots & \vdots & \\ \gamma_\nu^{(1)} & \dots & \gamma_\nu^{(n)} & \gamma_\nu^{(n+1)} & \dots \\ \vdots & & \vdots & \vdots & \\ \gamma_\mu^{(1)} & \dots & \gamma_\mu^{(n)} = ? & \dots & \dots \\ \vdots & & & & \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

To get started, we have to define the first row of the matrix. We construct sequences $\beta^{(n)}$, $n \in \mathbb{N} \cup \{0\}$, and $\gamma_0^{(n)}$, $n \in \mathbb{N}$, in Γ' . We take an arbitrary $\beta^{(0)} \in \Gamma'$. Having constructed $\beta^{(n)}$, we choose $\gamma_0^{(n+1)}$ and $\beta^{(n+1)}$ as follows. Since Γ' has no last element, we can choose $\mu^{(n)}, \nu^{(n)} \in \Gamma'$ such that $\beta^{(n)} < \mu^{(n)} < \nu^{(n)}$. Hence,

$$\iota\beta^{(n)} < \iota\mu^{(n)} < \iota\nu^{(n)}.$$

Let $\sigma^{(n)} \in \Gamma$ be the least element in $\text{supp } \iota\beta^{(n)} \cup \text{supp } \iota\mu^{(n)}$ for which

$$(2) \quad (\iota\beta^{(n)})_{\sigma^{(n)}} < (\iota\mu^{(n)})_{\sigma^{(n)}},$$

and $\tau^{(n)} \in \Gamma$ the least element in $\text{supp } \iota\mu^{(n)} \cup \text{supp } \iota\nu^{(n)}$ for which

$$(3) \quad (\iota\mu^{(n)})_{\tau^{(n)}} < (\iota\nu^{(n)})_{\tau^{(n)}}.$$

Since Γ' is cofinal in Γ , we can choose $\beta^{(n+1)} \in \Gamma'$ such that

$$\beta^{(n+1)} \geq \max\{\sigma^{(n)}, \tau^{(n)}\}.$$

Further, we set

$$d^{(n+1)} := (d_\gamma^{(n+1)})_{\gamma \in \Gamma} \quad \text{with} \quad d_\gamma^{(n+1)} = \begin{cases} (\iota\mu^{(n)})_\gamma & \text{for } \gamma \leq \beta^{(n+1)}, \\ 0_\gamma & \text{for } \gamma > \beta^{(n+1)}. \end{cases}$$

Then by (2) and (3),

$$\iota\beta^{(n)} < d^{(n+1)} < \iota\nu^{(n)}.$$

Thus, $d^{(n+1)} \in \iota\Gamma'$ by convexity, and we can set

$$\gamma_0^{(n+1)} := \iota^{-1}d^{(n+1)}.$$

Now for every $n \in \mathbb{N}$ we have that $\beta^{(n)} < \gamma_0^{(n+1)}$, hence every well-ordered set $S \subset \Gamma'$ with smallest element $\gamma_0^{(n+1)}$ has the property that $(\iota\gamma_0^{(n)})_\gamma = d_\gamma^{(n)} = 0_\gamma$ for all $\gamma \in S$; and moreover,

$$\iota\gamma_0^{(n)} < \iota\gamma_0^{(n)} \oplus S < \iota\nu^{(n-1)}.$$

Thus, $\iota\gamma_0^{(n)} \oplus S \in \iota\Gamma'$ by convexity. Suppose now that for some ordinal number $\mu \geq 1$ we have chosen elements $\gamma_\nu^{(n)} \in \Gamma'$, $\nu < \mu$, $n \in \mathbb{N}$, such that for every fixed n , the sequence $(\gamma_\nu^{(n)})_{\nu < \mu}$ is strictly increasing. Then we set

$$\gamma_\mu^{(n)} := \iota^{-1}(\iota\gamma_0^{(n)} \oplus \{\gamma_\nu^{(n+1)} \mid \nu < \mu\}) \in \Gamma'$$

for every $n \in \mathbb{N}$. If $\lambda < \mu$, then $\{\gamma_\nu^{(n+1)} \mid \nu < \lambda\} \subsetneq \{\gamma_\nu^{(n+1)} \mid \nu < \mu\}$ and thus, $\gamma_\lambda^{(n)} < \gamma_\mu^{(n)}$ by (1). So for every ordinal number μ , the sequences $(\gamma_\nu^{(n)})_{\nu < \mu}$ can be extended. We obtain strictly increasing sequences of arbitrary length, contradicting the fact that their length is bounded by the cardinality of Γ . $\square$

Corollary 3. *Assume that 0_Δ is not the last element of Δ . If there is an embedding of Γ in Δ^Γ with convex image, then Γ has a last element.*

3. SOLUTIONS TO THE FUNCTORIAL EQUATIONS

We start with a few easy remarks and lemmas. Throughout, fix a chain Δ with distinguished element 0_Δ .

Remark 4. 1) If 0_Δ is last in Δ (respectively, least), then 0 is last in Δ^Γ (respectively, least), for any non-empty chain Γ .

2) Let I be chain, and C a non-empty convex subset of I . Let $c \in C$. Then the initial segment determined by c in C is a final segment of the initial segment determined by c in I .

Remark 5. If $\Delta^{<0_\Delta}$ has no last element, then also $(\Delta^\Gamma)^{<0}$ has no last element, for any chain Γ : If not, let s be last in $(\Delta^\Gamma)^{<0}$ and set $\gamma = \min \text{supp}(s)$. Then $s(\gamma) = \delta < 0_\Delta$. Take $\delta < \delta' < 0_\Delta$. Consider s' defined by $s'(\gamma) = \delta'$ and $s'(\gamma') = 0_\Delta$ if $\gamma' \neq \gamma$. Then $s' \in (\Delta^\Gamma)^{<0}$, but $s' > s$, a contradiction.

Lemma 6. *Let Γ and Γ' be chains, and suppose that $\phi : \Gamma \rightarrow \Gamma'$ is a chain embedding. Then ϕ lifts to a chain embedding*

$$\hat{\phi} : \Delta^\Gamma \rightarrow \Delta^{\Gamma'}.$$

Proof. For $s \in \Delta^\Gamma$ and $x \in \Gamma'$, set

$$\hat{\phi}(s)(x) = \begin{cases} 0_\Delta & \text{if } x \notin \text{Im } \phi, \\ s(\phi^{-1}(x)) & \text{if } x \in \text{Im } \phi \end{cases}$$

(here, $\text{Im } \phi$ denotes the image of ϕ). Now, it is straightforward to check the assertion of the lemma. $\square$

In view of this lemma, if F is a subchain of a chain Γ , then there is a natural identification of Δ^F as a subchain of Δ^Γ .

Lemma 7. *Let Γ be a chain and F a non-empty final segment of Γ . Then Δ^F is convex in Δ^Γ (and $0 \in \Delta^F$).*

Proof. Let $s_i \in \Delta^F$, and set $\gamma_i = \min \text{supp}(s_i) \in F$, for $i = 1, 2$. Let $s \in \Delta^\Gamma$ be such that $s_1 < s < s_2$. If $s = 0$, then $s \in \Delta^F$. So assume $s \neq 0$ and set $\gamma = \min \text{supp}(s)$. Suppose that $\gamma \notin F$. If $s > 0$, then $s(\gamma) > 0_\Delta$. On the other hand, $\gamma < \gamma_2$ (otherwise, $\gamma \in F$). Thus, $s > s_2$, a contradiction. Similarly, we argue that if $s < 0$, then $s < s_1$, a contradiction. Hence, $\min \text{supp}(s) \in F$. Since F is a final segment of Γ , this implies that $s \in \Delta^F$, which proves our assertion. $\square$

Corollary 8. *Assume that Γ has a last element. Then Δ embeds convexly in Δ^Γ , so that 0_Δ is mapped to $0 \in \Delta^\Gamma$. If moreover 0_Δ is last in Δ , then Δ^F embeds as a final segment in Δ^Γ , for any non-empty final segment F of Γ . Consequently, if Γ has a last element, and 0_Δ is last in Δ , then Δ embeds as a final segment in Δ^Γ .*

Proof. The first assertion follows from Lemma 7, applied to the final segment consisting of the single last element of Γ . For the second assertion use Remark 4, parts 1) and 2). $\square$

We now give a complete solution to the **first functorial equation**, and a sufficient condition for the existence of solutions Γ to the third functorial equation:

Theorem 9. *There is always a non-empty solution Γ for the functorial equation $(\Delta^\Gamma)^{\leq 0} \simeq \Gamma$. If $\Delta^{<0_\Delta}$ has a last element, then there is also a non-empty solution Γ for $(\Delta^\Gamma)^{<0} \simeq \Gamma$.*

Proof. Set $\Gamma_0 := \Delta^{\leq 0_\Delta}$. Since Γ_0 has a last element, Δ embeds convexly in Δ^{Γ_0} . Consequently, Γ_0 embeds as a final segment in $\Gamma_1 := (\Delta^{\Gamma_0})^{\leq 0}$. By induction on $n \in \mathbb{N}$ we define $\Gamma_n := (\Delta^{\Gamma_{n-1}})^{\leq 0}$, and obtain an embedding of Γ_{n-1} as a final segment in Γ_n . We set $\Gamma := \bigcup_{n \in \mathbb{N}} \Gamma_n$.

Since every Γ_n is a final segment of Γ , every well-ordered subset S of Γ is already contained in some Γ_n (just take n such that the first element of S lies in Γ_n). Hence, an element of $(\Delta^\Gamma)^{\leq 0}$ with support S is actually an element of $\Gamma_{n+1} = (\Delta^{\Gamma_n})^{\leq 0}$, for some n . This fact gives rise to an order isomorphism of $(\Delta^\Gamma)^{\leq 0}$ onto Γ .

To prove the second assertion, we set $\Gamma_0 := \Delta^{<0_\Delta}$. Since Γ_0 has a last element by assumption, Δ embeds convexly in Δ^{Γ_0} , and the same arguments as above work if we define $\Gamma_n := (\Delta^{\Gamma_{n-1}})^{<0}$. $\square$

Remark 10. Note that Γ_0 has a last element and embeds as a final segment in the constructed solution Γ (in both cases considered in the proof). Thus, Γ has a last element, and there is no contradiction to Theorem 2.

Note that if 0_Δ is least in Δ , then the first equation has the trivial solution $\Gamma = \{0_\Delta\}$.

We next turn to the **second functorial equation**.

Remark 11. Suppose that 0_Δ is last in Δ . Then the solution to the first equation given in Theorem 9 also solves the second equation. Indeed, in this case, 0 is last in Δ^Γ , so $(\Delta^\Gamma)^{\leq 0} = \Delta^\Gamma$.

We also have the converse:

Corollary 12. *Assume Δ is a chain such that the functorial equation $\Delta^\Gamma \simeq \Gamma$ has a non-empty solution Γ . Then 0_Δ is last in Δ . Thus, the functorial equation $\Delta^\Gamma \simeq \Gamma$ has a non-empty solution if and only if 0_Δ is last in Δ .*

Proof. Assume 0_Δ is not last, and choose some element $1_\Delta > 0_\Delta$. This provides us with characteristic functions. If $S \subset \Gamma$ is well ordered, then let $\chi_S \in \Delta^\Gamma$ denote the characteristic function on S defined by:

$$\chi_S(\gamma) = \begin{cases} 1_\Delta & \text{if } \gamma \in S, \\ 0_\Delta & \text{if } \gamma \notin S. \end{cases}$$

Note that these characteristic functions reflect inclusion: if S is a proper well-ordered subset of S' , then $\chi_S < \chi_{S'}$. Now assume for a contradiction that $i : \Gamma \simeq \Delta^\Gamma$, and let $\kappa = \text{card}(\Gamma)$. We shall construct a strictly increasing sequence $\{\gamma_\mu; \mu < \kappa^+\}$ in Γ .

Set $\gamma_0 = i^{-1}(0)$, and assume by induction that $\{\gamma_\nu; \nu < \mu\}$ is defined, and strictly increasing in Γ . Then define

$$\gamma_\mu = i^{-1}(\chi_{\{\gamma_\nu; \nu < \mu\}}).$$

It follows that $\chi_{\{\gamma_\lambda; \lambda < \nu\}} < \chi_{\{\gamma_\lambda; \lambda < \mu\}}$, whenever $\nu < \mu$. Since i^{-1} is order preserving, it follows that $\gamma_\nu < \gamma_\mu$ as required. $\square$

We now turn to the **third functorial equation**. We deduce a simple criterion for the existence of solutions:

Corollary 13. *Assume that 0_Δ is not the last element of Δ . Then the functorial equation $(\Delta^\Gamma)^{<0} \simeq \Gamma$ has a non-empty solution Γ if and only if $\Delta^{<0_\Delta}$ has a last element.*

Proof. The “if” direction is just the second assertion of Theorem 9. So assume now that Γ is a non-empty solution. Assume for a contradiction that $\Delta^{<0_\Delta}$ has no last element. Then by Remark 5 $(\Delta^\Gamma)^{<0}$ has no last element as well. Thus, the same holds for the solution Γ . This contradicts Theorem 2. $\square$

4. SIMULTANEOUS SOLUTIONS

Recall that by Remark 11, the chain Γ given in Theorem 9 solves the first *and* the second functorial equations, if 0_Δ is last in Δ . By ω^* we denote the ordinal ω with the reverse ordering.

Theorem 14. *Assume that 0_Δ is last in Δ and that ω^* embeds as a final segment in Δ . Then the solution Γ given in Theorem 9 to the first and second functorial equations solves $(\Delta^\Gamma)^{<0} \simeq \Gamma$ as well.*

Proof. Recall that Δ embeds as a final segment in the given solution Γ . Thus, ω^* embeds as a final segment in Γ as well. In particular, Γ has a last element 0. Since $\Delta^\Gamma = (\Delta^\Gamma)^{<0} \cup \{0\}$ and $\Delta^\Gamma \simeq \Gamma$, we find that $(\Delta^\Gamma)^{<0} \simeq \Gamma \setminus \{0\}$. But $\Gamma \simeq \Gamma \setminus \{0\}$, since ω^* is a final segment of Γ . $\square$

We now turn to the question of whether the sufficient conditions given in this last theorem are also necessary. We need to introduce a definition: Say that a solution Γ (to any of the three equations) is **special** if Δ embeds as a final segment in Γ . Note that special solutions are necessarily non-empty.

Proposition 15. *Every non-empty solution to $\Gamma \simeq \Delta^\Gamma$ is special.*

Proof. Necessarily, 0_Δ is last in Δ (by Corollary 12). Thus, Γ has a last element, so by Corollary 8, Δ embeds as a final segment in Δ^Γ , and thus in Γ . $\square$

Corollary 16. *Assume that Δ is infinite and Γ is any non-empty chain which solves simultaneously*

$$(\Delta^\Gamma)^{<0} \simeq \Gamma \simeq \Delta^\Gamma.$$

Then 0_Δ is last in Δ and ω^* embeds as a final segment in Δ .

Proof. Since $\Gamma \simeq \Delta^\Gamma$, 0_Δ is last in Δ (Corollary 12). Therefore, 0 is last in Δ^Γ by Remark 4, and so also Γ has a last element 0. The assumptions imply that $\Gamma \setminus \{0\} \simeq \Gamma$. This is equivalent to the assertion that ω^* embeds as a final segment in Γ . Now note that Γ is a special solution by Proposition 15, i.e., Δ embeds as a final segment of Γ . Since Δ is infinite this implies that ω^* embeds as a final segment in Δ , as required. $\square$

Corollary 17. *Assume that Δ is infinite. Then the following are equivalent:*

- (a) 0_Δ is last in Δ and ω^* embeds as a final segment in Δ .
- (b) There exists a (special) simultaneous solution to all three equations.
- (c) There exists a (special) simultaneous solution to the second and third equations.

Proof. (a) implies (b) by Theorem 14. (b) implies (c) trivially. Finally, (c) implies (a) by Corollary 16. $\square$

We conclude with the following question: *Are special solutions unique up to isomorphism?* We can give a partial answer to this last question:

Proposition 18. *Assume that 0_Δ is last in Δ . Let $\Gamma = \bigcup \Gamma_n$ be the solution to the second equation given in Theorem 9. Then Γ embeds as a final segment in any other solution.*

Proof. Let Γ' be another solution. Then it is a special solution, by Proposition 15. So $\Delta = \Gamma_0$ embeds as a final segment in Γ' . Since 0_Δ is last in Δ , $\Gamma_1 = \Delta^{\Gamma_0}$ embeds as a final segment in $\Delta^{\Gamma'}$. By induction, Γ_n is a final segment of Γ' for every $n \in \mathbb{N}$. Thus, Γ embeds as a final segment in Γ' as well. $\square$

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