$C^{2,\alpha}$ ESTIMATE OF A PARABOLIC MONGE-AMPÈRE EQUATION ON S^n

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ABSTRACT. We consider a special type of parabolic Monge-Ampère equation on S^n arising from convex hypersurfaces expansion in Euclidean spaces. We obtained a $C^{2,\alpha}$ parabolic estimate of the support functions for the convex hypersurfaces assuming that we have already had a C^2 parabolic estimate.

1. INTRODUCTION

Consider the evolution of a smooth closed convex hypersurface M_0 (we may assume it encloses the origin in its interior by translation) in $\mathbb{R}^{n+1}$ by a function $F\left(\frac{1}{K}\right)$ of its Gauss curvature K . Parametrize M_0 by a smooth map $X_0 : S^n \rightarrow \mathbb{R}^{n+1}$. The evolution is equivalent to the following initial value problem for $X : S^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$:

$$(1.1) \quad \begin{cases} \frac{\partial X}{\partial t}(x, t) = F\left(\frac{1}{K(x, t)}\right) \cdot N, & (x, t) \in S^n \times [0, T), \\ X(x, 0) = X_0(x), & x \in S^n, \end{cases}$$

where K is the Gauss curvature, N is the outward unit normal, and $F : (0, \infty) \rightarrow \mathbb{R}$ is an arbitrary smooth increasing function, i.e., $F' > 0$ everywhere, in order to satisfy the parabolic condition.

Assuming that the hypersurface $M_t := X(\cdot, t)$ remains convex for all $t \in [0, T)$, where $T > 0$ is some short time, it is well known that, under the evolution equation (1.1), the evolution of the support function, which is defined as

$$(1.2) \quad u(N, t) = \langle P, N \rangle : N \in S^n \rightarrow \mathbb{R},$$

where $P \in M_t$ is the unique point on the convex hypersurface M_t such that the outward unit normal at P is N , is given by the equation

$$(1.3) \quad \frac{\partial u}{\partial t} = F\left(\frac{1}{K}\right).$$

Recall that the Gauss curvature is given in terms of u by the formula

$$(1.4) \quad \frac{1}{K} = \frac{\det h_{ij}}{\det g_{ij}}, \quad h_{ij} = \nabla_i \nabla_j u + u g_{ij} > 0,$$

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where g_{ij} is the standard metric on S^n and ∇ is the covariant differentiation. The eigenvalues $\frac{1}{\kappa_1}, \frac{1}{\kappa_2}, \dots, \frac{1}{\kappa_n}$ of the symmetric 2-tensor h_{ij} are the principal radii of the convex hypersurface M_t . Hence (1.3) becomes the following scalar parabolic Monge-Ampère equation for $u(x, t)$ (see [U]):

$$(1.5) \quad \begin{cases} \frac{\partial u(x, t)}{\partial t} = F \left(\frac{\det(\nabla_i \nabla_j u + u g_{ij})}{\det g_{ij}} \right), & (x, t) \in S^n \times [0, T), \\ u(x, 0) = u_0(x) > 0, & x \in S^n. \end{cases}$$

Here $u_0(x) > 0$ is the support function of the initial convex hypersurface M_0 satisfying

$$\nabla_i \nabla_j u_0 + u_0 g_{ij} > 0$$

on S^n .

The first result regarding equation (1.5) is the following gradient estimate in Chow [C] and [CG]. Its significance is that it holds for any arbitrary smooth increasing function F .

Proposition 1.1. *There exists a positive constant C depending only on M_0 such that, as long as the solution to (1.5) exists and $h_{ij} > 0$ on $S^n \times [0, T)$, the support function satisfies the uniform gradient estimate*

$$(1.6) \quad |\nabla u(x, t)| \leq C,$$

for all $(x, t) \in S^n \times [0, T)$.

From now on we shall focus more on the analytic aspects of equation (1.5). Long-time behavior of the solution $u(x, t)$ (or the behavior of the expanding geometry of M_t) has been studied in [CT] under suitable assumptions on F in the case when $F > 0$ on $(0, \infty)$.

There has been a long history on the study of Gauss curvature flows (contracting or expanding), i.e., on the study of equation (1.1) for variable choices of F . We refer the readers to the recent papers by Ben Andrews [AN] for literature.

The purpose of this paper is to answer a Schauder-type question posed in [CT]. The question is: assuming we have already had a $\tilde{C}^2(S^n \times [0, T))$ estimate on solution u to equation (1.5), does it imply a $\tilde{C}^{2, \alpha}(S^n \times [0, T))$ estimate? As we shall see below, it is equivalent to the following question:

Question. *Assume $u \in \tilde{C}^2(S^n \times [0, T))$ and*

$$(1.7) \quad \det(\nabla_i \nabla_j u + u g_{ij}) \in \tilde{C}^{0, \alpha}(S^n \times [0, T)).$$

Does it imply $u \in \tilde{C}^{2, \alpha}(S^n \times [0, T))$?

Throughout this paper we shall assume the following convexity property: there exist a constant $C > 0$ such that

$$(1.8) \quad \sup_{S^n \times [0, T)} |u| < C \quad \text{and} \quad 0 < \frac{1}{C} g_{ij} \leq h_{ij}(x, t) \leq C g_{ij},$$

on $S^n \times [0, T)$, where $h_{ij}(x, t) = \nabla_i \nabla_j u(x, t) + u(x, t) g_{ij}$.

As a consequence of assumption (1.8) and Proposition 1.1, we have the $\tilde{C}^2$ estimate for u :

$$(1.9) \quad \|u\|_{\tilde{C}^2(S^n \times [0, T))} \leq C,$$

where C is a positive constant depending on F , the initial condition $u_0(x)$, and the constant C in (1.8).

We want to show that under assumption (1.8), we have the $\tilde{C}^{2,\alpha}(S^n \times [0, T])$ estimate of u . As a consequence of the standard parabolic theory, all the higher derivatives of u remain bounded as $t \rightarrow T$.

2. STATEMENT AND PROOF OF THE MAIN THEOREM

2.1. Definition of $C^{k,\alpha}$ and $\tilde{C}^{k,\alpha}$ spaces. In the following discussion, we first introduce the Banach spaces of k -times (Hölder) continuously differentiable functions on S^n and $S^n \times I$ and their corresponding $C^{k,\alpha}$, $\tilde{C}^{k,\alpha}$ Hölder norms, where $I \subset \mathbb{R}$ is an interval.

Let $k \geq 0$ be a nonnegative integer and $\alpha \in (0, 1]$. The space $C^k(S^n)$ is the Banach space of real valued functions $u(x)$ on S^n which are k -times continuously differentiable, equipped with the norm

$$(2.10) \quad \|u\|_{C^k(S^n)} = \sup_{0 \leq |\beta| \leq k} \sup_{S^n} |\nabla^\beta u|.$$

$C^{k,\alpha}(S^n)$ is the subspace of functions in $C^k(S^n)$ such that the norm

$$(2.11) \quad \|u\|_{C^{k,\alpha}(S^n)} = \|u\|_{C^k(S^n)} + \sup_{|\beta|=k} [\nabla^\beta u]_{C^\alpha(S^n)}$$

is finite. Here

$$(2.12) \quad [\nabla^\beta u]_{C^\alpha(S^n)} = \sup_{x, y \in S^n, x \neq y} \frac{|\nabla^\beta u(x) - \nabla^\beta u(y)|}{|x - y|^\alpha}$$

and $|x - y|$ denote the distance between x and y in S^n .

We denote by $\tilde{C}^k(S^n \times I)$ the space of real valued functions $u(x, t)$ on $S^n \times I$ which are k -times continuously differentiable with respect to $x \in S^n$, and $[k/2]$ -times continuously differentiable with respect to $t \in I$, such that the norm

$$(2.13) \quad \|u\|_{\tilde{C}^k(S^n \times I)} = \sup_{0 \leq |\beta| + 2r \leq k} \sup_{S^n \times I} \left| \nabla^\beta \left(\frac{\partial^r u}{\partial t^r} \right) \right|$$

is finite. We denote by $\tilde{C}^{k,\alpha}(S^n \times I)$ the subspace of functions in $\tilde{C}^k(S^n \times I)$ such that the norm

$$(2.14) \quad \|u\|_{\tilde{C}^{k,\alpha}(S^n \times I)} = \|u\|_{\tilde{C}^k(S^n \times I)} + \sup_{|\beta| + 2r = k} \left[\nabla^\beta \left(\frac{\partial^r u}{\partial t^r} \right) \right]_{\tilde{C}^\alpha(S^n \times I)}$$

is finite, where

$$(2.15) \quad \left[\nabla^\beta \left(\frac{\partial^r u}{\partial t^r} \right) \right]_{\tilde{C}^\alpha(S^n \times I)} = \sup_{\substack{(x,s), (y,t) \in S^n \times I, \\ (x,s) \neq (y,t)}} \frac{|\nabla^\beta \left(\frac{\partial^r u}{\partial t^r} \right)(x, s) - \nabla^\beta \left(\frac{\partial^r u}{\partial t^r} \right)(y, t)|}{(|x - y|^2 + |s - t|)^{\frac{\alpha}{2}}}.$$

By the above definition, if we have $\|u\|_{\tilde{C}^2(S^n \times I)} \leq C$ and

$$(2.16) \quad \left\| \frac{\partial u}{\partial t} \right\|_{\tilde{C}^{0,\alpha}(S^n \times I)} + \|\nabla^2 u\|_{\tilde{C}^{0,\alpha}(S^n \times I)} \leq C,$$

then $\|u\|_{\tilde{C}^{2,\alpha}(S^n \times I)} \leq C$.

The following two lemmas are elementary:

Lemma 2.2. (1) Let I be an interval and let C_1, C_2 be constants. If for some $\alpha > 0$ we have $[f(\cdot, t)]_{C^\alpha(S^n)} \leq C_1$, for all $t \in I$ and for some $\beta > 0$ we have $[f(x, \cdot)]_{C^{\beta/2}(I)} \leq C_2$, for all $x \in S^n$, then we have

$$(2.17) \quad [f]_{\tilde{C}^\gamma(S^n \times I)} \leq C(C_1 + C_2),$$

where $\gamma = \min(\alpha, \beta)$ and $C = C(\alpha, \beta, \text{diam}(S^n), |I|)$ is a constant. In particular, if $\alpha = \beta$, then $C = 1$.

(2) If we have

$$(2.18) \quad \|f(\cdot, t)\|_{C^{0,\alpha}(S^n)} \leq C_1 \quad \text{and} \quad \|g(\cdot, t)\|_{C^{0,\alpha}(S^n)} \leq C_2$$

for all $t \in I$, then we have

$$(2.19) \quad \|fg(\cdot, t)\|_{C^{0,\alpha}(S^n)} \leq 2C_1C_2 \quad \text{for each } t \in I.$$

A similar statement holds when we fix the space variable.

By (2.17), to estimate the $\tilde{C}^{0,\alpha}$ norm of a function, we can fix either space variable or time variable.

Lemma 2.3. If $f(x, t) \in \tilde{C}^{0,\alpha}(S^n \times [0, T])$ for some $\alpha > 0$, then

$$(2.20) \quad g_h(x, t) := \frac{f(x, t+h) - f(x, t)}{|h|^{\frac{\alpha}{4}}} \in \tilde{C}^{0, \frac{\alpha}{4}}(S^n \times [0, T-h]),$$

where $0 < h < 1$ is a fixed small positive number. Moreover there is a constant C independent of h such that

$$\|g_h\|_{\tilde{C}^{0, \frac{\alpha}{4}}(S^n \times [0, T-h])} \leq C.$$

As a consequence, we also have

$$\|g_h(\cdot, t)\|_{C^{0, \frac{\alpha}{4}}(S^n)} \leq C,$$

where C is a constant independent of $t \in [0, T-h]$ and h .

The readers may consult the book [LSU] for the definition of parabolic norms and the proof of Lemmas 2.2 and 2.3.

2.2. $\tilde{C}^{2,\alpha}$ estimate of u . From now on, we shall use C to denote any positive constant which may depend on the final time T and other quantities (say $F, u_0, \sup_{S^n \times [0, T]} |u|, \dots$, etc.). The point is that C remains finite even when t approaches the final time T . C may change from line to line.

Recall that in [CT] the speed function $F = F(1/K)$ satisfies the equation

$$(2.21) \quad \frac{\partial F}{\partial t} = \frac{F'}{K} \square F + \frac{HF'}{K} F,$$

where $\square = h^{ij} \nabla_i \nabla_j$ is an elliptic operator on S^n and H is the mean curvature of the hypersurface. By (1.8), (2.21) may be considered as a uniformly parabolic linear equation with bounded coefficients on $S^n \times [0, T]$. Hence by Krylov-Safonov [KS], we have the Hölder estimate

$$(2.22) \quad F(1/K) \in \tilde{C}^{0,\alpha}(S^n \times [\varepsilon, T])$$

for some $0 < \alpha < 1$ and some small $\varepsilon > 0$. Since F is a smooth function and K is bounded away from zero, it implies

$$(2.23) \quad \|K\|_{\tilde{C}^{0,\alpha}(S^n \times [\varepsilon, T])} \leq C \quad \text{and} \quad \left\| \frac{1}{K} \right\|_{\tilde{C}^{0,\alpha}(S^n \times [\varepsilon, T])} \leq C,$$

and hence

$$(2.24) \quad \frac{1}{K} = \det(\nabla_i \nabla_j u + u g_{ij}) \in \tilde{C}^{0,\alpha}(S^n \times [\varepsilon, T]).$$

Theorem 2.4. *We have the estimate*

$$(2.25) \quad \left\| \frac{\partial u}{\partial t} \right\|_{\tilde{C}^{0,\alpha}(S^n \times [\varepsilon, T])} + \|\nabla^2 u\|_{\tilde{C}^{0,\alpha}(S^n \times [\varepsilon, T])} \leq C,$$

for some constants $\alpha \in (0, 1)$ and C . Here C depends on F , $\|u\|_{\tilde{C}^2(S^n \times [0, T])}$, T and $\frac{1}{\varepsilon}$.

Note that the constant α in (2.25) may not be the same as the constant α in (2.24), but this is not important. As long as we have (2.25) for certain $\alpha > 0$, we are able to apply the standard parabolic theory to conclude that all the higher derivatives of u remain bounded as $t \rightarrow T$.

Proof. Let $I = [\varepsilon, T]$ throughout the proof. Since we already have

$$\left\| \frac{\partial u}{\partial t} \right\|_{\tilde{C}^{0,\alpha}(S^n \times I)} = \left\| F\left(\frac{1}{K}\right) \right\|_{\tilde{C}^{0,\alpha}(S^n \times I)} \leq C,$$

it suffices to prove that $\|\nabla^2 u\|_{\tilde{C}^\alpha(S^n \times I)} \leq C$. Since we know $1/K \in \tilde{C}^{0,\alpha}(S^n \times I)$, we have

$$(2.26) \quad \begin{aligned} & \left\| \frac{1}{K} \right\|_{\tilde{C}^\alpha(S^n \times I)} \\ &= \sup_{\substack{(x,s), (y,t) \in S^n \times I, \\ (x,s) \neq (y,t)}} \frac{|\det(\nabla_i \nabla_j u + u g_{ij})(x, s) - \det(\nabla_i \nabla_j u + u g_{ij})(y, t)|}{(|x - y|^2 + |s - t|)^{\frac{\alpha}{2}}} \\ &\leq C. \end{aligned}$$

(2.26) implies that for each fixed time $t \in I$, we have the space norm estimate

$$(2.27) \quad \|\det(\nabla_i \nabla_j u + u g_{ij})(\cdot, t)\|_{C^{0,\alpha}(S^n)} \leq C,$$

where C is a constant independent of $t \in [\varepsilon, T]$. As K is uniformly bounded, we can apply Theorem 4, p. 149 of Caffarelli [CA] to obtain the space norm estimate

$$(2.28) \quad \|u(\cdot, t)\|_{C^{2,\alpha}(S^n)} \leq C,$$

where C is independent of $t \in I$. Therefore in view of (1) of Lemma 2.2, it remains to show that

$$(2.29) \quad \sup_{\substack{s, t \in I \\ s \neq t}} \frac{|\nabla_i \nabla_j u(x, s) - \nabla_i \nabla_j u(x, t)|}{(|s - t|)^{\frac{\alpha}{2}}} \leq C,$$

where C is independent of $x \in S^n$. For a small $h > 0$, we consider the function $U_{s,h}(x, t)$ defined as

$$U_{s,h}(x, t) = su(x, t) + (1 - s)u(x, t + h), \quad 0 \leq s \leq 1,$$

where now the domain of $U_{s,h}(x, t)$ is $S^n \times [\varepsilon, T - h]$. We know

$$\begin{aligned} & \nabla_i \nabla_j U_{s,h}(x, t) + U_{s,h}(x, t) g_{ij} \\ &= s [\nabla_i \nabla_j u(x, t) + u(x, t) g_{ij}] + (1 - s) [\nabla_i \nabla_j u(x, t + h) + u(x, t + h) g_{ij}] > 0, \end{aligned}$$

for all $s \in [0, 1]$ and $h > 0$ small, and moreover we still have

$$(2.30) \quad 0 < \frac{1}{C} g_{ij} \leq \nabla_i \nabla_j U_{s,h}(x, t) + U_{s,h}(x, t) g_{ij} \leq C g_{ij}, \quad \text{on } S^n \times [\varepsilon, T - h],$$

where C is independent of s and h . Differentiation gives

$$\frac{d}{ds} U_{s,h}(x, t) = u(x, t) - u(x, t + h), \quad \text{on } S^n \times [\varepsilon, T - h],$$

for all $s \in [0, 1]$. Since we have

$$\|u(\cdot, t)\|_{C^{2,\alpha}(S^n)} \leq C,$$

where C is independent of h and $t \in [\varepsilon, T - h]$, by (2) of Lemma 2.2 it implies that for each fixed t

$$\begin{aligned} & \det [s (\nabla_i \nabla_j u(\cdot, t) + u(\cdot, t) g_{ij}) + (1 - s) (\nabla_i \nabla_j u(\cdot, t + h) + u(\cdot, t + h) g_{ij})] \\ &= \det [\nabla_i \nabla_j U_{s,h}(\cdot, t) + U_{s,h}(\cdot, t) g_{ij}] \in C^{0,\alpha}(S^n) \end{aligned}$$

and the $C^{0,\alpha}(S^n)$ norm of $\det (\nabla_i \nabla_j U_{s,h}(\cdot, t) + U_{s,h}(\cdot, t) g_{ij})$ is bounded by a constant C independent of $s \in [0, 1]$, h and $t \in [\varepsilon, T - h]$.

Now

$$\begin{aligned} (2.31) \quad & \log \det [\nabla_i \nabla_j u(x, t) + u(x, t) g_{ij}] - \log \det [\nabla_i \nabla_j u(x, t + h) + u(x, t + h) g_{ij}] \\ &= \log K(x, t + h) - \log K(x, t) \\ &= \int_0^1 \frac{d}{ds} (\log \det [\nabla_i \nabla_j U_{s,h}(x, t) + U_{s,h}(x, t) g_{ij}]) ds \\ &= \left(\int_0^1 \omega_{s,h}^{ij}(x, t) ds \right) [\nabla_i \nabla_j (u(x, t) - u(x, t + h)) + (u(x, t) - u(x, t + h)) g_{ij}], \end{aligned}$$

where

$$\begin{aligned} \omega_{s,h}^{ij}(x, t) &= \text{the inverse of } (\nabla_i \nabla_j U_{s,h}(x, t) + U_{s,h}(x, t) g_{ij}) > 0, \quad h > 0 \text{ small}, \\ \omega_{s,0}^{ij}(x, t) &= \text{the inverse of } (\nabla_i \nabla_j u(x, t) + u(x, t) g_{ij}) = h^{ij}(x, t). \end{aligned}$$

Moreover the $C^{0,\alpha}(S^n)$ norm of each $\omega_{s,h}^{ij}$ is also bounded by a constant C independent of $s \in [0, 1]$, h and $t \in [\varepsilon, T - h]$ and we still have

$$(2.32) \quad 0 < \frac{1}{C} g_{ij} \leq \omega_{s,h}^{ij}(x, t) \leq C g_{ij}, \quad \text{on } S^n \times [\varepsilon, T - h],$$

where C is independent of s and h . Let

$$\Omega_h^{ij}(x, t) = \int_0^1 \omega_{s,h}^{ij}(x, t) ds > 0, \quad \Omega_0^{ij}(x, t) = \int_0^1 \omega_{s,0}^{ij}(x, t) ds = h^{ij}(x, t) > 0.$$

We also have the $C^{0,\alpha}(S^n)$ norm of each Ω_h^{ij} bounded by a constant C independent of h and $t \in [\varepsilon, T-h]$. To see this, we use the inequality

$$(2.33) \quad \left[\Omega_h^{ij}(\cdot, t) \right]_{C^\alpha(S^n)} = \sup_{x, y \in S^n, x \neq y} \frac{\left| \int_0^1 \omega_{s,h}^{ij}(x, t) ds - \int_0^1 \omega_{s,h}^{ij}(y, t) ds \right|}{|x - y|^\alpha} \leq \int_0^1 \left[\omega_{s,h}^{ij}(\cdot, t) \right]_{C^\alpha(S^n)} ds \leq C.$$

Define the elliptic operator $L_h(u)$ at (x, t) as

$$(2.34) \quad L_h(u) = \Omega_h^{ij}(x, t) \nabla_i \nabla_j u(x, t).$$

We know that it is uniformly elliptic and its ellipticity is between two positive constants $\frac{1}{C}$ and C independent of x , t and h .

Rewrite (2.31) as the following uniformly elliptic equation on S^n :

$$(2.35) \quad \begin{aligned} & L_h \left(\frac{u(x, t) - u(x, t+h)}{|h|^{\alpha/4}} \right) + Tr(\Omega_h) \left(\frac{u(x, t) - u(x, t+h)}{|h|^{\alpha/4}} \right) \\ &= \frac{\log K(x, t+h) - \log K(x, t)}{|h|^{\alpha/4}}, \end{aligned}$$

where $Tr(\Omega_h) = \Omega_h^{ij}(x, t) g_{ij} \in C^{0,\alpha}(S^n)$ and the $C^{0,\alpha}(S^n)$ norm of $Tr(\Omega_h)(\cdot, t)$ is again bounded by a constant C independent of h and $t \in [\varepsilon, T-h]$. By Lemma 2.3 and the fact $K(x, t) \in \tilde{C}^{0,\alpha}(S^n \times [\varepsilon, T])$, we know

$$(2.36) \quad \frac{\log K(x, t+h) - \log K(x, t)}{|h|^{\alpha/4}} \in \tilde{C}^{0, \frac{\alpha}{4}}(S^n \times [\varepsilon, T-h]).$$

Moreover the $\tilde{C}^{0, \frac{\alpha}{4}}$ norm of $(\log K(x, t+h) - \log K(x, t)) / |h|^{\alpha/4}$ over the space $S^n \times [\varepsilon, T-h]$ remains bounded as $h \rightarrow 0$. As a consequence, the $C^{0, \frac{\alpha}{4}}(S^n)$ norm of $(\log K(\cdot, t+h) - \log K(\cdot, t)) / |h|^{\alpha/4}$ is also bounded by a constant C independent of $t \in [\varepsilon, T-h]$ and h .

The elliptic Schauder estimate applied to equation (2.35) on S^n yields

$$(2.37) \quad \sup_{x \in S^n} \left| \frac{\nabla_i \nabla_j u(x, t) - \nabla_i \nabla_j u(x, t+h)}{|h|^{\alpha/4}} \right| \leq C$$

for all $h > 0$ small, where C is independent of $t \in [\varepsilon, T-h]$ and h . Hence we obtain

$$(2.38) \quad \sup_{\substack{s, t \in [\varepsilon, T] \\ s \neq t}} \frac{|\nabla_i \nabla_j u(x, s) - \nabla_i \nabla_j u(x, t)|}{(|s - t|)^{\frac{\alpha}{4}}} \leq C,$$

where C is independent of $x \in S^n$. The proof of Theorem 1.1 is finished. $\square$

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