A NOTE ON THE SPECTRUM OF AN UPPER TRIANGULAR OPERATOR MATRIX

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ABSTRACT. Let $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ be a 2×2 upper triangular operator matrix acting on the Banach space $E \oplus F$. We investigate the set of the operators C for which $\sigma(M_C) = \sigma(A) \cup \sigma(B)$, where $\sigma(\cdot)$ denotes the spectrum.

Let E, F be complex Banach spaces and let $\mathcal{L}(F, E)$ be the space of all bounded linear operators from F to E . We abbreviate $\mathcal{L}(E, E)$ by $\mathcal{L}(E)$. When $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(F)$ and $C \in \mathcal{L}(F, E)$ are given, we denote by M_C an operator on $E \oplus F$ of the form $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$. Also, we let L_A and R_B denote the left (respectively right) multiplication by A (respectively B). The generalized derivation induced by the operators A and B is defined by $\delta_{A,B} = L_A - R_B$. When $E = F$ and $A = B$, we simply write $\delta_{A,A} = \delta_A$.

For $T \in \mathcal{L}(E)$, we let $\sigma_l(T)$, $\sigma_r(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, $r(T)$, $N(T)$ and $R(T)$ denote the left spectrum, right spectrum, left essential spectrum, right essential spectrum, spectral radius, null space and range of T respectively. If G is a subset of $\mathcal{L}(F, E)$, we denote $cl[G]$ as its closure.

For bounded linear operators A, B and C , the equality

$$(1) \quad \sigma(M_C) = \sigma(A) \cup \sigma(B)$$

was studied by numerous authors. In [4], it was shown that if $\sigma(A) \cap \sigma(B)$ has no interior points, then (1) is satisfied for every $C \in \mathcal{L}(F, E)$. In the Hilbert space setting, equality (1) was considered in [1, 4], where it was shown that if in particular A and B are normal operators, then (1) holds for every $C \in \mathcal{L}(F, E)$.

The aim of this note is the investigation of the set of operators $C \in \mathcal{L}(F, E)$ for which (1) holds.

Our main result is the following

Theorem 1. *If $A \in \mathcal{L}(E)$ and $B \in \mathcal{L}(F)$, then*

$$\sigma(M_C) = \sigma(A) \cup \sigma(B)$$

for every $C \in cl[R(\delta_{A,B}) + N(\delta_{A,B}) + \bigcup_{\lambda \in \mathbb{C}} N(L_{A-\lambda}) + \bigcup_{\lambda \in \mathbb{C}} N(R_{B-\lambda})]$.

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As a corollary of Theorem 1, we have

Corollary 2. *Assume that E and F are Hilbert spaces. If $A \in \mathcal{L}(E)$ and $B \in \mathcal{L}(F)$ satisfy the following conditions:*

- i) $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$,
- ii) *there exists no nonzero trace class operator X such that $BX = XA$,*

then

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \quad \text{for every } C \in \mathcal{L}(F, E).$$

Proof. Since $R(\delta_{A,B})$ is norm dense (see [2]), the result follows from Theorem 1. $\square$

For the proof of Theorem 1, we need some auxiliary results. We begin with the following well known lemma.

Lemma 3. *If $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(F)$ and $C \in \mathcal{L}(F, E)$, then*

$$(2) \quad \sigma_l(A) \cup \sigma_r(B) \subseteq \sigma(M_C) \subseteq \sigma(A) \cup \sigma(B).$$

Proof. For every $\lambda \in \mathbb{C}$, we have

$$(3) \quad M_C - \lambda = \begin{bmatrix} I & 0 \\ 0 & B - \lambda \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda & 0 \\ 0 & I \end{bmatrix}.$$

Since $\begin{bmatrix} I & C \\ 0 & I \end{bmatrix}$ is invertible, we derive that

$$\lambda \notin \sigma(A) \cup \sigma(B) \implies \lambda \notin \sigma(M_C) \implies \lambda \notin \sigma_l(A) \cup \sigma_r(B).$$

This completes the proof. $\square$

The inclusions in (2) are proper in general. To see this, consider the following example from [1].

Example 4. Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n \geq 1}$. Define the operators S and C by

$$Se_i = e_{i+1}, \quad i = 1, 2, \dots \text{ and } C = e_1 \otimes e_1 = \langle \cdot, e_1 \rangle e_1.$$

If S^* is the adjoint of S , then $\begin{bmatrix} S & C \\ 0 & S^* \end{bmatrix}$ is a unitary operator. Thus

$$\sigma\left(\begin{bmatrix} S & C \\ 0 & S^* \end{bmatrix}\right) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\} \subsetneq \sigma(S) \cup \sigma(S^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

Also, we have

$$\sigma_l(S) \cup \sigma_r(S^*) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \subsetneq \sigma\left(\begin{bmatrix} S & 0 \\ 0 & S^* \end{bmatrix}\right).$$

For operators $A \in \mathcal{L}(E)$ and $B \in \mathcal{L}(F)$, let

$$M(A, B) = \{C \in \mathcal{L}(F, E) : \sigma(M_C) = \sigma(A) \cup \sigma(B)\}.$$

It is clear that $0 \in M(A, B)$ and $M(A, B) = M(A - \lambda, B - \lambda)$ for all complex number λ . By using the operators S , S^* and C in Example 4, we see that the equality $M(A, B) = M(B, A)$ fails to be true in general.

The set $M(A, B)$ is norm closed. Indeed, let $\{C_n\}_n$ be a sequence in $M(A, B)$ such that $C_n \rightarrow C$ as $n \rightarrow \infty$ for some $C \in \mathcal{L}(F, E)$. Then $M_{C_n} \rightarrow M_C$ as $n \rightarrow \infty$. Thus, the upper semi-continuity of the spectrum (see [3, Problem 86]) implies that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$, that is, $C \in M(A, B)$.

The following result will be useful in the sequel.

Lemma 5. *Let $T, N \in \mathcal{L}(E)$ such that $N^2 = 0$ and $N \neq 0$. If either $TN = 0$ or $NT = 0$, then*

$$\sigma(T + N) = \sigma(T).$$

Proof. The proofs for the statements $TN = 0$ and $NT = 0$ are similar, therefore we may assume that $TN = 0$. Observe first that since $(T + N)N = 0$ and $N \neq 0$, then $0 \in \sigma(T + N) \cap \sigma(N)$. Next, let $\lambda \in \mathbb{C} \setminus \{0\}$. Since $TN = N^2 = 0$, we have that $N - \lambda$ is invertible, and

$$-\lambda^{-1}(T - \lambda)(N - \lambda) = T + N - \lambda.$$

From this, we conclude that $\lambda \in \sigma(T + N)$ if and only if $\lambda \in \sigma(T)$. Consequently,

$$\sigma(T + N) = \sigma(T).$$

The proof is finished. $\square$

The next lemma is well known, so its proof will be omitted.

Lemma 6. *Let $T, N \in \mathcal{L}(E)$ such that $\sigma(N) = \{0\}$ and $TN = NT$. Then $\sigma(T + N) = \sigma(T)$.*

Using Lemma 5 and Lemma 6 we prove

Lemma 7. *Let $C \in M(A, B)$ and $D \in \bigcup_{\lambda \in \mathbb{C}} N(L_{A-\lambda}) \cup \bigcup_{\lambda \in \mathbb{C}} N(R_{B-\lambda})$. Then $C + D \in M(A, B)$.*

Proof. Suppose $D \in N(L_{A-\lambda})$ for some λ ; then $(A - \lambda)D = 0$. Write $M_{C+D} - \lambda = M_C - \lambda + \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}$. Since $\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}^2 = 0$ and $(M_C - \lambda)\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} = 0$, we deduce from Lemma 5 that

$$\sigma(M_{C+D} - \lambda) = \sigma(M_C - \lambda) = \sigma(A - \lambda) \cup \sigma(B - \lambda),$$

that is, $C + D \in M(A, B)$.

By a similar argument, we show that if $D \in \bigcup_{\lambda \in \mathbb{C}} N(R_{B-\lambda})$, then $C + D \in M(A, B)$.

The proof is complete. $\square$

Lemma 8. *If $A \in \mathcal{L}(E)$ and $B \in \mathcal{L}(F)$, then*

$$R(\delta_{A,B}) + N(\delta_{A,B}) \subseteq M(A, B).$$

Proof. Let $C \in R(\delta_{A,B})$ and $D \in N(\delta_{A,B})$. A straightforward computation shows that $M_C \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} M_C$. Since $\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}^2 = 0$, Lemma 6 implies that

$$(4) \quad \sigma(M_{C+D}) = \sigma(M_C).$$

On the other hand, since $C \in R(\delta_{A,B})$, there is an operator $X \in \mathcal{L}(E)$ such that $C = AX - XB$. Therefore, M_C is similar to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, where the similarity is implemented by the invertible operator $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$. Consequently, we have

$$(5) \quad \sigma(M_C) = \sigma(A) \cup \sigma(B).$$

Combining (4) and (5) shows that $C + D \in M(A, B)$, which completes the proof. $\square$

Proof of Theorem 1. Let $C = C_1 + C_2 + C_3 + C_4$, where $C_1 \in R(\delta_{A,B})$, $C_2 \in N(\delta_{A,B})$, $C_3 \in \bigcup_{\lambda \in \mathbb{C}} N(L_{A-\lambda})$ and $C_4 \in \bigcup_{\lambda \in \mathbb{C}} N(R_{B-\lambda})$. By Lemma 8, $C_1 + C_2 \in M(A, B)$. Now Lemma 7 can be used twice. It shows that

$$\sigma(M_C) = \sigma(M_{C_1+C_2+C_3}) = \sigma(A) \cup \sigma(B).$$

Therefore the result follows by the closedness of $M(A, B)$. This ends the proof. $\square$

Remark 1. If given A, B and C such that (1) holds, then using the same reasoning as in the above we check that $C + D \in M(A, B)$ for every

$$D \in cl[R(\delta_{A,B}) + N(\delta_{A,B}) + \bigcup_{\lambda \in \mathbb{C}} N(L_{A-\lambda}) \cup \bigcup_{\lambda \in \mathbb{C}} N(R_{B-\lambda})].$$

As an application of Theorem 1 we have the next example.

Example 9. Let $A_1 \in \mathcal{L}(E)$ and $B_1 \in \mathcal{L}(F)$. If $A \in \mathcal{L}(E \oplus E)$ and $B \in \mathcal{L}(F \oplus F)$ are the operators defined by $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix}$, then $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{L}(F \oplus F, E \oplus E)$ of the form $\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$ such that

$$C_2 \in cl[R(\delta_{A_1, B_1}) + N(\delta_{A_1, B_1})].$$

Indeed, write

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & C_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C_3 & C_4 \end{bmatrix}.$$

Since

$$\begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

we have

$$\begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \in N(R_B).$$

Also, since $C_2 \in cl[R(\delta_{A_1, B_1}) + N(\delta_{A_1, B_1})]$, it follows that

$$\begin{bmatrix} 0 & C_2 \\ 0 & 0 \end{bmatrix} \in cl[R(\delta_{A,B}) + N(\delta_{A,B})].$$

Next, we have

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ C_3 & C_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 0 & 0 \\ C_3 & C_4 \end{bmatrix} \in N(L_A).$$

Consequently, Theorem 1 implies that $C \in M(A, B)$.

If, in particular, $A_1 = S$ and $B_1 = S^*$, where S is the operator defined in Example 4, we see that $\sigma_r(A) \cap \sigma_l(B)$ may be large in a certain sense.

We conclude with the next proposition.

Proposition 10. Let $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(F)$ and $C \in \mathcal{L}(F, E)$. Assume that there exists some $\lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C)$. Then $\sigma_r(A) \cap \sigma_l(B) \setminus (\sigma_l(A) \cup \sigma_r(B))$ contains a neighborhood of λ .

Proof. Without loss of generality we may assume that $\lambda = 0$. Since $0 \notin \sigma(M_C)$, Lemma 3 implies that $0 \notin \sigma_l(A)$. Hence there is an operator $U \in \mathcal{L}(E)$ such that $UA = I$ (I : the identity operator). Let $\epsilon_1 = \frac{1}{r(U)}$ and let $|\lambda| < \epsilon_1$. Then $I - \lambda U$ is invertible, and

$$(6) \quad U(I - \lambda U)^{-1}(A - \lambda) = I.$$

On the other hand we deduce from (3) that $AU \neq I$. Thus

$$(7) \quad (A - \lambda)U(I - \lambda U)^{-1} = (AU - \lambda U)(I - \lambda U)^{-1} \neq I.$$

From (6) and (7) we derive that

$$\lambda \in \sigma_r(A) \setminus \sigma_l(A).$$

Consequently, we have

$$B(0, \epsilon_1) \subseteq \sigma_r(A) \setminus \sigma_l(A).$$

By a similar argument, we claim that there exists $\epsilon_2 > 0$ such that

$$B(0, \epsilon_2) \subseteq \sigma_l(B) \setminus \sigma_r(B).$$

Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then

$$B(0, \epsilon) \subseteq \sigma_r(A) \cap \sigma_l(B) \setminus (\sigma_l(A) \cup \sigma_r(B)).$$

This ends the proof. $\square$

Corollary 11. *If $\sigma_r(A) \cap \sigma_l(B) \setminus (\sigma_l(A) \cup \sigma_r(B))$ has no interior points, then*

$$M(A, B) = \mathcal{L}(F, E).$$

If in particular $E = F$ and $A = B$, then

$$M(A, A) = \mathcal{L}(E).$$

Remark 2. Suppose E and F are Hilbert spaces. It is clear that, for given operators A and B , $\sigma_r(A) \cap \sigma_l(B) \setminus (\sigma_l(A) \cup \sigma_r(B)) \subseteq \sigma_p(A^*)^* \cap \sigma_p(B)$, where $\sigma_p(A^*)^*$ denotes the set of the complex conjugates of points of $\sigma_p(A^*)$. We conclude from Proposition 10 that if $\sigma_p(A^*)^* \cap \sigma_p(B)$ has no interior points, then $M(A, B) = \mathcal{L}(F, E)$.

Let us give an example showing that the converse of Theorem 1 is not true.

Example 12. Let $A \in \mathcal{L}(E)$ such that $A^2 = 0$ and $A \neq 0$. A simple computation shows that $\delta_A^3 = 0$. Thus $R(\delta_A) \subseteq N(\delta_A^2)$. If $\lambda \in \mathbb{C}$ and $X \in N(L_{A-\lambda})$, then $AX = \lambda X$. Since $A^2 = 0$, we conclude that $AX = 0$. Hence $X \in N(\delta_A^2)$. Therefore

$$\bigcup_{\lambda \in \mathbb{C}} N(L_{A-\lambda}) \subseteq N(\delta_A^2).$$

A similar reasoning shows that

$$\bigcup_{\lambda \in \mathbb{C}} N(R_{A-\lambda}) \subseteq N(\delta_A^2).$$

Consequently, we have

$$\begin{aligned} cl[R(\delta_A) + N(\delta_A) + \bigcup_{\lambda \in \mathbb{C}} N(L_{A-\lambda}) + \bigcup_{\lambda \in \mathbb{C}} N(R_{A-\lambda})] \\ \subseteq N(\delta_A^2) \subsetneq M(A, A) = \mathcal{L}(E). \end{aligned}$$

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