HARDY SPACES OF SPACES OF HOMOGENEOUS TYPE

XUAN THINH DUONG AND LIXIN YAN

(Communicated by Andreas Seeger)

ABSTRACT. Let X be a space of homogeneous type, and L be the generator of a semigroup with Gaussian kernel bounds on $L^2(X)$. We define the Hardy spaces $H_s^p(X)$ of X for a range of p , by means of area integral function associated with the Poisson semigroup of L , which is proved to coincide with the usual atomic Hardy spaces $H_{at}^p(X)$ on spaces of homogeneous type.

1. INTRODUCTION

We begin by recalling the definitions necessary for introducing Hardy spaces on spaces of homogeneous type. A quasi-metric d on a set X is a function

$$d : X \times X \mapsto [0, \infty)$$

satisfying:

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) there exists a constant $A < \infty$ such that for all x, y , and $z \in X$, $d(x, y) \leq A(d(x, z) + d(z, y))$.

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in X : d(y, x) < r\}$ form a base. However, the balls themselves need not be open when $A > 1$.

Definition 1.1 ([CW]). A space of homogeneous type (X, d, μ) is a set together with a quasi-metric d and a nonnegative measure μ on X such that $\mu(B(x, r)) < \infty$ for all $x \in X$ and all $r > 0$, and there exists $A' < \infty$ such that for all $x \in X$ and all $r > 0$,

$$\mu(B(x, 2r)) \leq A' \mu(B(x, r)).$$

Here μ is assumed to be defined on a σ -algebra which contains all Borel sets and all balls $B(x, r)$.

Received by the editors January 24, 2002 and, in revised form, May 16, 2002.

2000 *Mathematics Subject Classification.* Primary 42B20, 42B30, 47G10.

Key words and phrases. Spaces of homogeneous type, Hardy spaces, semigroup, Calderón-type reproducing formula, atomic decomposition.

Both authors were partially supported by a grant from Australia Research Council, and the second author was also partially supported by the NSF of China.

Macias and Segovia [MS] have shown that one can replace d by another quasi-metric ρ such that there exist c and some $\theta, 0 < \theta < 1$,

$$\rho(x, y) \approx \inf\{\mu(B) : B \text{ is a ball containing } x \text{ and } y\},$$

$$|\rho(x, y) - \rho(x', y)| \leq c\rho(x, x')^\theta [\rho(x, y) + \rho(x', y)]^{1-\theta} \quad \text{for all } x, x' \in X \text{ and } y \in X.$$

We will suppose that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$. A function $a(x)$ defined on X is called a p -atom if there exists a ball $B(x_0, r)$ for some $x_0 \in X$ and $r > 0$ such that $\text{supp } a(x) \subset B(x_0, r)$,

$$\|a\|_2 \leq \mu(B(x_0, r))^{1/2-1/p} \quad \text{and} \quad \int_X a(x) d\mu(x) = 0.$$

Definition 1.2 ([CW]). We say that a function $f(x) \in H_{at}^p(X)$ for $(1+\theta)^{-1} < p \leq 1$, if $f(x)$ possesses an atomic decomposition $f(x) = \sum_k \lambda_k a_k(x)$, where the a_k 's are p -atoms and $\sum_k |\lambda_k|^p < \infty$. The norm of f is then defined by

$$\|f\|_{H_{at}^p(X)} = \inf \left\{ \left(\sum_k |\lambda_k|^p \right)^{1/p} : \text{for all } f(x) = \sum_k \lambda_k a_k(x) \right\}.$$

It is well-known that Hardy spaces $H^p(\mathbb{R}^n)$ on $\mathbb{R}^n$ can be characterized by means of the square function associated with the Poisson semigroup $e^{-t\sqrt{A}}$ or the heat semigroup e^{-tA} , where $A = -\Delta$ is the Laplace operator; see [FeS]. In this paper, our concern is to establish similar results on spaces of homogeneous type under the following assumptions about the operator L :

(1.1) L is a self-adjoint, positive definite operator on $L^2(X)$;

(1.2) the kernel of e^{-tL} , denoted by $K_t(x, y)$, is a measurable function on $X \times X$ and there exists $\beta > 0$ such that for all $0 < t < \infty$ and almost every $x, y \in X$,

$$|K_t(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \exp\left\{-\frac{\beta d^2(x, y)}{t}\right\};$$

(1.3) for all $y \in X$ and all $0 < t < \infty$, the function $x \rightarrow K_t(x, y)$ is Hölder continuous in X and there exists $\gamma > 0$ such that for all $0 < t < \infty$ and all $x, x', y \in X$,

$$|K_t(x, y) - K_t(x', y)| \leq \frac{C}{\mu(B(x, \sqrt{t}))} \left(\frac{d(x, x')}{\sqrt{t}}\right)^\gamma;$$

(1.4) for all $t > 0$, $e^{-tL}(1) = 1$.

We define the square function $S_\alpha(f)$, $0 < \alpha < \infty$ by the area integral

$$S_\alpha(f)(x) = \left(\int_0^\infty \int_{\rho(x, y) < \alpha t} \mu(B(x, \alpha t))^{-1} \left| t \frac{\partial}{\partial t} e^{-t\sqrt{L}}(f) \right|^2 \frac{d\mu(y) dt}{t} \right)^{1/2}, \quad x \in X.$$

$S_1(f)(x)$ will be denoted by $S(f)(x)$. By a standard argument, for any $0 < \alpha < \infty$ we have

$$\|S_\alpha(f)\|_{L^p(X)} \approx \|S_1(f)\|_{L^p(X)}, \quad 0 < p < \infty.$$

Definition 1.3. We say that a function $f(x) \in H_s^p(X)$, $(1+\gamma)^{-1} < p \leq 1$, if $S(f)(x)$ belongs to $L^p(X)$, and its norm is defined by $\|f\|_{H_s^p(X)} = \|S(f)\|_{L^p(X)}$.

The main result of this paper is the following theorem.

Theorem 1.4. *Assume that L satisfies the conditions (1.1), (1.2), (1.3) and (1.4). Then, for any p such that $\max((1+\theta)^{-1}, (1+\gamma)^{-1}) < p \leq 1$, $H_s^p(X) = H_{at}^p(X)$ with equivalent norms.*

The paper is organized as follows. In Section 2, we introduce a Calderón-type reproducing formula associated with the self-adjoint operator L . Our reproducing formula (2.5) below is based on the functional calculus of L . For other constructions of Calderón-type reproducing formula on spaces of homogeneous type, we refer readers to [HS]. Using our reproducing formula we obtain atomic decomposition of Hardy spaces $H_s^p(X)$, and prove Theorem 1.4 in Section 3. We give some applications in Section 4.

2. CALDERÓN-TYPE REPRODUCING FORMULA

Recall that for any positive self-adjoint operator L on $L^2(X)$ and for every bounded Borel function $F : [0, \infty) \mapsto \mathcal{C}$, we define an operator $F(L) : L^2(X) \mapsto L^2(X)$ by the formula

$$F(L) = \int_0^\infty F(\lambda) dE(\lambda),$$

where $E(\lambda)$ is the spectral decomposition of the operator L . Therefore, the operators $\cos(t\sqrt{L})$ and $(t\sqrt{L})^{-1} \sin(t\sqrt{L})$ are well-defined on $L^2(X)$. The functional calculus for L gives us the relationship

$$(2.1) \quad (t\sqrt{L})^{-1} \sin(t\sqrt{L}) = (2t)^{-1} \int_{-t}^t \cos(\xi\sqrt{L}) d\xi.$$

See Theorem 2 in [Si]. Throughout the paper, we denote

$$(2.2) \quad S_t(\sqrt{L}) = (t\sqrt{L})^{-1} (2 \sin(t\sqrt{L}/2) - \sin(t\sqrt{L})).$$

Lemma 2.1. *Assume that L satisfies (1.1) and (1.2). Then the Schwartz kernel $K_{S_t(\sqrt{L})}$ of $S_t(\sqrt{L})$ has support contained in*

$$\{(x, y) \in X^2 : d(x, y) \leq t\}.$$

Proof. An argument of Davies, as adapted in [DR], shows that the semigroup e^{-tL} can be extended to an analytic semigroup for all z such that $\operatorname{Re} z \geq 0$, and

$$|K_z(x, y)| \leq \frac{C}{\mu(B(x, \sqrt{\operatorname{Re} z}))} \exp\left\{-\frac{\beta d^2(x, y)}{\operatorname{Re} z}\right\}.$$

See Lemma 2.4 in [CD]. Then it follows from Theorem 3 in [Si] that we have

$$\operatorname{supp} (K_{\cos(t\sqrt{L})}) \subseteq \{(x, y) \in X^2 : d(x, y) \leq t\}.$$

Using the formula (2.1), we get

$$\operatorname{supp} (K_{(t\sqrt{L})^{-1} \sin(t\sqrt{L})}) \subseteq \{(x, y) \in X^2 : d(x, y) \leq t\},$$

which implies this lemma.

Lemma 2.2. *Assume that L satisfies (1.1), (1.2) and (1.4). For all $t > 0$, we have $S_t(\sqrt{L})(1) = 0$.*

Proof. By (2.2) it suffices to prove $(t\sqrt{L})^{-1}\sin(t\sqrt{L})(1) = 1$ for all $t > 0$. For each $\epsilon > 0$, denote $\theta = 2\arctan(\frac{\epsilon}{2})$. Let $\Gamma_b^\epsilon = \Gamma_{0,b}^\epsilon \cup \Gamma_{1,b}^\epsilon$ be the oriented contour, which is given by $\Gamma_{0,b}^\epsilon = \{\lambda = be^{i\phi}, \phi \in [-\pi, \pi) \setminus (-\theta, \theta)\}$ for some $b > 0$, and

$$\Gamma_{1,b}^\epsilon := \begin{cases} -te^{-i\theta}, & -\infty < t \leq -b, \\ te^{i\theta}, & b \leq t < \infty. \end{cases}$$

We can write (see e.g. [Mc])

$$e^{it\sqrt{L}}e^{-\epsilon t\sqrt{L}} = (2\pi i)^{-1} \int_{\Gamma_b^\epsilon} (\lambda I - L)^{-1} e^{it\sqrt{\lambda}} \cdot e^{-\epsilon t\sqrt{\lambda}} d\lambda.$$

The integral is absolutely convergent in the norm topology on $L^2(X)$. It is also independent of the choice of b .

The assumption (1.4) implies that $(\lambda I - L)^{-1}(1) = \lambda^{-1}$. As a consequence, we have

$$\begin{aligned} (2.3) \quad \lim_{\epsilon \rightarrow 0} e^{it\sqrt{L}}e^{-\epsilon t\sqrt{L}}(1) &= \lim_{\epsilon \rightarrow 0} \lim_{b \rightarrow 0} (2\pi i)^{-1} \int_{\Gamma_b^\epsilon} \lambda^{-1} e^{it\sqrt{\lambda}} \cdot e^{-\epsilon t\sqrt{\lambda}} d\lambda \\ &= \lim_{\epsilon \rightarrow 0} \lim_{b \rightarrow 0} (2\pi i)^{-1} \left(\int_{\Gamma_{0,b}^\epsilon} \lambda^{-1} e^{it\sqrt{\lambda}} \cdot e^{-\epsilon t\sqrt{\lambda}} d\lambda + \int_{\Gamma_{1,b}^\epsilon} \lambda^{-1} e^{it\sqrt{\lambda}} \cdot e^{-\epsilon t\sqrt{\lambda}} d\lambda \right) \\ &= 1 + \lim_{\epsilon \rightarrow 0} \lim_{b \rightarrow 0} (2\pi i)^{-1} \int_{\Gamma_{1,b}^\epsilon} \lambda^{-1} e^{it\sqrt{\lambda}} \cdot e^{-\epsilon t\sqrt{\lambda}} d\lambda. \end{aligned}$$

Since $\theta = 2\arctan(\frac{\epsilon}{2})$, it can be verified that

$$\begin{aligned} (2.4) \quad &\lim_{\epsilon \rightarrow 0} \lim_{b \rightarrow 0} \operatorname{Re}[(2\pi i)^{-1} \int_{\Gamma_{1,b}^\epsilon} \lambda^{-1} e^{it\sqrt{\lambda}} \cdot e^{-\epsilon t\sqrt{\lambda}} d\lambda] \\ &= \lim_{\epsilon \rightarrow 0} \pi^{-1} \int_0^\infty e^{-x} \cdot \left[\sin\left(\frac{2}{\epsilon} + \epsilon\right)x - \sin\left(\frac{2}{3\epsilon} - \frac{\epsilon}{3}\right)x \right] \frac{dx}{x} \\ &= -1/2 + 1/2 = 0. \end{aligned}$$

Substituting (2.4) into (2.3) yields

$$\cos(t\sqrt{L})(1) = \operatorname{Re}[e^{it\sqrt{L}}(1)] = \lim_{\epsilon \rightarrow 0} \operatorname{Re}[e^{it\sqrt{L}}e^{-\epsilon t\sqrt{L}}(1)] = 1$$

for all $t > 0$. Hence,

$$\begin{aligned} (t\sqrt{L})^{-1}\sin(t\sqrt{L})(1) &= (2t)^{-1} \int_{-t}^t \cos(\xi\sqrt{L})(1) d\xi \\ &= (2t)^{-1} \int_{-t}^t d\xi = 1, \end{aligned}$$

which implies $S_t(\sqrt{L})(1) = 0$ for all $t > 0$.

Lemma 2.3. Assume that L satisfies (1.1). Then for any $f \in L^2(X)$, there exists a constant C such that

$$\left(\int_0^\infty \|S_t(\sqrt{L})(f)\|_2^2 \frac{dt}{t} \right)^{1/2} \leq C \|f\|_{L^2(X)}.$$

Proof. Let $\psi(t) = 2(\sqrt{t})^{-1} \sin(\sqrt{t}/2) - (\sqrt{t})^{-1} \sin(\sqrt{t})$. Then

$$q = (\sqrt{2})^{-1} \left(\int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty.$$

Since L is a self-adjoint operator, we have

$$\begin{aligned} \left(\int_0^\infty \|S_t(\sqrt{L})(f)\|_2^2 \frac{dt}{t} \right)^{1/2} &= \left(\int_0^\infty \|\psi(t^2 L)(f)\|_2^2 \frac{dt}{t} \right)^{1/2} \\ &\leq q \|f\|_{L^2(X)}. \end{aligned}$$

Next, we establish a new Calderón-type reproducing formula associated with the semigroup generated by L , which will be used in the next section.

Lemma 2.4. *Assume that L satisfies (1.1). Then, for any $f(x) \in L^2(X)$, we have*

$$(2.5) \quad f(x) = C_s \int_0^\infty S_t(\sqrt{L})(t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f) \frac{dt}{t}(x),$$

where the integral converges strongly in $L^2(X)$, and $C_s = (\arctan(\frac{4}{3}) - \frac{\pi}{4})^{-1}$.

Proof. Note that for any $\lambda > 0$, we have

$$\int_0^\infty S_t(\sqrt{\lambda})(t\sqrt{\lambda}) e^{-t\sqrt{\lambda}} \frac{dt}{t} = \int_0^\infty (2e^{-2t} - e^{-t}) \sin t \frac{dt}{t} = C_s^{-1}.$$

As a consequence, one has

$$f(x) = C_s \int_0^\infty S_t(\sqrt{L})(t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f) \frac{dt}{t}(x),$$

where $C_s = (\arctan(\frac{4}{3}) - \frac{\pi}{4})^{-1}$, and the integral converges strongly in $L^2(X)$.

3. PROOF OF THEOREM 1.4

To prove Theorem 1.4, we first recall a result of Christ [Ch] which gives an analogue of the Euclidean dyadic cubes.

Lemma 3.1. *There exist a collection of open subsets $\{Q_\tau^k \subset X : k \in \mathbf{Z}, \tau \in I_k\}$, where I_k denotes some (possibly finite) index set depending on k , and constants $\delta \in (0, 1)$, $a_0 > 0$, $\eta > 0$ and $0 < c_1, c_2 < \infty$ such that*

- (i) $\mu(X \setminus \cup Q_\tau^k) = 0$ for all $k \in \mathbf{Z}$;
- (ii) if $j \geq k$, then either $Q_{\tau'}^j \subset Q_\tau^k$ or $Q_{\tau'}^j \cap Q_\tau^k = \emptyset$;
- (iii) for all (k, τ) and $j < k$, there is a unique τ' such that $Q_\tau^k \subset Q_{\tau'}^j$;
- (iv) $\text{diam}(Q_\tau^k) \leq c_1 \delta^k$;
- (v) each Q_τ^k contains some ball $B(z_\tau^k, a_0 \delta^k)$.

We fix such a collection of open subsets as in Lemma 3.1 and call all Q_τ^k in Lemma 3.1 the dyadic cubes in X . Without loss of generality, we may assume that $\delta = 1/2$ in Lemma 3.1. It is easy to check that our results and proofs are independent of the choice of open subsets which satisfy the hypotheses of Lemma 3.1.

We first prove that $H_s^p(X) \subset H_{at}^p(X)$. We follow the idea in [CF]. Let $\Omega_k = \{x \in X : S_{c_0}(f)(x) > 2^k\}$, where c_0 is a constant to be chosen later. Set

$$\mathcal{B}_k = \left\{ Q : \mu(Q \cap \Omega_k) > \mu(Q)/2 \quad \text{and} \quad \mu(Q \cap \Omega_{k+1}) < \mu(Q)/2 \right\}.$$

Note that for each dyadic cube Q in X there is a unique $k \in \mathbf{Z}$ such that $Q \in \mathcal{B}_k$. For each dyadic cube $Q \in \mathcal{B}_k$, there is a unique maximal dyadic cube $Q' \in \mathcal{B}_k$ such that $Q \subseteq Q'$. Denote the collection of all maximal dyadic cubes in $\mathcal{B}_k$ by $Q_k^i, i \in I_k$, an index set which depends on k (it is possibly finite). We then have

$$\bigcup_{Q: \text{dyadic cube}} Q = \bigcup_k \bigcup_{i \in I_k} \bigcup_{Q \subseteq Q_k^i, Q \in \mathcal{B}_k} Q.$$

For dyadic cube Q with $\mu(Q) \sim 2^{-k}$, we write $\hat{Q} = \{(y, t) \in X \times \mathbb{R}^+, y \in Q, 2^k < t < 2^{k+1}\}$, and for any k and i , let $\hat{Q}_k^i = \bigcup_{\substack{Q \subseteq Q_k^i \\ Q \in \mathcal{B}_k}} \hat{Q}$. If $f \in H_s^p(X) \cap L^2(X)$, applying the Calderón-type reproducing formula (2.5) we have

$$\begin{aligned} f(x) &= \lim_{A \rightarrow \infty} \lim_{\epsilon \rightarrow 0} C_s \int_{\epsilon}^A S_t(\sqrt{L})(t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f) \frac{dt}{t}(x) \\ &= \sum_k \sum_{\mu(Q) \sim 2^{-k}} C_s \int_{2^k}^{2^{k+1}} \int_Q S_t(x, y) (t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f)(y) \frac{d\mu(y)dt}{t} \\ (3.1) \quad &= \sum_{k,i} \sum_{Q \subseteq Q_k^i, Q \in \mathcal{B}_k} C_s \int_{\hat{Q}} S_t(x, y) (t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f)(y) \frac{d\mu(y)dt}{t} \\ &= \sum_{k,i} C_s \int_{\hat{Q}_k^i} S_t(x, y) (t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f)(y) \frac{d\mu(y)dt}{t}. \end{aligned}$$

Set

$$\lambda_{k,i} = c\mu(Q_k^i)^{1/p-1/2} \left(\int_{\hat{Q}_k^i} |(t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f)|^2 \frac{dydt}{t} \right)^{1/2}.$$

Then one can write (3.1) as

$$(3.2) \quad f(x) = \sum_{k,i} \lambda_{k,i} a_{k,i}(x),$$

where

$$a_{k,i}(x) = C_s \lambda_{k,i}^{-1} \int_{\hat{Q}_k^i} S_t(x, y) (t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f)(y) \frac{d\mu(y)dt}{t}.$$

We now show that this sum (3.2) gives a decomposition of f into atoms. Since

$$\text{supp}(K_{S_t(\sqrt{L})}) \subset \{(x, y) \in X^2 : d(x, y) < t\}$$

by Lemma 2.1, we have

$$\text{supp } a_{k,i}(x) \subset \bigcup_{\substack{Q \subseteq Q_k^i \\ Q \in \mathcal{B}_k}} Q \subset 2Q_k^i.$$

Using Lemma 2.3, we have

$$\begin{aligned}
\|a_{k,i}\|_2 &\leq \sup_{\|b\|_2 \leq 1} \left| \int_X a_{k,i}(x)b(x)d\mu(x) \right| \\
&\leq C(\lambda_{k,i})^{-1} \sup_{\|b\|_2 \leq 1} \left(\int_0^\infty \int_X \left(t \frac{\partial}{\partial t}\right) e^{-t\sqrt{L}}(f)(y) \chi_{\hat{Q}_{k,i}}(y,t) (S_t(\sqrt{L})^*)(b)(y) \frac{d\mu(y)dt}{t} \right) \\
&\leq C\mu(Q_k^i)^{1/2-1/p} \sup_{\|b\|_2 \leq 1} \left(\int_0^\infty \int_X |S_t(\sqrt{L})(b)(y)|^2 \frac{d\mu(y)dt}{t} \right)^{1/2} \\
&\leq C\mu(Q_k^i)^{1/2-1/p}.
\end{aligned}$$

It follows from Lemma 2.2 that

$$\int_X a_{k,i}(x)d\mu(x) = \lambda_{k,i}^{-1} \int_{\hat{Q}_k^i} S_t^*(\sqrt{L})(1)(t \frac{\partial}{\partial t}) e^{-t\sqrt{L}}(f)(y) \frac{d\mu(y)dt}{t} = 0.$$

This proves that for each k, i , the function $a_{k,i}(x)$ is an atom.

Now we estimate $\sum_{k,i} |\lambda_{k,i}|^p$. We need the following lemma, whose proof is similar to that of Lemma 7.18 in [CF].

Lemma 3.2. *Suppose that $\mathcal{B}_k$ is as above. Then there exists a constant C such that*

$$\begin{aligned}
\sum_{Q \in \mathcal{B}_k} \int_{\hat{Q}} \left| \left(t \frac{\partial}{\partial t}\right) e^{-t\sqrt{L}}(f)(y) \right|^2 \frac{d\mu(y)dt}{t} &= \sum_i \int_{\hat{Q}_k^i} \left| \left(t \frac{\partial}{\partial t}\right) e^{-t\sqrt{L}}(f)(y) \right|^2 \frac{d\mu(y)dt}{t} \\
&\leq C2^{2k} \mu(\Omega_k).
\end{aligned}$$

Using Lemma 3.2 and the Hölder inequality we have

$$\begin{aligned}
\sum_{k,i} |\lambda_{k,i}|^p &\leq \sum_k \sum_i \mu(Q_k^i)^{1-p/2} \left(\int_{\hat{Q}_k^i} \left| \left(t \frac{\partial}{\partial t}\right) e^{-t\sqrt{L}}(f) \right|^2 \frac{d\mu(y)dt}{t} \right)^{p/2} \\
&\leq \sum_k \left(\sum_i \mu(Q_k^i) \right)^{1-p/2} \left(\sum_i \int_{\hat{Q}_k^i} \left| \left(t \frac{\partial}{\partial t}\right) e^{-t\sqrt{L}}(f) \right|^2 \frac{d\mu(y)dt}{t} \right)^{p/2} \\
&\leq C \sum_k \mu(\Omega_k)^{1-p/2} (2^{2k} \mu(\Omega_k))^{p/2} \leq C \sum_k 2^{kp} \mu(\Omega_k) \\
&\leq C \|S_{c_0}(f)\|_{L^p(X)}^p \leq C \|S(f)\|_{L^p(X)}^p.
\end{aligned}$$

This proves that a function $f \in H_s^p(X) \cap L^2$ possesses an atomic decomposition. A standard density argument gives an atomic decomposition for $f \in H_s^p(X)$. See, for example, [CF].

We now prove that $H_{at}^p(X) \subset H_s^p(X)$. Let $a(x)$ be an atom supported in a cube Q . Note that L satisfies assumptions (1.2) and (1.3), a standard argument shows that $\|a\|_{H_s^p} \leq C$, hence $H_{at}^p(X) \subset H_s^p(X)$. So, the proof of Theorem 1.4 is completed.

4. APPLICATIONS

Example 4.1. Let $\Phi : \mathbb{R}^{n-1} \mapsto \mathbb{R}^n$ be a Lipschitz function, i.e., $\|\nabla \Phi\|_\infty \leq M < \infty$ for some constant M . Define $\Omega = \{(X, y) \mid (X, y) \in \mathbb{R}^{n-1} \times \mathbb{R}^n; y > \Phi(X)\}$; then Ω is a Lipschitz domain of $\mathbb{R}^n$.

We assume that $a_{ij}(x) = a_{ji}(x)$ are bounded, real-valued, Lebesgue measurable functions on Ω , which satisfy the ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2$$

for some constant $\lambda \in (0, 1)$, and for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. A function u belonging to the Sobolev space $W^{1,2}(\Omega)$ is called a solution of the Neumann boundary value problem

$$\begin{aligned} Lu &= f && \text{in } \Omega, \\ \sum_{i,j=1}^n a_{ij}\nu_j \frac{\partial u}{\partial x_i} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\nu_j, 1 \leq j \leq n$, are the components of the unit outer normal vector field on $\partial\Omega$, if u satisfies

$$\int_{\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \int_{\Omega} f v dx$$

for all nonnegative functions $v \in C^1(\Omega)$. See, for example, [D].

Then the operator L satisfies all assumptions in Theorem 1.4. Recall that Hardy spaces $H_z^p(\Omega)$ of D-C. Chang, S.G. Krantz and E.M. Stein ([CKS]) on the Lipschitz domain of $\mathbb{R}^n$ is given by

$$H_z^p(\Omega) = H^p(\mathbb{R}^n) \cap \{f \in H^p(\mathbb{R}^n) : f = 0 \text{ on } {}^c(\bar{\Omega})\} / \{f \in H^p(\mathbb{R}^n) : f = 0 \text{ on } \Omega\}.$$

By Theorem 3.3 in [CKS], Theorem 1.4 gives a new proof of $H_s^p(\Omega) = H_z^p(\Omega)$. See also [AR].

Example 4.2. Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra. Assume that

$$\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i$$

as a vector space, where $[\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+j}$ for all i, j , and $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra.

Let G be the associated connected, simply connected Lie group. Then G has homogeneous dimension d given by the formula

$$d = \sum_{j=1}^m j \dim(\mathfrak{g}_j),$$

where $\dim(\mathfrak{g}_j)$ denotes the dimension of $\mathfrak{g}_j$.

Consider any finite subset $\{X_k\}$ of $\mathfrak{g}_1$ which spans $\mathfrak{g}_1$. Each X_k can be identified with a unique left invariant vector field on G . Let d be the control distance corresponding to the vector fields and ρ the Haar measure. Then the space (G, d, ρ) is of polynomial growth, hence it is a space of homogeneous type. Define

$$L = - \sum_k X_k^2.$$

The operator L is a left invariant second order differential operator, which is a non-negative self-adjoint operator in $L^2(G)$. Moreover, the operator L then has a heat kernel of e^{-tL} which satisfies all assumptions in Theorem 1.4. See [DR],

and [Sc]. Hence, Theorem 1.4 gives an equivalent characterization of Hardy spaces $H^p(G)$ on G (see [FoS]) by means of the square function associated with the Poisson semigroup.

ACKNOWLEDGMENT

We thank A. McIntosh and A. Sikora for their useful suggestions.

REFERENCES

- [AR] P. Auscher and E. Russ, *Hardy spaces and divergence operators on strongly Lipschitz domain of $\mathbb{R}^n$* , preprint.
- [CD] T. Coulhon and X.T. Duong, *Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss*, Adv. Differential Equations, **5** (2000), 343–368. MR **2001d**:34087
- [CF] S.Y.A. Chang and R. Fefferman, *The Calderón-Zygmund decomposition on product domain*, Amer. J. Math., **104** (1982), 445–468. MR **84a**:42028
- [Ch] M. Christ, *A Tb theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math., **61** (1990), 601–628. MR **92k**:42020
- [CKS] D-C. Chang, S.G. Krantz, and E.M. Stein, *H^p theory on a smooth domain in $\mathbb{R}^n$ and elliptic boundary value problems*, J. Funct. Anal., **114** (1993), 286–347. MR **94j**:46032
- [CW] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., **83** (1977), 569–645. MR **56**:6264
- [D] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, 1989. MR **90e**:35123
- [DR] X.T. Duong and D.W. Robinson, *Semigroup kernels, Poisson bounds, and holomorphic functional calculus*, J. Funct. Anal., **142** (1996), 89–128. MR **97j**:47056
- [FeS] C. Fefferman and E.M. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137–195. MR **56**:6263
- [FoS] G.B. Folland and E.M. Stein, *Hardy spaces on homogeneous group*, Math. Notes, **28**, Princeton University Press, (1982). MR **84h**:43027
- [HS] Y.S. Han, E. Sawyer, *Littlewood-Paley theory on spaces of homogeneous type and the classical function spaces*, Mem. Amer. Math. Soc. **110** (1994), no. 530. MR **96a**:42016
- [Mc] A. McIntosh, *Operators which have an H_∞ -calculus*, Miniconference on Operator Theory and Partial Differential Equations (Proc. Centre Math. Analysis, 14, A.N.U., Canberra, 1986), 210–231. MR **88k**:47019
- [MS] R.A. Maciac and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math., **33** (1979), 257–270.
- [Sc] L. Saloff-Coste, *Analyse sur les groupes de Lie à croissance polynômiale*, Arkiv för Mat., **28** (1990), 315–331. MR **92d**:22014
- [Si] A. Sikora, *Sharp pointwise estimates on heat kernels*, Quart. J. Math. Oxford, **47** (1996), 371–382. MR **97m**:58189

DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NEW SOUTH WALES 2109, AUSTRALIA

E-mail address: duong@ics.mq.edu.au

DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NEW SOUTH WALES 2109, AUSTRALIA – AND – DEPARTMENT OF MATHEMATICS, ZHONGSHAN UNIVERSITY, GUANGZHOU, 10275, PEOPLE'S REPUBLIC OF CHINA

E-mail address: lixin@ics.mq.edu.au