WEAK COMPACTNESS OF CERTAIN SETS OF MEASURES

SURJIT SINGH KHURANA

(Communicated by Claudia M. Neuhauser)

ABSTRACT. For a compact Hausdorff space X and a Montel Hausdorff locally convex space E , let $F = (C(X, E), u)$, u being the uniform topology. We determine the necessary and sufficient conditions for an equicontinuous $H \subset F'$ to be $\sigma(F', F'')$ -compact. Special results are obtained when X is an F -space or a σ -Stonian space.

1. INTRODUCTION AND NOTATIONS

For locally convex spaces, the notations and results from [14] are used and for general topology we refer to [6]. All vector spaces are taken over K , the field of real or complex numbers (we call K the scalar field). For measures and vector measures, we will follow the notations of and results from [2], [7], [8], [9] [13], [16]. For a locally convex space G , G' will be its dual. For a completely regular Hausdorff space Y , $\hat{Y}$ will denote the Stone-Cech compactification of Y . N will denote the set of natural numbers. For two vector spaces in duality, $\langle, \rangle$ will denote the bilinear form on their product.

In this paper, X is a compact Hausdorff space and E a Montel Hausdorff locally convex space. Recall that a locally convex space is called a Montel space if it is barreled and every bounded set is relatively compact ([10], pp. 369-378). As stated by Köthe ([10], p. 372), some of the most important spaces in analysis are Montel (the space $H(D)$, of analytic functions on the open unit disc, with the topology of uniform convergence on the compact subsets of D is a Fréchet-Montel space ([10], p. 373)). We denote by $C(X, E)$ the space of all E -valued continuous functions on X . If $E = K$, then we denote $C(X, E)$ by $C(X)$. Let $\{\|\cdot\|_p : p \in P\}$ be the collections of all continuous semi-norms on E . For a $p \in P$, E_p denotes the normed space arising from E by the semi-norm $\|\cdot\|_p$. E'_p can be considered a subspace of E' : $E'_p = \{f \in E' : f(p^{-1}[0, 1]) \text{ is bounded}\}$. We take the uniform topology u on $C(X, E)$ and denote the locally convex space $(C(X, E), u)$ by F . The dual of F is denoted by $M(X, E')$. If E is a normed space, then with norm topology on E' , $M(X, E')$ consists of regular Borel E' -valued measures, of finite variations, on X . If E is locally convex, then $(C(X, E), u)$ can be considered a subspace of $\prod_{p \in P} (C(X, E_p), u)$, with induced topology. So for every $\mu \in M(X, E')$, there

Received by the editors January 2, 2002 and, in revised form, April 28, 2002.

2000 *Mathematics Subject Classification.* Primary 60B10, 46G10, 46G15, 28C15; Secondary 47B38, 28A51, 54C35.

Key words and phrases. Montel locally convex space, Stonian space, Fréchet space, lifting of measures.

is a $p \in P$ such that $\mu \in (C(X, E_p), u)'$. Thus every $\mu \in F'$ gives a $p \in P$ such that μ is an E'_p -valued regular Borel measure, of finite variation $|\mu|_p$, on X ([8]). If $E = K$, then $M(X, E')$ is denoted by $M(X)$ and its positive elements are denoted by $M^+(X)$. If $H \subset F'$ is equicontinuous, then there is a $p \in P$ such that $|\mu|_p(X) \leq 1$, $\forall \mu \in H$, and all $\mu \in H$ are E'_p -valued. We denote by γ the canonical mapping $E \rightarrow E_p$, by $\mathcal{B}$ all closed, absolutely convex bounded subsets of E (since E is Montel, these subsets are compact), and by $\mathcal{B}_0$ the collection $\{\gamma(B) : B \in \mathcal{B}\}$; the elements of $\mathcal{B}_0$ are absolutely convex and compact in E_p .

In the celebrated paper of Grothendieck [5], some remarkable characterizations of weakly compact subsets of $M(X)$ are obtained and then are applied to the special case when X is a Stonian space. These results are extended to the case of $C(X, E)$ by many authors ([4], [1], [17]). In this paper we give necessary and sufficient conditions for an equicontinuous subset of F' to be $\sigma(F', F'')$ -compact (i.e. weakly compact in F'_β —note, in the notations of [14], that F'_β is the space F' with topology of uniform convergence on the $\sigma(F, F')$ -bounded subsets of F and $(F'_\beta)' = F''$) and then apply the results to the special cases when X is a σ -Stonian or an F -space. Recall that X is Stonian if the closure of every open set is open, is σ -Stonian if the closure of every open F_σ -set is open, and is F -space if disjoint open F_σ -sets have disjoint closures.

We will also need the following lemmas:

Lemma 1. *Let $\lambda_n : 2^N \rightarrow K$ be a sequence of countably additive functions (on all subsets of N). If $\lim \lambda_n(A)$ exists for all subsets $A \subset N$, then the convergence is uniform on 2^N .*

Proof. This is a particular case of classical Phillips' Lemma ([2], p. 33).

Lemma 2. *Let T be a Hausdorff topological space having a σ -compact dense subset, $C(T)$ all scalar-valued continuous functions on T with the topology of pointwise convergence, and $A \subset C(T)$ such that every sequence in A has a cluster point in $C(T)$. Let g be in the closure of A in K^T . Then there exists a sequence, in A , which converges to g pointwise.*

Proof. This is proved in [12].

2. MAIN RESULTS

Theorem 3. *Let X be a compact set, E a Montel locally convex space and $F = (C(X, E), u)$ (here u denotes the uniform topology). Suppose $H \subset F'$ is equicontinuous. Then H is relatively weakly compact in F'_β , if and only if, $\forall x \in E$, $\{\mu_x : \mu \in H\}$ is relatively weakly compact in $(M(X), \|\cdot\|)$.*

Proof. Suppose H is relatively weakly compact in F'_β . Fix an $x \in E$. Since the mapping $\phi : (C(X), \|\cdot\|) \rightarrow (C(X, E), u)$, $f \rightarrow f \otimes x$ is continuous, the adjoint $\phi' : F'_\beta \rightarrow (M(X), \|\cdot\|)$ is continuous and so the result follows.

Now we come to the converse. Since H is equicontinuous, H_0 , the closed convex hull of H in F'_σ , is compact. From this it is easy to verify that H_0 is complete in F'_β . So to prove the weak-compactness of H in F'_β , it is enough to prove that every sequence in H has a cluster point ([14], Theorem 11.2). Take a sequence $\{\mu_n\} \subset H$ and let $\bar{H}$ be the closure of H in F'_σ . Let $\mu_0 \in \bar{H}$ be a cluster point $\{\mu_n\}$ in F'_σ . There exists a $p \in P$ such that $|\mu_n|_p(X) \leq 1$, $0 \leq n < \infty$. We fix a

lifting ρ for the finite measure $\lambda = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} |\mu_n|_p$ ([16], [13]). We take the lifting topology $\mathcal{T}_\rho$ on X (this topology is denoted by $\mathcal{T}_1$ in [13]) and denote the associated completely regular Hausdorff space by $(X_0, \mathcal{T}_\rho)$. $\tilde{X}_0$ will denote the Stone-Cech compactification of X_0 .

We state some notations and results from [9]:

M_0 is the linear span of μ_n , $0 \leq n < \infty$, in F' .

$V = \{x \in E : \|x\|_p \leq 1\}$.

$L_1 = \{\phi : (X_0, \mathcal{T}_\rho) \rightarrow E'_\sigma, \phi \text{ continuous and } \phi(X_0) \subset \alpha V^0, \alpha > 0\}$; here V^0 is the polar of V in the duality $\langle E, E' \rangle$. Since E is Montel, ϕ is also continuous in E'_β . Also every $\phi \in L_1$ is E'_p -valued.

For a $B \in \mathcal{B}$, $\phi \in L_1$, $|\phi|_B(t) = \sup\{|\langle x, \phi(t) \rangle| : x \in B\}$. $|\phi|_B$ is continuous on X_0 . Since ϕ is E'_p -valued, if we put $B_0 = \gamma(B)$, then we also have $|\phi|_B = \sup\{|\langle x, \phi(t) \rangle| : x \in B_0\}$.

$\{f \in C(X, E) : f(X) \subset B\}$, as B varies in $\mathcal{B}$, form a fundamental system of bounded sets in F .

For a $\mu \in M_0$, there exists a $q \in L_1$ such that $\mu = q\lambda$, in the sense that $\mu_x = q_x\lambda$ (here $q_x(t) = \langle x, q(t) \rangle$; also $\sup\{|\mu(f)| : f(X) \subset B\} = \int |q|_B d\lambda, \forall B \in \mathcal{B}$). Further $|\mu|_p(X) = \int |q|_V d\lambda$ where $|q|_V = \sup\{|\langle x, q \rangle| : x \in V\}$. From this it follows that $\mu \rightarrow q$ is a linear isomorphism if the topology on M_0 is the one induced by F'_β and the topology on L_1 is generated by $\mathcal{B}$ (each $B \in \mathcal{B}$ gives a semi-norm on L_1 , $q \rightarrow \int |q|_B d\lambda$).

$L_\infty = \{f : X_0 \rightarrow E_p, f(X_0) \subset \gamma(B), B \in \mathcal{B}, f \text{ continuous with norm topology on } E_p\}$.

$L_\infty = (L_1, \mathcal{B})'$, algebraically.

Now we continue with the proof. Take an $f \in L_\infty$. This means there is a $B \in \mathcal{B}$ such that if $B_0 = \gamma(B)$, then $f(X_0) \subset B_0$. By continuity, extend f to $\tilde{f} : \tilde{X}_0 \rightarrow B_0$. Fix $c > 0$ and take an $\tilde{f}_0 \in C(\tilde{X}_0) \otimes E_p$ such that $\|\tilde{f} - \tilde{f}_0\|_p < c$. Put $\tilde{f}_0|_{X_0} = f_0$. Take a net $\{\mu_\alpha\} \subset \{\mu_n\}$ such that $\mu_\alpha \rightarrow \mu_0$, in F'_σ . Also take a net $\{q_\alpha\} \subset L_1$ such that $\mu_\alpha = q_\alpha\lambda$; further take $q_0 \in L_1$, satisfying $\mu_0 = q_0\lambda$. Since $\int |q_\alpha|_V d\lambda \leq 1$ and $\int |q_0|_V d\lambda \leq 1$, we have $\int |\langle f - f_0, q_\alpha - q_0 \rangle| d\lambda \leq c$. Also since $\{(\mu_\alpha)_x\}$ is relatively weakly compact in $(M(X), \|\cdot\|)$ for every $x \in E$, $\int \langle f_0, q_\alpha - q_0 \rangle d\lambda \rightarrow 0$. The result now follows.

Corollary 4. *Let X be a compact Hausdorff space, E a Montel-Fréchet locally convex space and $F = (C(X, E), u)$. Suppose $H \subset F'$ is bounded in F'_σ . Then H is relatively weakly compact, in F'_β , if and only if, $\forall x \in E$, $\{\mu_x : \mu \in H\}$ is relatively weakly compact in $(M(X), \|\cdot\|)$.*

Proof. In this case F is barreled and so H will be equicontinuous. The result follows from Theorem 3.

The following theorem is proved in [17]; we give below a different proof.

Theorem 5 ([17, Theorem 3.3, Theorem 3.4]). *Let X be a compact Hausdorff space. Assume X is either an F -space or a σ -Stonian space. Suppose H is a compact subset of $(M(X), \sigma(M(X), C(X)))$. The H is weakly compact iff there is a $\lambda \in M^+(X)$ such that $\mu \ll \lambda$ for every $\mu \in H$.*

Proof. Assume that there is a $\lambda \in M^+(X)$ such that $\mu \ll \lambda$ for every $\mu \in H$. Suppose H is not weakly compact. By ([11], Theorem 2(ii), p. 4846) there is a sequence $\{\mu_n\} \subset H$, a disjoint sequence $\{V_n\}$ of open subsets of X , and a $c > 0$

such that $|\mu_n(V_n)| > c$, $\forall n$. By regularity, we can assume that each V_n is F_σ . Take a sequence $\{f_n\} \subset C(X)$, $0 \leq f_n \leq \chi_{V_n}$ such that $|\mu_n(f_n)| > c$, $\forall n$. Now the $\text{supp}(|\mu|) \subset \text{supp}(\lambda)$, $\forall \mu \in H$. If X is an F -space, for notational convenience, we take $X = \text{supp}(\lambda)$ (note that every continuous scalar-valued function on a closed subset of X can be extended to a continuous function on the whole of X); this means X is Stonian ([15], Theorem 2.2). In both the cases X is σ -Stonian and so the real-valued functions in $C(X)$ are an order σ -complete vector lattice. For any $M \subset N$, let

$$f|_M = \sup_{\text{in } C(X)} \left\{ \sum_{i \in M, i \leq n} f_i : n \in N \right\}$$

(for $M = \emptyset$, $f|_M = 0$). It is easy to verify that if $M_1 \cap M_2 = \emptyset$, then $f|_{M_1} \wedge f|_{M_2} = 0$. Define $\lambda_0 : 2^N \rightarrow R^+$, $\lambda_0(M) = \lambda(f|_M)$. λ_0 is finitely additive and exhaustive ([3]) and $\lambda_0(\{n\}) = \lambda(f_n)$, $\forall n$. By ([3], Proposition 1, p. 728), there is an infinite subset of N , which again we denote by N , such that λ_0 is countably additive on 2^N . Bringing this information on λ , we get $\lambda(f|_M - \sum_{i \in M} f_i) = 0$, $\forall M \subset N$, and so for every $\mu \in H$, $|\mu|(f|_M - \sum_{i \in M} f_i) = 0$, $\forall M \subset N$. Define $\nu_n : 2^N \rightarrow K$, $\nu_n(M) = \mu_n(\sum_{i \in M} f_i) = \mu(f|_M)$. These measures are countably additive. Let $Y = \{f|_M : M \subset N\}$ be given the topology induced by K^H with product topology. This means the countable set $\{f|_M : M \text{ a finite subset of } N\}$ is dense in Y . Applying Lemma 2 to $H \subset C(Y)$ (note that H is playing the role of A in Lemma 2), we get a subsequence of $\{\mu_n\}$, which again we denote by $\{\mu_n\}$, and a $\mu_0 \in H$ such that $\mu_n \rightarrow \mu_0$, pointwise on Y . This implies that $\lim \nu_n(M)$ exists for any $M \subset N$. By Lemma 1, $\mu_n(f_n) \rightarrow 0$ which is a contradiction.

The converse is trivial.

Theorem 6. *Let X be a compact Hausdorff space. Assume that X is also either an F -space or a σ -Stonian space. Let E be a Montel locally convex space and $F = (C(X, E), u)$ (here u denotes the uniform topology). Suppose $H \subset F'$ is equicontinuous and F'_σ -closed. Then H is relatively weakly compact in F'_β , if and only if, for every $x \in E$, there exists a $\lambda_x \in M^+(X)$ such that $\mu_x \ll \lambda_x$, for every $\mu \in H$.*

Proof. Suppose the condition of the theorem is satisfied. Fix an $x \in E$. By Theorem 3, we have only to prove that $\{\mu_x : \mu \in H\}$ is relatively weakly compact in $M(X)$. The mapping $F'_\sigma \rightarrow (M(X), \sigma(M(X), C(X)))$, $(\mu \rightarrow \mu_x)$ is continuous and so $\{\mu_x : \mu \in H\}$ is $\sigma(M(X), C(X))$ -compact. The result follows from Theorem 5.

We are very grateful to the referee for some very useful suggestions which have improved the paper.

REFERENCES

- [1] Brooks, J. K., Lewis, P. W., Linear operators and vector measures, Trans. Amer. Math. Soc. 192(1974), 139-162. MR **49**:3585
- [2] Diestel, J., Uhl, J. J., Vector Measures, Amer. Math. Soc. Surveys, Vol. 15, Amer. Math. Soc., 1977. MR **56**:12216
- [3] Drewnowski, L., Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodym Theorems, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. 20(1972), 725-731. MR **47**:431
- [4] Freniche, F. J., Grothendieck locally convex spaces of continuous vector-valued functions, Pac. J. Math. 120(1985), 345-355. MR **87a**:46059
- [5] Grothendieck, A., Sur les applications lineaires faiblement compactes d'espace du type $C(K)$, Can. J. Math. 5(1953), 129-173. MR **15**:438b

- [6] Kelley, J. L., General Topology, Van Nostrand, Princeton, N. J., 1955. MR **16**:1136c
- [7] Khurana, S. S., Topologies on spaces of vector-valued continuous functions, Trans. Amer. Math. Soc. 241(1978), 195-211. MR **81f**:46053
- [8] Khurana, S. S., Topologies on spaces of vector-valued continuous functions II, Math. Ann. 234(1978), 159-166. MR **82f**:46042
- [9] Khurana, S. S., Grothendieck Spaces, II, J. Math. Anal. Appl. 159(1991), 202-207. MR **92i**:46042
- [10] Köthe, G., Topological vector spaces I, Springer-Verlag, New York, 1969. MR **40**:1750
- [11] Panchapagesan, T. V., Baire and σ -Borel characterizations of weakly compact sets in $M(T)$, Trans. Amer. Math. Soc. 350(1998), 4839-4847. MR **99h**:28008
- [12] Pryce, J. D., A device of R. J. Whitley applied to pointwise compactness in spaces of continuous functions, Proc. Lon. Math. Soc. 23(1971), 532-546. MR **45**:5729
- [13] Rao, M. M., Measure Theory and integration, John Wiley & Sons, New York, 1987. MR **89k**:28001
- [14] Schaefer, H. H., Topological vector spaces, Springer-Verlag, 1970. MR **49**:7722
- [15] Seever, G. L., Measures on F -spaces, Trans. Amer. Math. Soc. 133(1968), 267-280. MR **37**:1976
- [16] Ionescu Tulcea, A., Ionescu Tulcea, C., Topics in the theory of liftings, Springer-Verlag, New York, 1969. MR **43**:2185
- [17] Zhang, Xiao-Dong, On weak compactness in spaces of measures, J. Funct. Anal. 143(1997), 1-9. MR **98d**:46030

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242

E-mail address: `skhurana@blue.weeg.uiowa.edu`