NUMBER OF SYLOW SUBGROUPS IN p -SOLVABLE GROUPS

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ABSTRACT. If G is a finite group and p is a prime number, let $\nu_p(G)$ be the number of Sylow p -subgroups of G . If H is a subgroup of a p -solvable group G , we prove that $\nu_p(H)$ divides $\nu_p(G)$.

1. INTRODUCTION

Let G be a finite group, p a prime number, and write $\nu_p(G)$ for the number of Sylow p -subgroups of G . If H is a subgroup of G , then it is well known that $\nu_p(H) \leq \nu_p(G)$. In this note, we prove the following elementary fact which perhaps has not been noticed up to now.

Theorem A. *Suppose that G is a finite p -solvable group. If H is a subgroup of G , then $\nu_p(H)$ divides $\nu_p(G)$.*

If G is not p -solvable, then Theorem A is simply not true. For instance, if $G = A_5$, $H = A_4$ and $p = 3$, then $\nu_p(H) = 4$ and $\nu_p(G) = 10$.

2. PROOF

We start with the following result.

(2.1) **Lemma.** *Suppose that A is a finite group acting coprimely on a finite group G , and let H be an A -invariant subgroup of G . Let $C = \mathbf{C}_G(A)$. Then $|C : C \cap H|$ divides $|G : H|$.*

Proof. By standard arguments, recall that in any coprime action, if q is a prime, then every A -invariant q -subgroup of G is contained in an A -invariant Sylow q -subgroup of G . Furthermore, any two A -invariant Sylow q -subgroups are C -conjugate and if $P \in \text{Syl}_q(G)$ is A -invariant, then $P \cap C \in \text{Syl}_q(C)$. Now, let Q be an A -invariant Sylow q -subgroup of H . Then $Q \cap C \cap H = Q \cap C$ is a Sylow q -subgroup of $C \cap H$. Now, Q is contained in an A -invariant Sylow q -subgroup P of G . Also, $C \cap P$ is a Sylow q -subgroup of C . Now, $|C \cap P : C \cap Q| \leq |P : Q|$, and therefore $|C \cap P : C \cap Q|$ divides $|P : Q|$. We deduce that $|C|_q / |C \cap H|_q$ divides $|G|_q / |H|_q$. Hence, $|C : C \cap H|$ divides $|G : H|$. $\square$

(2.2) **Lemma.** *Suppose that G is p -solvable and let $P \in \text{Syl}_p(G)$. If $P \subseteq M \subseteq G$, then $|\mathbf{N}_G(P) : \mathbf{N}_M(P)|$ divides $|G : M|$.*

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Proof. We argue by induction on $|G|$. Suppose first that there is a proper normal subgroup N of G such that G/N is a p' -group. Let $V = M \cap N$. Now, by induction, we have that $|\mathbf{N}_N(P) : \mathbf{N}_V(P)|$ divides $|N : V|$. Now, by the Frattini argument, we have that $G = N\mathbf{N}_G(P)$. Hence, $G = (NM)\mathbf{N}_G(P)$ and thus $|G : NM| = |\mathbf{N}_G(P) : \mathbf{N}_{NM}(P)|$. Now, we claim that $\mathbf{N}_{NM}(P) = \mathbf{N}_N(P)\mathbf{N}_M(P)$. By the Frattini argument, we have that $M = (M \cap N)\mathbf{N}_M(P)$. Thus, $NM = N\mathbf{N}_M(P)$. Now, $\mathbf{N}_{NM}(P) = \mathbf{N}_{NM}(P) \cap \mathbf{N}_M(P)N = \mathbf{N}_M(P)\mathbf{N}_N(P)$, as claimed. Now,

$$\begin{aligned} |\mathbf{N}_G(P) : \mathbf{N}_M(P)| &= |\mathbf{N}_G(P) : \mathbf{N}_{NM}(P)| |\mathbf{N}_{NM}(P) : \mathbf{N}_M(P)| \\ &= |G : NM| |\mathbf{N}_N(P) : \mathbf{N}_V(P)| \end{aligned}$$

divides $|N : V| |G : NM|$. Now, $|N : V| = |NM : M|$, and we are easily done.

So we may assume that there is a proper normal subgroup N of G such that G/N is a p -group. Let $H = N \cap M$. By induction, we have that $|\mathbf{N}_N(P \cap N) : \mathbf{N}_H(P \cap N)|$ divides $|N : H| = |G : M|$. Now, let $G_0 = \mathbf{N}_G(P \cap N)$. We have that $\mathbf{N}_G(P) \subseteq G_0$. Let $M_0 = G_0 \cap M$. Suppose that G_0 is proper in G . Then $|\mathbf{N}_{G_0}(P) : \mathbf{N}_{M_0}(P)| = |\mathbf{N}_G(P) : \mathbf{N}_M(P)|$ divides $|G_0 : M_0|$. Now, since $P \subseteq \mathbf{N}_M(P \cap N)$ and $PN = G$, we have that

$$|G_0 : M_0| = |\mathbf{N}_G(P \cap N) : \mathbf{N}_M(P \cap N)| = |\mathbf{N}_N(P \cap N) : \mathbf{N}_H(P \cap N)|$$

divides $|N : H| = |G : M|$, and the result easily follows. Hence, we may assume that $P \cap N \triangleleft G$. Let $\bar{N} = N/P \cap N$. Then P acts coprimely on $\bar{N}$. Let $\bar{C} = \mathbf{N}_G(P) \cap N$, and notice that $\bar{C} = \mathbf{C}_{\bar{N}}(\bar{P})$. Then

$$|\mathbf{N}_G(P) : \mathbf{N}_M(P)| = |\bar{C} : \bar{C} \cap \bar{H}|.$$

Also,

$$|G : M| = |\bar{N} : \bar{H}|.$$

Now, the result follows by Lemma (2.1). $\square$

Proof of Theorem A. We argue by induction on $|G : H|$. Suppose that $H < K < G$. By induction, $\nu_p(H)$ divides $\nu_p(K)$, $\nu_p(K)$ divides $\nu_p(G)$, and we are done. So we may assume that H is a maximal subgroup of G . Since G is p -solvable, we have that $|G : H|$ is a power of p or not divisible by p .

Assume first that $|G : H|$ is a power of p , and let $P \in \text{Syl}_p(G)$. Since $(|G : H|, |G : P|) = 1$, we have that $G = HP$ and $Q = P \cap H \in \text{Syl}_p(H)$. Also, $G = H\mathbf{N}_G(P)$ and $|G : \mathbf{N}_G(P)| = |H : \mathbf{N}_H(P)|$. Now, $\mathbf{N}_H(P) \subseteq \mathbf{N}_H(P \cap H)$ and therefore $\nu_p(G) = |H : \mathbf{N}_H(P)|$ divides $|H : \mathbf{N}_H(P \cap H)| = \nu_p(H)$, and we are done in this case.

Now, assume that $|G : H|$ is not divisible by p and let $P \in \text{Syl}_p(H) \subseteq \text{Syl}_p(G)$. By Lemma (2.2), we have that $|\mathbf{N}_G(P) : \mathbf{N}_H(P)|$ divides $|G : H|$, and therefore $|H : \mathbf{N}_H(P)| = \nu_p(H)$ divides $|G : \mathbf{N}_G(P)| = \nu_p(G)$, as desired. $\square$

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