CONVERGENCE OF SEQUENCES OF SETS OF ASSOCIATED PRIMES

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ABSTRACT. It is a well-known result of M. Brodmann that if $\mathfrak{a}$ is an ideal of a commutative Noetherian ring A , then the set of associated primes $\text{Ass}(A/\mathfrak{a}^n)$ of the n -th power of $\mathfrak{a}$ is constant for all large n . This paper is concerned with the following question: given a prime ideal $\mathfrak{p}$ of A which is known to be in $\text{Ass}(A/\mathfrak{a}^n)$ for all large integers n , can one identify a term of the sequence $(\text{Ass}(A/\mathfrak{a}^n))_{n \in \mathbb{N}}$ beyond which $\mathfrak{p}$ will subsequently be an ever-present? This paper presents some results about convergence of sequences of sets of associated primes of graded components of finitely generated graded modules over a standard positively graded commutative Noetherian ring; those results are then applied to the above question.

0. INTRODUCTION

Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring A . It is a well-known result of M. Brodmann [1] that the sequences $(\text{Ass}(A/\mathfrak{a}^n))_{n \in \mathbb{N}}$ and $(\text{Ass}(\mathfrak{a}^n/\mathfrak{a}^{n+1}))_{n \in \mathbb{N}_0}$ have ultimate constant values, denoted here by $\text{Ass}^*(\mathfrak{a})$ and $\text{Bss}^*(\mathfrak{a})$, respectively. (We use $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of positive, non-negative and all integers, respectively.) This paper is about the way in which these sequences converge.

Neither sequence need be monotonic; see S. McAdam and P. Eakin [4, p. 80]. In that paper, McAdam and Eakin obtained a great deal of positive information about the convergence of the two sequences, some of which can be phrased in terms of the (ordinary) Rees ring $\mathcal{R}(\mathfrak{a})$ or the associated graded ring $\mathcal{G}(\mathfrak{a})$ of $\mathfrak{a}$. Let $\mathcal{R}(\mathfrak{a})_+$ denote the irrelevant ideal of $\mathcal{R}(\mathfrak{a})$; recall that $\text{Proj}(\mathcal{R}(\mathfrak{a}))$ denotes the set of graded prime ideals of $\mathcal{R}(\mathfrak{a})$ which do not contain the irrelevant ideal. Also, for a graded $\mathcal{R}(\mathfrak{a})$ -module H , let $a_{\mathcal{R}(\mathfrak{a})_+}^0(H)$ denote the supremum of the degrees of non-zero homogeneous elements of the graded submodule

$$\Gamma_{\mathcal{R}(\mathfrak{a})_+}(H) = \bigcup_{n=1}^{\infty} (0 :_H \mathcal{R}(\mathfrak{a})_+^n)$$

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of H . (Interpret $a_{\mathcal{R}(\mathfrak{a})_+}^0(H)$ as $-\infty$ if $\Gamma_{\mathcal{R}(\mathfrak{a})_+}(H) = 0$. If H is finitely generated, then $a_{\mathcal{R}(\mathfrak{a})_+}^0(H)$ is an integer or $-\infty$.) We denote by $\text{Var}(\mathfrak{a})$ the *variety* $\text{Supp}_A(A/\mathfrak{a})$ of $\mathfrak{a}$. The following results of McAdam and Eakin are fundamental.

0.1. Theorem (McAdam and Eakin). *We have*

- (i) $\text{Ass}^*(\mathfrak{a}) = \text{Bss}^*(\mathfrak{a}) \cup (\text{Var}(\mathfrak{a}) \cap \text{Ass } A)$ ([4, Proposition 10 and Corollary 13]);
- (ii) $(\text{Ass}(\mathfrak{a}^n/\mathfrak{a}^{n+1}))_{n \in \mathbb{N}_0}$ is increasing for $n > a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{G}(\mathfrak{a}))$ (see [4, pp. 71–72]);
- (iii) $\text{Bss}^*(\mathfrak{a}) = \{\mathfrak{P} \cap A : \mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a})) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))\}$ (see [4, Proposition 16]).

(In connection with Theorem 0.1(ii), it should be noted that if the i -th component of the graded module $(0 :_{\mathcal{G}(\mathfrak{a})} \mathcal{R}(\mathfrak{a})_+)$ is zero for all i greater than an integer b , then $a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{G}(\mathfrak{a})) \leq b$. This can be established by an easy inductive argument.)

This paper is concerned with the following question: given a prime ideal $\mathfrak{p}$ of A which is known to be in $\text{Ass}^*(\mathfrak{a})$, can one identify an integer $n_{\mathfrak{p}}$ with the property that $\mathfrak{p} \in \text{Ass}(A/\mathfrak{a}^n)$ for all $n > n_{\mathfrak{p}}$? In other words, can one identify a term of the sequence $(\text{Ass}(A/\mathfrak{a}^n))_{n \in \mathbb{N}}$ beyond which $\mathfrak{p}$ will subsequently be an ‘ever-present’? The main results of this paper in this direction are as follows. Let $\text{reg}(\mathcal{G}(\mathfrak{a}))$ denote the Castelnuovo regularity of $\mathcal{G}(\mathfrak{a})$.

It is shown in §2 that $\mathfrak{p} \in \text{Ass}(A/\mathfrak{a}^n)$ for all $n > \text{reg}(\mathcal{G}(\mathfrak{a})) + 1$ if

- $\mathfrak{p} = \mathfrak{P} \cap A$ where $\mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}))$ and $\text{ht}((\mathfrak{P} + \mathcal{R}(\mathfrak{a})_+)/\mathfrak{P}) = 1$,

or

- $\mathfrak{p}$ is a minimal member of the set $\{\mathfrak{Q} \cap A : \mathfrak{Q} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a})) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))\}$,

or

- $\mathfrak{p} \in \text{Ass}^*(\mathfrak{a}) \setminus \text{Bss}^*(\mathfrak{a})$.

These results are essentially consequences of results in §1 about the convergence of the sequence $(\text{Ass}_{R_0} M_n)_n$, where $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a finitely generated graded module over a positively graded commutative Noetherian ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ which is standard (that is, such that $R = R_0[R_1]$). These results are perhaps of independent interest. Applications to the convergence of the sequences $(\text{Ass}(A/\mathfrak{a}^n))_{n \in \mathbb{N}}$ and $(\text{Ass}(\mathfrak{a}^n/\mathfrak{a}^{n+1}))_{n \in \mathbb{N}_0}$ are presented in §2.

1. GRADED MODULES OVER A STANDARD POSITIVELY GRADED RING

1.1. Notation. Throughout the paper, $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ will denote a positively graded commutative Noetherian ring which is *standard* in the sense that $R = R_0[R_1]$. We denote the set of graded prime ideals of R by ${}^*\text{Spec}(R)$, and, for a graded ideal $\mathfrak{b}$ of R , we will use ${}^*\text{Var}(\mathfrak{b})$ as an abbreviation for $\text{Var}(\mathfrak{b}) \cap {}^*\text{Spec}(R)$. Also, $M = \bigoplus_{n \in \mathbb{Z}} M_n$ will denote a graded R -module; M will only be assumed to be finitely generated when this is explicitly stated.

We use, for $j \in \mathbb{Z}$, the notation L_j to denote the j -th component of a $\mathbb{Z}$ -graded module L , and $(\bullet)(j)$ to denote the j -th shift functor on the category of graded R -modules and homogeneous homomorphisms (by ‘homogeneous’ here, we mean ‘homogeneous of degree zero’). It will be convenient to have available the concepts of the *end* and *beginning* ($\text{beg}(L)$) of the graded R -module $L = \bigoplus_{n \in \mathbb{Z}} L_n$, which

are defined by

$$\text{end}(L) := \sup \{n \in \mathbb{Z} : L_n \neq 0\} \quad \text{and} \quad \text{beg}(L) := \inf \{n \in \mathbb{Z} : L_n \neq 0\}.$$

(Note that $\text{end}(L)$ could be ∞ , and that the supremum of the empty set of integers is to be taken as $-\infty$; similar comments apply to $\text{beg}(L)$.) For $i \in \mathbb{N}_0$, we denote $\text{end}(H_{R_+}^i(M))$ by $a_{R_+}^i(M)$; when M is finitely generated, this is an integer or $-\infty$.

1.2. Lemma. *We have*

$$a_{R_+}^0(M) = \sup \{n \in \mathbb{Z} : \text{there exists } m_n \in M_n \text{ with } (0 :_R m_n) \in \text{Ass}_R M \cap {}^*\text{Var}(R_+)\}.$$

Proof. If $\mathfrak{p} \in \text{Ass}_R M \cap {}^*\text{Var}(R_+)$ and $n \in \mathbb{Z}$, $m_n \in M_n$ are such that $(0 :_R m_n) = \mathfrak{p}$, then $0 \neq m_n \in \Gamma_{R_+}(M)$, and so $n \leq a_{R_+}^0(M)$. On the other hand, for any integer $b \leq a_{R_+}^0(M)$, the submodule $N := \bigoplus_{j \geq b} \Gamma_{R_+}(M)_j$ is a non-zero graded R -module and so must have an associated prime $\mathfrak{q}$; in fact, $\mathfrak{q} = (0 : g_j)$ for some homogeneous element g_j of N of degree j , say. Since g_j is annihilated by some power of R_+ , we must have $\mathfrak{q} \in \text{Ass}_R M \cap {}^*\text{Var}(R_+)$. $\square$

1.3. Lemma. *Let $\mathfrak{p} \in \text{Ass}_R M \cap \text{Proj}(R)$. Then there exists $w_{\mathfrak{p}} \in \mathbb{Z}$ such that $\mathfrak{p} \cap R_0 \in \text{Ass}_{R_0} M_n$ for all $n \geq w_{\mathfrak{p}}$.*

Proof. There exists a homogeneous element $m_{w_{\mathfrak{p}}}$, of degree $w_{\mathfrak{p}}$, say, such that $(0 :_R m_{w_{\mathfrak{p}}}) = \mathfrak{p}$. Since $\mathfrak{p} \in \text{Proj}(R)$ and R is standard, there exists $a_1 \in R_1 \setminus \mathfrak{p}$, and $(0 :_R a_1^k m_{w_{\mathfrak{p}}}) = \mathfrak{p}$ for all $k \in \mathbb{N}_0$. Therefore $\mathfrak{p} \cap R_0 \in \text{Ass}_{R_0} M_{k+w_{\mathfrak{p}}}$ for all $k \in \mathbb{N}_0$. $\square$

The statement of the next theorem involves the 0-th level anchor points for M ; see [5, §0].

1.4. Theorem. *Let $\mathfrak{p}_0 \in \text{Ass}_{R_0} M_n$, where $n \in \mathbb{Z}$. Then*

- (i) $\mathfrak{p}_0 = \mathfrak{p} \cap R_0$ for some $\mathfrak{p} \in \text{Ass}_R M \cap \text{Proj}(R)$, and $\mathfrak{p}_0 \in \text{Ass}_{R_0} M_i$ for all $i \geq n$;

or

- (ii) $\mathfrak{p}_0 = \mathfrak{p} \cap R_0$ for some $\mathfrak{p} \in \text{Ass}_R M \cap {}^*\text{Var}(R_+)$, there exists a 0-th level anchor point h of $\mathfrak{p}$ for M with $n \leq h \leq a^0 := a_{R_+}^0(M)$, and $\mathfrak{p}_0 \in \text{Ass}_{R_0} M_i$ for all $i = n, n+1, \dots, h$.

Note. It should be noted that the conclusions in (i) and (ii) of the above statement could occur simultaneously.

Proof. One can use homogeneous localization at $\mathfrak{p}_0 + R_+$ to see that it is enough to prove the theorem under the additional hypotheses that R is ${}^*\text{local}$ with unique ${}^*\text{maximal ideal } \mathfrak{m}$, and $\mathfrak{m}_0 := \mathfrak{m} \cap R_0 = \mathfrak{p}_0$.

There exists $g_n \in M_n$ such that $(0 :_{R_0} g_n) = \mathfrak{m}_0$. The graded R -submodule Rg_n of M must have an associated prime: thus there exists $k \in \mathbb{N}_0$ and a homogeneous element $r_k \in R_k$ such that $(0 :_R r_k g_n) = \mathfrak{p}$, a graded prime ideal of R with $\mathfrak{p} \cap R_0 = \mathfrak{m}_0$.

First consider the case where $\mathfrak{p} \in \text{Proj}(R)$. Let $j \in \mathbb{N}$. Then, since R is standard, there exists $a_1 \in R_1 \setminus \mathfrak{p}$, and $\mathfrak{m}_0 \subseteq (0 :_{R_0} a_1^j g_n)$ and $a_1^j g_n \neq 0$ because $a_1^j r_k g_n \neq 0$. Hence $\mathfrak{m}_0 = (0 :_{R_0} a_1^j g_n)$, so that $\mathfrak{m}_0 \in \text{Ass}_{R_0} M_{n+j}$. This is true for all $j \in \mathbb{N}$.

Now consider the remaining case, where $\mathfrak{p} \in {}^*\mathrm{Var}(R_+)$. We use ${}^*E_R(N)$ to denote the * injective envelope of a graded R -module N . Set $h := k + n$. In this case, ${}^*E_R(Rr_k g_n)$ is a direct summand of ${}^*E_R(M)$, and so the latter graded R -module has a direct summand homogeneously isomorphic to ${}^*E_R(R/\mathfrak{p})(-h)$. Thus h is a 0-th level anchor point of $\mathfrak{p}$ for M .

Also, it follows from the fact that R is standard that, for each $j = 1, \dots, k$, there exists a homogeneous element $r_j \in R_j$ such that $r_j g_n \neq 0$. We must therefore have $\mathfrak{m}_0 = (0 :_{R_0} r_j g_n)$; note also that $r_j g_n \in M_{n+j}$. Hence $\mathfrak{m}_0 \in \mathrm{Ass}_{R_0} M_i$ for all $i = n, n+1, \dots, n+k = h$. This completes the proof. $\square$

In the special case when M is finitely generated, the fact that $\mathrm{Ass}_{R_0} M_n \subseteq \mathrm{Ass}_{R_0} M_{n+1}$ for all $n > a_{R_+}^0(M)$ is (essentially) proved in [4, pp. 71–72]. The following corollary presents, among other things, the corresponding statement in the general case.

1.5. Corollary. *Let the situation be as in Theorem 1.4, and let n be an integer.*

- (i) $\mathrm{Ass}_{R_0} M_n \subseteq \mathrm{Ass}_{R_0} M_{n+1}$ provided n is not a 0-th level anchor point for M . In particular, $\mathrm{Ass}_{R_0} M_n \subseteq \mathrm{Ass}_{R_0} M_{n+1}$ if $n < \inf\{h : h \text{ is a 0-th level anchor point for } M\}$ or if $n > a_{R_+}^0(M)$, and $\mathrm{Ass}_{R_0} M_n \subseteq \mathrm{Ass}_{R_0} M_{n+1}$ for every value of n if $\Gamma_{R_+}(M) = 0$.

- (ii) We always have $\mathrm{Ass}_{R_0} M_n \subseteq \{\mathfrak{p} \cap R_0 : \mathfrak{p} \in \mathrm{Ass}_R M\}$, while

$$\mathrm{Ass}_{R_0} M_n \subseteq \{\mathfrak{p} \cap R_0 : \mathfrak{p} \in \mathrm{Proj}(R) \cap \mathrm{Ass}_R M\} \quad \text{if } n > a_{R_+}^0(M).$$

- (iii) If $\mathrm{Proj}(R) \cap \mathrm{Ass}_R M$ is finite (this is the case if M is finitely generated, of course), then

$$\mathrm{Ass}_{R_0} M_n = \{\mathfrak{p} \cap R_0 : \mathfrak{p} \in \mathrm{Proj}(R) \cap \mathrm{Ass}_R M\} \quad \text{for all } n \gg 0.$$

Proof. (i) This all follows from Theorem 1.4 once it has been recalled from [5, Theorem 2.4] that $a_{R_+}^0(M)$ is the least upper bound of all the 0-th level anchor points for M .

(ii) The first claim follows from Theorem 1.4. Furthermore, if $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0} M_n$, where $n > a_{R_+}^0(M)$, then it follows from that theorem that $\mathfrak{p}_0 = \mathfrak{p} \cap R_0$ for some $\mathfrak{p} \in \mathrm{Ass}_R M \cap \mathrm{Proj}(R)$, because the conclusion of Theorem 1.4(ii) cannot hold.

(iii) By part (ii), $\mathrm{Ass}_{R_0} M_n \subseteq \{\mathfrak{p} \cap R_0 : \mathfrak{p} \in \mathrm{Proj}(R) \cap \mathrm{Ass}_R M\}$ for all $n > a_{R_+}^0(M)$. Now consider $\mathfrak{p} \in \mathrm{Proj}(R) \cap \mathrm{Ass}_R M$. Then $\mathfrak{p} \cap R_0 \in \mathrm{Ass}_{R_0} M_n$ for all $n \gg 0$, by Lemma 1.3. Since $\mathrm{Proj}(R) \cap \mathrm{Ass}_R M$ is finite, this is enough to establish the claim. $\square$

1.6. Remark. In the special case of Corollary 1.5 in which M is finitely generated, there are only finitely many 0-th level anchor points for M , and so there are at most finitely many integers n for which $\mathrm{Ass}_{R_0} M_n \not\subseteq \mathrm{Ass}_{R_0} M_{n+1}$.

The example in which $R = K[X]$, the ring of polynomials over a field K , graded in the usual way, and

$$M = (K[X]/(X^t))(t+1) \oplus K[X],$$

where $t \in \mathbb{N}$, illustrates both possibilities described in the conclusion of Theorem 1.4. Note that -2 is a 0-th level anchor point of (X) for M . Also, the sequence

$$(\mathrm{Ass}_{R_0} M_n \cap \{\mathfrak{p} \cap R_0 : \mathfrak{p} \in \mathrm{Proj}(R) \cap \mathrm{Ass}_R M\})_{n=\mathrm{beg}(M), \mathrm{beg}(M)+1, \dots}$$

is not increasing in this example.

The next result uses the ideal transform $D_{R_+}(M)$ of M with respect to R_+ ; see [2, §2.2 and 12.4.2].

1.7. Theorem. *Let the situation be as in Theorem 1.4. Then*

- (i) $\text{Ass}_R D_{R_+}(M) = \text{Proj}(R) \cap \text{Ass}_R M$;
- (ii) $\text{Ass}_{R_0}(D_{R_+}(M)_n) \subseteq \text{Ass}_{R_0}(D_{R_+}(M)_{n+1})$ for all $n \in \mathbb{Z}$;
- (iii) if M is finitely generated,

$$\text{Ass}_{R_0}(D_{R_+}(M)_n) = \{\mathfrak{p} \cap R_0 : \mathfrak{p} \in \text{Proj}(R) \cap \text{Ass}_R M\} \quad \text{for all } n \gg 0.$$

Proof. Set $\overline{M} := M/\Gamma_{R_+}(M)$, and note that, by [2, 2.1.12],

$$\text{Ass}_R(\overline{M}) = \text{Proj}(R) \cap \text{Ass}_R M.$$

The exact sequence

$$0 \longrightarrow \overline{M} \longrightarrow D_{R_+}(M) \longrightarrow H_{R_+}^1(M) \longrightarrow 0$$

of graded R -modules and homogeneous homomorphisms, together with the fact ([2, 2.2.8(iv)]) that $\Gamma_{R_+}(D_{R_+}(M)) = 0$, now shows that $\text{Ass}_R D_{R_+}(M) = \text{Ass}_R \overline{M} = \text{Proj}(R) \cap \text{Ass}_R M$. Note that each associated prime ideal of the R -module $H_{R_+}^1(M)$ must contain R_+ . This proves part (i).

Since $a_{R_+}^0(D_{R_+}(M)) = \text{end}(H_{R_+}^0(D_{R_+}(M))) = -\infty$, the claim in part (ii) follows from Corollary 1.5(i), while part (iii) follows from part (i) and Corollary 1.5(iii). $\square$

For the remainder of this section, we shall suppose that the graded R -module M is finitely generated. In the situation of Corollary 1.5 it would be interesting to have some estimate of the point of the sequence $(\text{Ass}_{R_0} M_n)_{n=\text{beg}(M), \text{beg}(M)+1, \dots}$ at which the ultimate constant value is first attained. A related, but perhaps easier, question is the following: given $\mathfrak{p} \in \text{Proj}(R) \cap \text{Ass}_R M$, so that, by Corollary 1.5(iii), $\mathfrak{p}_0 := \mathfrak{p} \cap R_0$ belongs to all except finitely many terms of the sequence, can one estimate an $n_0 \in \mathbb{Z}$ such that $\mathfrak{p}_0 \in \text{Ass}_{R_0} M_n$ for all $n > n_0$? We next present two results of this type. For the first, recall the definition of the Castelnuovo regularity $\text{reg}(M)$ of M .

It is well known that $a_{R_+}^i(M) := \text{end}(H_{R_+}^i(M)) < \infty$ for all $i \in \mathbb{N}_0$. Also, $H_{R_+}^i(M) = 0$ for $i > \text{ara}(R_+)$, the arithmetic rank of R_+ ; see [2, §3.3]. The Castelnuovo regularity $\text{reg}(M)$ of M can be defined by

$$\text{reg}(M) := \sup \left\{ a_{R_+}^i(M) + i : i \in \mathbb{N}_0 \right\};$$

this is either an integer or $-\infty$. (See [2, Chapter 15].) Note that $a_{R_+}^0(M) \leq \text{reg}(M)$.

1.8. Proposition. *Assume that the graded R -module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is finitely generated. Suppose that $\mathfrak{p}_0$ is a minimal member of the set*

$$\{\mathfrak{q} \cap R_0 : \mathfrak{q} \in \text{Proj}(R) \cap \text{Ass}_R M\}.$$

Then $\mathfrak{p}_0 \in \text{Ass}_{R_0} M_n$ for all $n > \text{reg}(M)$.

Proof. One can use homogeneous localization at $\mathfrak{p}_0 R + R_+$ to see that it is enough to prove the lemma under the additional hypothesis that R is $\ast$ local with unique $\ast$ maximal ideal $\mathfrak{m}$, and that $\mathfrak{m}_0 := \mathfrak{m} \cap R_0 = \mathfrak{p}_0$.

Suppose that there exists $i \in \mathbb{N}$ such that $\mathfrak{m}_0 \notin \text{Ass}_{R_0} M_{\text{reg}(M)+i}$. Then, by Corollary 1.5, we have

$$\text{Ass}_{R_0} M_{\text{reg}(M)+i} = \emptyset,$$

so that $M_{\text{reg}(M)+i} = 0$. But M can be generated by homogeneous elements of degrees not exceeding $\text{reg}(M)$ (by [2, 15.3.1]). Since R is homogeneous, it follows from this that, since $M_{\text{reg}(M)+i} = 0$, we must have $M_{\text{reg}(M)+n} = 0$ for every $n \geq i$. However, Corollary 1.5 shows that $\text{Ass}_{R_0} M_{\text{reg}(M)+n} = \{\mathfrak{m}_0\}$ for all $n \gg 0$, and so we have arrived at a contradiction. $\square$

1.9. Proposition. *Assume that the graded R -module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is finitely generated.*

Let $\mathfrak{p} \in \text{Proj}(R) \cap \text{Ass}_R M$; set $\mathfrak{p}_0 = \mathfrak{p} \cap R_0$ and suppose that

$$\text{ht}((\mathfrak{p} + R_+)/\mathfrak{p}) = 1.$$

Then $\mathfrak{p}_0 \in \text{Ass}_{R_0} M_n$ for all $n > \max\{a_{R_+}^0(M), a_{R_+}^1(M)\}$.

Proof. Again, one can use homogeneous localization at $\mathfrak{p}_0 R + R_+$ to see that it is enough to prove the lemma under the additional hypothesis that R is $\ast$ local with unique $\ast$ maximal ideal $\mathfrak{m}$, and that $\mathfrak{m}_0 := \mathfrak{m} \cap R_0 = \mathfrak{p} \cap R_0$.

Set $b := \sup\{a_{R_+}^0(M), a_{R_+}^1(M)\}$ and $\overline{M} := M/\Gamma_{R_+}(M)$. Let $\pi : M \rightarrow \overline{M}$ denote the natural epimorphism. Since $\overline{M}$ is R_+ -torsion-free, it follows (from [3, 1.5.10], for example) that there exists $d \in \mathbb{N}$ and $r_d \in R_d \setminus \mathfrak{p}$ such that r_d is a non-zerodivisor on $\overline{M}$.

Since $\text{end}(\Gamma_{R_+}(M)) \leq b$, it follows from an elementary calculation that both the kernel and the cokernel of the homogeneous homomorphism

$$\ast \text{Hom}_R(R/\mathfrak{p}, \pi) : \ast \text{Hom}_R(R/\mathfrak{p}, M) \rightarrow \ast \text{Hom}_R(R/\mathfrak{p}, \overline{M})$$

have ends not exceeding b . We wish to show that the kernel and the cokernel of the homogeneous homomorphism $\ast \text{Hom}_R(R/\mathfrak{p}, M) \xrightarrow{r_d} \ast \text{Hom}_R(R/\mathfrak{p}, M)(d)$ both have ends not exceeding b . It is therefore sufficient, in order to achieve this, to show that the kernel K and the cokernel C of the homogeneous homomorphism $\ast \text{Hom}_R(R/\mathfrak{p}, \overline{M}) \xrightarrow{r_d} \ast \text{Hom}_R(R/\mathfrak{p}, \overline{M})(d)$ both have ends not exceeding b . In this case, $K = 0$, because r_d is a non-zerodivisor on $\overline{M}$.

The exact sequence

$$0 \rightarrow \overline{M} \xrightarrow{r_d} \overline{M}(d) \rightarrow (\overline{M}/r_d \overline{M})(d) \rightarrow 0$$

induces an exact sequence of graded R -modules and homogeneous homomorphisms

$$0 \rightarrow \ast \text{Hom}_R(R/\mathfrak{p}, \overline{M}) \xrightarrow{r_d} \ast \text{Hom}_R(R/\mathfrak{p}, \overline{M})(d) \rightarrow \ast \text{Hom}_R(R/\mathfrak{p}, \overline{M}/r_d \overline{M})(d).$$

Hence $\text{end}(C) \leq \text{end}(\ast \text{Hom}_R(R/\mathfrak{p}, \overline{M}/r_d \overline{M})(d)) \leq \text{end}(\Gamma_{\mathfrak{p}}(\overline{M}/r_d \overline{M})(d))$. Now

$$\Gamma_{\mathfrak{p}}(\overline{M}/r_d \overline{M}) = \Gamma_{\mathfrak{p} + Rr_d}(\overline{M}/r_d \overline{M}).$$

But $\sqrt{\mathfrak{p} + Rr_d} = \mathfrak{m}$, in view of the hypotheses about $\mathfrak{p}$. Hence

$$\text{end}(C) \leq \text{end}(\Gamma_{R_+}(\overline{M}/r_d \overline{M})(d)).$$

Use again the exact sequence $0 \rightarrow \overline{M} \xrightarrow{r_d} \overline{M}(d) \rightarrow (\overline{M}/r_d \overline{M})(d) \rightarrow 0$, this time in conjunction with the fact that $\Gamma_{R_+}(\overline{M}) = 0$, to see that $\Gamma_{R_+}(\overline{M}/r_d \overline{M})(d)$ is homogeneously isomorphic to a graded submodule of $H_{R_+}^1(\overline{M})$. Since the latter graded module is homogeneously isomorphic to $H_{R_+}^1(M)$, it follows that $\text{end}(C) \leq b$.

It therefore follows that

$$\ast \text{Hom}_R(R/\mathfrak{p}, M)_{n+d} = r_d \ast \text{Hom}_R(R/\mathfrak{p}, M)_n \quad \text{for all } n > b.$$

Since $\mathfrak{p} \in \text{Ass}_R M$, there exists a homogeneous element g_j of M_j (for some $j \in \mathbb{Z}$) with annihilator exactly $\mathfrak{p}$; since R is standard, there exists $r'_1 \in R_1 \setminus \mathfrak{p}$. Multiplication of g_j by r'_1 produces another homogeneous element, of degree $j+1$, also with annihilator exactly $\mathfrak{p}$. The last display therefore shows that there is an element of M_{b+1} with annihilator exactly $\mathfrak{p}$, and the claim follows easily from these observations. $\square$

2. APPLICATION TO ASYMPTOTIC PRIMES

In this section, we apply the results of §1 to obtain some results about the convergence of the sequences $(\text{Ass}(N/\mathfrak{a}^n N))_{n \in \mathbb{N}}$ and $(\text{Ass}(\mathfrak{a}^n N/\mathfrak{a}^{n+1} N))_{n \in \mathbb{N}_0}$, where N is a finitely generated module over the commutative Noetherian ring A and $\mathfrak{a}$ is an ideal of A . (This notation will be in force for this entire section.) Denote the ultimate constant values of these sequences by $\text{Ass}^*(\mathfrak{a}, N)$ and $\text{Bss}^*(\mathfrak{a}, N)$, respectively. Denote by $\mathcal{G}(\mathfrak{a}, N)$ the associated graded module of $\mathfrak{a}$ with respect to N . For each $t \in \mathbb{N}$, let $\mathcal{G}(\mathfrak{a}, N, t)$ denote the graded $\mathcal{R}(\mathfrak{a})$ -module

$$\cdots \oplus 0 \oplus 0 \oplus N/\mathfrak{a}N \oplus N/\mathfrak{a}^2 N \oplus \cdots \oplus N/\mathfrak{a}^t N \oplus \mathfrak{a}N/\mathfrak{a}^{t+1} N \oplus \mathfrak{a}^2 N/\mathfrak{a}^{t+2} N \oplus \cdots,$$

where it is to be understood that $N/\mathfrak{a}N$ is the component of degree 1. (Thus $\mathcal{G}(\mathfrak{a}, N, 1) = \mathcal{G}(\mathfrak{a}, N)(-1)$.)

Also, let $N[X]$ denote the $A[X]$ -module $N \otimes_A A[X]$ (where X is an indeterminate over A); in $N[X]$, denote $y \otimes X^j$, where $y \in N$ and $j \in \mathbb{N}_0$, by yX^j . Note that $N[X]$ can be regarded as an $\mathcal{R}(\mathfrak{a})$ -module by restriction of scalars. Let

$$\mathcal{R}(\mathfrak{a}, N) = \left\{ \sum_{j=0}^t n_j X^j : t \in \mathbb{N}_0, n_j \in \mathfrak{a}^j N \ (0 \leq j \leq t) \right\},$$

an $\mathcal{R}(\mathfrak{a})$ -submodule of $N[X]$ which we refer to as the *Rees module of $\mathfrak{a}$ with respect to N* .

We begin by applying results from §1 to the graded $\mathcal{R}(\mathfrak{a})$ -modules $\mathcal{G}(\mathfrak{a}, N)$ and $N[X]/\mathcal{R}(\mathfrak{a}, N)$ to obtain results about the convergence of the sequences

$$(\text{Ass}(\mathfrak{a}^n N/\mathfrak{a}^{n+1} N))_{n \in \mathbb{N}_0} \quad \text{and} \quad (\text{Ass}(N/\mathfrak{a}^n N))_{n \in \mathbb{N}}.$$

The first two results are such consequences of Lemma 1.3.

2.1. Proposition. *Let $\mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N)) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))$. Then there exists $w_{\mathfrak{P}} \in \mathbb{N}_0$ such that*

$$\mathfrak{P} \cap A \in \text{Ass}(\mathfrak{a}^n N/\mathfrak{a}^{n+1} N) \text{ for all } n \geq w_{\mathfrak{P}}.$$

2.2. Proposition. *Let $\mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(N[X]/\mathcal{R}(\mathfrak{a}, N)) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))$. Then there exists $u_{\mathfrak{P}} \in \mathbb{N}$ such that $\mathfrak{P} \cap A \in \text{Ass}(N/\mathfrak{a}^n N)$ for all $n \geq u_{\mathfrak{P}}$.*

The next result is an immediate consequence of Theorem 1.4.

2.3. Theorem. *Let $\mathfrak{p} \in \text{Ass}(\mathfrak{a}^n N/\mathfrak{a}^{n+1} N)$, where $n \in \mathbb{N}_0$. Then*

(i) $\mathfrak{p} = \mathfrak{P} \cap A$ for some $\mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N)) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))$ and

$$\mathfrak{p} \in \text{Ass}(\mathfrak{a}^i N/\mathfrak{a}^{i+1} N) \text{ for all } i \geq n;$$

or

(ii) $\mathfrak{p} = \mathfrak{P} \cap A$ for some $\mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N)) \cap {}^*\text{Var}(\mathcal{R}(\mathfrak{a})_+)$, there exists a 0-th level anchor point h of $\mathfrak{P}$ for $\mathcal{G}(\mathfrak{a}, N)$ with $n \leq h \leq a^0 := a^0_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{G}(\mathfrak{a}, N))$, and $\mathfrak{p} \in \text{Ass}(\mathfrak{a}^i N/\mathfrak{a}^{i+1} N)$ for all $i = n, n+1, \dots, h$.

The next result is an immediate consequence of Corollary 1.5.

2.4. Proposition. (i) $\text{Ass}(\mathfrak{a}^n N / \mathfrak{a}^{n+1} N) \subseteq \text{Ass}(\mathfrak{a}^{n+1} N / \mathfrak{a}^{n+2} N)$ for all $n > a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{G}(\mathfrak{a}, N))$.

(ii) $\text{Ass}(\mathfrak{a}^n N / \mathfrak{a}^{n+1} N) = \{\mathfrak{P} \cap A : \mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N)) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))\}$ for all $n \gg 0$.

Given a $\mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N)) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))$, can one identify a term of the sequence

$$(\text{Ass}(\mathfrak{a}^n N / \mathfrak{a}^{n+1} N))_{n \in \mathbb{N}_0}$$

beyond which $\mathfrak{P} \cap A$ will subsequently be an ‘ever-present’? The next two results, which follow from Propositions 1.8 and 1.9, give such estimates in special cases.

2.5. Proposition. If $\mathfrak{p}$ is a minimal member of the set

$$\{\mathfrak{Q} \cap A : \mathfrak{Q} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N)) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))\},$$

then $\mathfrak{p} \in \text{Ass}(\mathfrak{a}^n N / \mathfrak{a}^{n+1} N)$ for all $n > \text{reg}(\mathcal{G}(\mathfrak{a}, N))$.

2.6. Proposition. Let $\mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N)) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))$ and suppose that $\text{ht}((\mathfrak{P} + \mathcal{R}(\mathfrak{a})_+)/\mathfrak{P}) = 1$. Then $\mathfrak{p} := \mathfrak{P} \cap A \in \text{Ass}(\mathfrak{a}^n N / \mathfrak{a}^{n+1} N)$ for all $n > \max\{a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{G}(\mathfrak{a}, N)), a_{\mathcal{R}(\mathfrak{a})_+}^1(\mathcal{G}(\mathfrak{a}, N))\}$.

2.7. Lemma. We have $\text{Ass}_A(N / \mathfrak{a}^n N) \subseteq \{\mathfrak{P} \cap A : \mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N))\}$ for all $n \in \mathbb{N}$.

Proof. When $t > 1$, there is an exact sequence

$$0 \longrightarrow \mathcal{G}(\mathfrak{a}, N, t-1) \longrightarrow \mathcal{G}(\mathfrak{a}, N, t) \longrightarrow \mathcal{G}(\mathfrak{a}, N)(-t) \longrightarrow 0$$

of graded $\mathcal{R}(\mathfrak{a})$ -modules and homogeneous homomorphisms, and so it follows easily by induction on t that, for all $t \in \mathbb{N}$, the graded $\mathcal{R}(\mathfrak{a})$ -module $\mathcal{G}(\mathfrak{a}, N, t)$ is finitely generated and such that

$$\text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N, t)) = \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a}, N)).$$

Since, for each $n \in \mathbb{N}$, the A -module $N / \mathfrak{a}^n N$ is the n -th component of $\mathcal{G}(\mathfrak{a}, N, n)$, the result follows from Corollary 1.5(ii). $\square$

2.8. Proposition. Let $\mathfrak{p} \in \text{Ass}^*(\mathfrak{a}, N) \setminus \text{Bss}^*(\mathfrak{a}, N)$. Then $\mathfrak{p} \in \text{Ass}(N / \mathfrak{a}^n N)$ for all positive integers n with $n > a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{G}(\mathfrak{a}, N))$.

Proof. Suppose there exists a positive integer n with $n > a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{G}(\mathfrak{a}, N))$ such that $\mathfrak{p} \notin \text{Ass}(N / \mathfrak{a}^n N)$. Since $\mathfrak{p} \in \text{Ass}^*(\mathfrak{a}, N)$, there exists $t \in \mathbb{N}$ such that $\mathfrak{p} \in \text{Ass}(N / \mathfrak{a}^{n+t} N)$ but $\mathfrak{p} \notin \text{Ass}(N / \mathfrak{a}^{n+t-1} N)$. Now we cannot have

$$\mathfrak{p} \in \text{Ass}(\mathfrak{a}^{n+t-1} N / \mathfrak{a}^{n+t} N)$$

since, by Proposition 2.4(i), that would imply that $\mathfrak{p} \in \text{Bss}^*(\mathfrak{a}, N)$. It therefore follows from the exact sequence

$$0 \longrightarrow \mathfrak{a}^{n+t-1} N / \mathfrak{a}^{n+t} N \longrightarrow N / \mathfrak{a}^{n+t} N \longrightarrow N / \mathfrak{a}^{n+t-1} N \longrightarrow 0$$

that $\mathfrak{p} \in \text{Ass}(N / \mathfrak{a}^{n+t-1} N)$, a contradiction. Hence $\mathfrak{p} \in \text{Ass}(N / \mathfrak{a}^m N)$ for all positive integers m with $m > a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{G}(\mathfrak{a}, N))$. $\square$

Actually, one can improve on Proposition 2.8. It is easy to modify the argument of N. V. Trung in [6, Theorem 3.1(i)] to show that $a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{R}(\mathfrak{a}, N)) \leq a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{G}(\mathfrak{a}, N))$.

2.9. Proposition. *Let $\mathfrak{p} \in \text{Ass}^*(\mathfrak{a}, N) \setminus \text{Bss}^*(\mathfrak{a}, N)$. Then $\mathfrak{p} \in \text{Ass}(N/\mathfrak{a}^n N)$ for all positive integers n with $n > a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{R}(\mathfrak{a}, N))$.*

Proof. An obvious modification of the arguments in [4, Lemma 11, Lemma 12, Corollary 13] will show that $\text{Ass}^*(\mathfrak{a}, N) \setminus \text{Bss}^*(\mathfrak{a}, N) \subseteq \text{Ass}_A N \cap \text{Var}(\mathfrak{a})$. Let $\mathfrak{p} \in \text{Ass}_A N \cap \text{Var}(\mathfrak{a})$, so that there exists $w \in N$ with $(0 :_A w) = \mathfrak{p} \supseteq \mathfrak{a}$. Then $(0 :_{A[X]} wX^j) = \mathfrak{p}A[X]$ for all $j \in \mathbb{N}_0$, and so $(0 :_{\mathcal{R}(\mathfrak{a})} wX^j) = \mathfrak{p}A[X] \cap \mathcal{R}(\mathfrak{a})$ for all $j \in \mathbb{N}_0$. However, $\mathfrak{p}A[X] \cap \mathcal{R}(\mathfrak{a}) \in {}^* \text{Var}(\mathcal{R}(\mathfrak{a})_+)$ because $\mathfrak{a} \subseteq \mathfrak{p}$. Hence, for $j \in \mathbb{N}$ with $j > a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{R}(\mathfrak{a}, N))$, the $\mathcal{R}(\mathfrak{a})$ -submodule $\mathcal{R}(\mathfrak{a})wX^j$ of $N[X]$ must have zero intersection with $\mathcal{R}(\mathfrak{a}, N)$. Consideration of the exact sequence

$$0 \longrightarrow \mathcal{R}(\mathfrak{a}, N) \longrightarrow N[X] \longrightarrow N[X]/\mathcal{R}(\mathfrak{a}, N) \longrightarrow 0$$

of graded $\mathcal{R}(\mathfrak{a})$ -modules and homogeneous homomorphisms therefore shows that the natural image of wX^j in $N[X]/\mathcal{R}(\mathfrak{a}, N)$ has annihilator $\mathfrak{p}A[X] \cap \mathcal{R}(\mathfrak{a})$ for all $j \in \mathbb{N}$ with $j > a_{\mathcal{R}(\mathfrak{a})_+}^0(\mathcal{R}(\mathfrak{a}, N))$. The claim follows from this. $\square$

We end the paper by drawing together some consequences of Propositions 2.5, 2.6 and 2.8 in order to list some circumstances in which a $\mathfrak{p} \in \text{Ass}^*(\mathfrak{a})$ is an ever-present from the $(r+2)$ -th term onwards of the sequence $(\text{Ass}(A/\mathfrak{a}^n))_{n \in \mathbb{N}}$, where $r := \text{reg}(\mathcal{G}(\mathfrak{a}))$.

2.10. Theorem. *Let $\mathfrak{p} \in \text{Ass}^*(\mathfrak{a})$.*

- (i) *Suppose $\mathfrak{p} = \mathfrak{P} \cap A$ where $\mathfrak{P} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a})) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))$ (see Theorem 0.1(iii)). Then $\mathfrak{p} \in \text{Ass}(A/\mathfrak{a}^n)$ for all $n > \text{reg}(\mathcal{G}(\mathfrak{a})) + 1$ if*
 - *$\mathfrak{p}$ is a minimal member of the set*

$$\{\mathfrak{Q} \cap A : \mathfrak{Q} \in \text{Ass}_{\mathcal{R}(\mathfrak{a})}(\mathcal{G}(\mathfrak{a})) \cap \text{Proj}(\mathcal{R}(\mathfrak{a}))\},$$

or

- $\text{ht}((\mathfrak{P} + \mathcal{R}(\mathfrak{a})_+)/\mathfrak{P}) = 1$.
- (ii) *If $\mathfrak{p} \in \text{Ass}^*(\mathfrak{a}) \setminus \text{Bss}^*(\mathfrak{a})$, then $\mathfrak{p} \in \text{Ass}(A/\mathfrak{a}^n)$ for all $n > \text{reg}(\mathcal{G}(\mathfrak{a}))$.*

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