LOCATING SUBSETS OF A NORMED SPACE

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ABSTRACT. Within the framework of Bishop's constructive mathematics, we give conditions under which a bounded convex subset of a uniformly smooth normed space over $\mathbf{R}$ is located, extending results presented recently by F. Richman and H. Ishihara for subsets of a Hilbert space.

1. INTRODUCTION

It is often pointed out that a constructive study of a mathematical subject provides more information than a classical one, and sometimes produces results about objects that no classical mathematician would ever consider, simply because their existence is taken for granted. Such an example is the notion of a located set, which was first introduced by Bishop [1] in the context of metric spaces, and which can be extended to the more general case of locally convex spaces [4]. A subset S of a metric space X is said to be *located* if we can compute the distance

$$\rho(x, S) := \inf\{\rho(x, y) : y \in S\}$$

from any point x of X to S . Since this distance is given by an infimum, classically there is no problem with its existence; but constructively the infimum in question is not necessarily computable.

To see how important locatedness is to constructive mathematics, it is enough to mention that results like the Hahn–Banach and separation theorems, or the existence of a projection on a closed subspace of a normed linear space, rely on the locatedness of certain subsets.

For more details on constructive mathematics and located sets, the reader is referred to [1], [2], and [3].

The aim of the present paper is to give conditions under which a bounded convex subset of a uniformly smooth normed space over $\mathbf{R}$ is located, extending the results presented in [8, 6] for subsets of a Hilbert space.

We begin with the fundamental definition.

Definition 1.1. A normed space E is said to be *smooth* if the limit

$$(1.1) \quad \lim_{t \rightarrow \infty} \frac{\|x + ty\| - \|x\|}{t}$$

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exists for all unit vectors $x, y \in E$. The norm of E is said to be *Fréchet differentiable* if the limit (1.1) is attained uniformly for all unit vectors $y \in E$. The space E is said to be *uniformly smooth* if the limit (1.1) is approached uniformly for all pairs (x, y) of unit vectors in $E \times E$.

With each vector x in a smooth normed space we associate the normable linear functional $J(x) : E \rightarrow \mathbf{R}$ defined by

$$(y, J(x)) = \|x\| \lim_{t \rightarrow \infty} \frac{\|x + ty\| - \|x\|}{t}.$$

A number of properties of this mapping can be found in [9]; for our purposes we only need the following:

- $(x, J(x)) = \|J(x)\|^2 = \|x\|^2$.
- $\|y\|^2 \leq \|x\|^2 - 2(x - y, J(y))$.

2. AN EXAMPLE OF UNIFORMLY SMOOTH NORMED SPACE

Examples of smooth normed spaces can be found in [7]. For uniformly smooth spaces, we have the following proposition.

A normed space E is said to *have a uniformly convex dual* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for all normable linear functionals f and g with $\|f\| = \|g\| = 1$, $(u, f + g) > 2 - \delta$ for some unit vector $u \in E$ implies that $(v, f - g) < \epsilon$ for all unit vectors $v \in E$.

Proposition 2.1. *Let E be a separable normed space. Then E is uniformly smooth if and only if E has a uniformly convex dual.*

Proof. Suppose that E has a uniformly convex dual. Then since for $t' \geq t > 0$ we have

$$\frac{\|x - t'y\| - \|x\|}{-t'} \leq \frac{\|x - ty\| - \|x\|}{-t} \leq \frac{\|x + ty\| - \|x\|}{t} \leq \frac{\|x + t'y\| - \|x\|}{t'},$$

it is enough to show that for each $\epsilon > 0$ there exists $t > 0$ such that for all unit vectors $x, y \in E$,

$$\frac{\|x + ty\| - \|x\|}{t} - \frac{\|x - ty\| - \|x\|}{-t} < \epsilon.$$

For a given $\epsilon > 0$, there exists $\delta > 0$ such that for all normable linear functionals f and g with $\|f\| = \|g\| = 1$, if $(u, f + g) > 2 - \delta$ for some unit vector $u \in E$, then $(w, f - g) < \epsilon/2$ for all unit vectors $w \in E$. Choose t with $t < 1/2$ and $8t(2 + t) < \delta/2$. Then for all unit vectors $x, y \in E$, we have $\|x + ty\| > 1/2$ and $\|x - ty\| > 1/2$, and hence, letting $u := (x + ty)/\|x + ty\|$ and $v := (x - ty)/\|x - ty\|$, we have

$$\begin{aligned} \|u - v\| &= \frac{1}{\|x + ty\|\|x - ty\|} \| \|x - ty\|(\|x + ty\| - \|x + ty\|) - \|x + ty\|(\|x - ty\|) \| \\ &\leq 4(\|x - ty\| - \|x + ty\| + t(\|x - ty\| + \|x + ty\|)) \\ &\leq 4(2t + 2t(1 + t)) = 8t(2 + t) < \frac{\delta}{2}. \end{aligned}$$

Taking $\gamma := \min\{t\epsilon/(4(1 + t)), \delta/4\}$, construct normable linear functionals f and g with $\|f\| = \|g\| = 1$ such that

$$(u, f) > 1 - \gamma \quad \text{and} \quad (v, g) > 1 - \gamma,$$

by [2, Corollary 7.4.5]. Then we have

$$\begin{aligned} (u, f + g) &= (u, f) + (v, g) + (u - v, g) \\ &> 2 - 2\gamma - \|u - v\| \\ &\geq 2 - 2\frac{\delta}{4} - \frac{\delta}{2} = 2 - \delta. \end{aligned}$$

Hence $(w, f - g) < \epsilon/2$ for all unit vectors $w \in E$; in particular $(y, f - g) < \epsilon/2$. Since

$$\|x + ty\| < (x + ty, f) + \|x + ty\|\gamma \leq (x + ty, f) + (1 + t)\gamma \leq (x + ty, f) + \frac{t\epsilon}{4}$$

and similarly

$$\|x - ty\| < (x - ty, g) + \frac{t\epsilon}{4},$$

we have

$$\begin{aligned} \frac{\|x + ty\| - \|x\|}{t} - \frac{\|x - ty\| - \|x\|}{-t} &= \frac{\|x + ty\| + \|x - ty\| - 2}{t} \\ &\leq \frac{(x + ty, f) + (x - ty, g) + t\epsilon/2 - 2}{t} \\ &\leq \frac{\epsilon}{2} + (y, f - g) < \epsilon. \end{aligned}$$

Conversely, suppose that E is uniformly smooth. Then for each $\epsilon > 0$ there exists $\gamma > 0$ such that for all unit vectors $x, y \in E$,

$$0 < |t| < \gamma \implies \left| \frac{\|x + ty\| - \|x\|}{t} - (y, J(x)) \right| < \frac{\epsilon}{4}.$$

Letting $t := \gamma/2$, we have

$$\begin{aligned} \|x + ty\| &< \frac{t\epsilon}{4} + t(y, J(x)) + 1, \\ \|x - ty\| &< \frac{t\epsilon}{4} - t(y, J(x)) + 1, \end{aligned}$$

and hence

$$\|x + ty\| + \|x - ty\| < 2 + \frac{t\epsilon}{2}$$

for all unit vectors $x, y \in E$. Let $\delta := \gamma\epsilon/4$, let f and g be normable linear functionals with $\|f\| = \|g\| = 1$, and assume that $(u, f + g) > 2 - \delta$ for some unit vector $u \in E$. Then for all unit vectors $v \in E$, we have

$$\begin{aligned} 2 - \frac{t\epsilon}{2} &= 2 - \delta < (u, f + g) \\ &= (u + tv, f) + (u + tv, g) - t(v, f - g) \\ &\leq \|u + tv\| + \|u - tv\| - t(v, f - g) \\ &< 2 + \frac{t\epsilon}{2} - t(v, f - g), \end{aligned}$$

and hence

$$(v, f - g) < \epsilon. \quad \square$$

Corollary 2.2. *For each $p > 1$, the space L_p is uniformly smooth.*

Proof. L_p has a uniformly convex dual L_q where $1/p + 1/q = 1$; see [2, Corollary 7.3.22, Theorem 7.3.25]. $\square$

3. THE MAIN RESULTS

Our setting will be a real Banach space E .

Lemma 3.1. *Let E be a normed space with a Fréchet differentiable norm. Then for each unit vector $u \in E$ and each $\epsilon > 0$ there exists $\delta > 0$ such that for all unit vectors $v, w \in E$ with $\|v\| = 1$,*

$$\|w\| < \delta \implies |(v, J(u)) - (v, J(u+w))| < \epsilon.$$

Proof. Let $u \in E$ with $\|u\| = 1$ and $\epsilon > 0$. Then there exists $\gamma > 0$ such that for all unit vectors $v \in E$,

$$0 < |t| < \gamma \implies \left| \frac{\|u+tv\| - \|u\|}{t} - (v, J(u)) \right| < \frac{\epsilon}{8}.$$

We may assume that $\gamma < 1$. Let $\delta := \min\{\gamma\epsilon/32, 1/2\}$, and suppose that $\|w\| < \delta$. Then letting $x := (u+w)/\|u+w\|$, we have

$$0 \leq 1 - (u, J(x)) = (x - u, J(x)) \leq \|x - u\| \leq \frac{2\|w\|}{\|u+w\|} < 4\delta.$$

On the other hand, letting $t := \gamma/2$, we have

$$\|u+tv\| + \|u-tv\| < 2 + \frac{t\epsilon}{4}$$

for all unit vectors $v \in E$. Therefore for a given unit vector $v \in E$,

$$\begin{aligned} 1 - \frac{t\epsilon}{4} &\leq 1 - 4\delta < (u, J(x)) = (u, J(u) + J(x)) - 1 \\ &= (u+tv, J(u)) + (u-tv, J(x)) - 1 - t(v, J(u) - J(x)) \\ &\leq \|u+tv\| + \|u-tv\| - 1 - t(v, J(u) - J(x)) \\ &< \frac{t\epsilon}{4} + 1 - t(v, J(u) - J(x)), \end{aligned}$$

and so

$$(v, J(u) - J(x)) < \frac{\epsilon}{2}.$$

Hence

$$\begin{aligned} (v, J(u) - J(u+w)) &= \|u+w\|(v, J(u) - J(x)) + (1 - \|u+w\|)(v, J(u)) \\ &\leq \frac{3}{2}(v, J(u) - J(x)) + \|w\| < \frac{3\epsilon}{4} + \frac{\epsilon}{32} < \epsilon. \end{aligned}$$

Replacing v with $-v$, we have $(v, J(u+w) - J(u)) < \epsilon$. Thus

$$|(v, J(u) - J(u+w))| < \epsilon. \quad \square$$

Lemma 3.2. *Let C be a bounded located convex subset of a normed space E with a Fréchet differentiable norm, whose diameter is bounded by $M > 0$, and let $x \in E$. Then for each $\epsilon > 0$ there exists $\tau > 0$ such that for all $y \in C$, either*

- (1) $(y, J(x)) + \tau\epsilon/6 < (y', J(x))$ for some $y' \in C$, or
- (2) $(z, J(x)) < (y, J(x)) + \epsilon$ for all $z \in C$.

Proof. If $\|x\|M < \epsilon$, then for all $z \in C$,

$$(z - y, J(x)) \leq \|x\|\|z - y\| \leq \|x\|M < \epsilon;$$

hence we may assume that $0 < \|x\|M$. Let

$$u := x/\|x\|.$$

Then there exists $\delta > 0$ such that for all unit vectors $v, w \in E$ with $\|v\| = 1$,

$$0 < |t| < \delta \implies \left| \frac{\|u + tv\| - \|u\|}{t} - (v, J(u)) \right| < \frac{2\epsilon}{3\|x\|M}$$

and

$$\|w\| < \delta \implies |(v, J(u)) - (v, J(u + w))| < \frac{2\epsilon}{3\|x\|M}.$$

Let $\tau := \min\{1, \delta\|x\|/M\}$. Then since $\rho(x + y, C)$ exists, either

$$\|x + y - y'\|^2 < \|x\|^2 - \tau\epsilon/3 \text{ for some } y' \in C$$

or

$$\|x\|^2 - 2\tau\epsilon/3 < \|x + y - z\|^2 \text{ for all } z \in C.$$

In the former case, we have

$$\|x\|^2 \leq \|x + y - y'\|^2 - 2(y - y', J(x)) < \|x\|^2 - \tau\frac{\epsilon}{3} - 2(y - y', J(x))$$

and hence

$$(y, J(x)) + \tau\frac{\epsilon}{6} < (y', J(x)).$$

In the latter case, for a given $z \in C$, if $\|x\|\|y - z\| < \epsilon$, then

$$(z - y, J(x)) \leq \|x\|\|z - y\| < \epsilon;$$

hence we may assume that $0 < \|x\|\|y - z\|$. Let

$$\begin{aligned} v &:= (y - z)/\|y - z\|, \\ w &:= \tau(y - z)/\|x\|, \\ t &:= \tau\|y - z\|/\|x\|. \end{aligned}$$

Then $\|w\| = |t| < \delta$. Hence

$$\begin{aligned} & \left| \|x\| \frac{\|x + \tau(y - z)\| - \|x\|}{\tau} - (y - z, J(x)) \right| \\ &= \|x\|\|y - z\| \left| \frac{\|u + tv\| - \|u\|}{t} - (v, J(u)) \right| \\ &< \|x\|\|y - z\| \frac{2\epsilon}{3\|x\|M} \leq \frac{2\epsilon}{3} \end{aligned}$$

and

$$\begin{aligned} & |(y - z, J(x)) - (y - z, J(x + \tau(y - z)))| \\ &= \|x\|\|y - z\| |(v, J(u)) - (v, J(u + w))| \\ &< \|x\|\|y - z\| \frac{2\epsilon}{3\|x\|M} \leq \frac{2\epsilon}{3}. \end{aligned}$$

Since C is convex and $0 \leq \tau \leq 1$, it follows that $(1 - \tau)y + \tau z \in C$; hence

$$\|x\|^2 - \tau\frac{2\epsilon}{3} < \|x + \tau(y - z)\|^2.$$

Therefore we have

$$\begin{aligned} 0 &< \|x + \tau(y - z)\|^2 - \|x\|^2 + 2\tau\epsilon/3 \\ &= (x, J(x + \tau(y - z))) + \tau(y - z, J(x + \tau(y - z))) - \|x\|^2 + 2\tau\epsilon/3 \\ &\leq \|x\|(\|x + \tau(y - z)\| - \|x\|) + \tau(y - z, J(x + \tau(y - z))) + 2\tau\epsilon/3. \end{aligned}$$

Hence

$$\begin{aligned} 0 &< \|x\| \frac{\|x + \tau(y - z)\| - \|x\|}{\tau} + (y - z, J(x + \tau(y - z))) + 2\epsilon/3 \\ &\leq ((y - z, J(x)) + 2\epsilon/3) + ((y - z, J(x)) + 2\epsilon/3) + 2\epsilon/3, \end{aligned}$$

which obviously implies that $(z, J(x)) < (y, J(x)) + \epsilon$. $\square$

Theorem 3.3. *Let C be a bounded located convex subset of a normed space E with a Fréchet differentiable norm. Then $\sup\{(z, J(x)) : z \in C\}$ exists for each $x \in E$.*

Proof. Let M be a bound for the diameter of C and let x be any element of E . We will prove that for each $\epsilon > 0$ there exists $y \in C$ such that $(z, J(x)) < (y, J(x)) + \epsilon$ for all $z \in C$. Choose $y_0 \in C$ and let τ be as in Lemma 3.2. Construct an increasing binary sequence (λ_n) with $\lambda_0 = 0$ and a sequence (y_n) in C such that

$$\begin{aligned} \lambda_n = 0 &\implies (y_{n-1}, J(x)) + \tau \frac{\epsilon}{6} < (y_n, J(x)) \\ \lambda_n = 1 &\implies (z, J(x)) < (y_{n-1}, J(x)) + \epsilon \text{ for all } z \in C. \end{aligned}$$

Assume we have constructed $\lambda_0, \dots, \lambda_n$ and $y_0, \dots, y_n$. If $\lambda_n = 1$, set $\lambda_{n+1} := 1$ and $y_{n+1} := y_n$. If $\lambda_n = 0$, then by Lemma 3.2, either $(y_n, J(x)) + \tau \frac{\epsilon}{6} < (y', J(x))$ for some $y' \in C$, or $(z, J(x)) < (y_n, J(x)) + \epsilon$ for all $z \in C$. In the former case, set $\lambda_{n+1} = 0$ and $y_{n+1} := y'$. In the latter case, set $\lambda_{n+1} = 1$ and $y_{n+1} := y_n$. Now choose N such that

$$M\|x\| < N\tau \frac{\epsilon}{6}.$$

In order to complete our proof, it will be enough to look at λ_N . If $\lambda_N = 0$, then

$$(y_0, J(x)) + N\tau \frac{\epsilon}{6} < (y_N, J(x)),$$

and hence

$$N\tau \frac{\epsilon}{6} < (y_N - y_0, J(x)) \leq M\|x\|$$

—a contradiction. Therefore $\lambda_N = 1$. $\square$

Corollary 3.4. *Let C be a bounded located convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Then $\sup\{(z, f) : z \in C\}$ exists for each normable linear functional f on E .*

Proof. For each normable linear functional f on E , there exists $x \in E$ such that $(x, f) = \|f\|^2 = \|x\|^2$ by [2, Proposition 7.3.23]. Hence $J(x) = f$ by the uniqueness of such a normable linear functional; see [5, Lemma 1]. $\square$

The following lemma will enable us to prove the converse of Theorem 3.3.

Lemma 3.5. *Let C be a bounded convex subset of a uniformly smooth normed space E such that for each $x \in E$, $\sup\{(z, J(x)) : z \in C\}$ exists. Then for each $\epsilon > 0$ there exists σ with $0 < \sigma < 1$ such that for every $x \in E$ and $y \in C$, either*

- (1) $\|x - y'\| < (1 - \sigma)\|x - y\|$ for some $y' \in C$, or
- (2) $\|x - y\| < \|x - z\| + \epsilon$ for all $z \in C$.

Proof. If $\|x - y\| < \epsilon$, then $\|x - y\| < \|x - z\| + \epsilon$ for all $z \in C$. Hence we may assume that $\epsilon/2 < \|x - y\|$. C being bounded, we can choose a positive integer

M such that $\|x - z\| < M$ for all $z \in C$ and $\epsilon/4 < M$. The space E is uniformly smooth; hence there exists δ with $0 < \delta < 2$ such that for all unit vectors $u, v \in E$,

$$0 < |t| < \delta \implies \left| \frac{\|u - tv\| - \|u\|}{-t} - (v, J(u)) \right| < \frac{\epsilon}{8M}.$$

Define $\sigma := \epsilon\delta/(8M)$. By our choice of M , we have $0 < \sigma < 1$. Then, since $\sup\{(z, J(x-y)) : z \in C\}$ exists, either

$$(z - y, J(x - y)) < \epsilon\|x - y\| \text{ for all } z \in C$$

or

$$\epsilon\|x - y\|/2 < (z_0 - y, J(x - y)) \text{ for some } z_0 \in C.$$

In the first case, for all $z \in C$ we have

$$\begin{aligned} \|x - y\|^2 &= (x - y, J(x - y)) = (x - z + z - y, J(x - y)) \\ &= (x - z, J(x - y)) + (z - y, J(x - y)) \\ &< \|x - z\|\|x - y\| + \epsilon\|x - y\|, \end{aligned}$$

and therefore

$$\|x - y\| < \|x - z\| + \epsilon.$$

In the second case, setting $\tau := \delta\|x - y\|/(2M)$, we have

$$0 < \tau = \delta \frac{\|x - y\|}{2M} < \frac{\delta}{2} < 1,$$

and hence

$$y' := y + \tau(z_0 - y) \in C.$$

Note that $\epsilon\|x - y\|/2 < (z_0 - y, J(x - y)) \leq \|z_0 - y\|\|x - y\|$, and therefore $0 < \|z_0 - y\|$. Then, letting

$$\begin{aligned} u &:= (x - y)/\|x - y\|, \\ v &:= (z_0 - y)/\|z_0 - y\|, \\ t &:= \tau\|z_0 - y\|/\|x - y\|, \end{aligned}$$

we have

$$t = \tau \frac{\|z_0 - y\|}{\|x - y\|} = \delta \frac{\|z_0 - y\|}{2M} < \delta,$$

and

$$\begin{aligned} \|x - y'\| &= \|x - y - \tau(z_0 - y)\| = \|x - y\|\|u - tv\| \\ &< \|x - y\| \left(\|u\| - t(v, J(u)) + t \frac{\epsilon}{8M} \right) \\ &= \|x - y\| - \tau \frac{(z_0 - y, J(x - y))}{\|x - y\|} + \tau \frac{\epsilon\|z_0 - y\|}{8M} \\ &< \|x - y\| - \tau \frac{\epsilon}{2} + \tau \frac{\epsilon}{4} \\ &= \|x - y\| - \frac{\delta}{2M} \frac{\epsilon}{4} \|x - y\| \\ &= (1 - \sigma)\|x - y\|. \end{aligned}$$

□

Theorem 3.6. *Let E be a uniformly smooth normed space and C a bounded convex subset of E . Suppose that $\sup\{(z, J(x)) : z \in C\}$ exists for each $x \in E$. Then C is located.*

Proof. Let x be an element of E and ϵ a positive number. We will prove that there exists $y \in C$ such that $\|x - y\| < \|x - z\| + \epsilon$ for all $z \in C$. Let σ be the corresponding positive number given by Lemma 3.5 and let y_0 be an element of C . If $\|x - y_0\| < \epsilon$, then $\|x - y_0\| < \|x - z\| + \epsilon$ for all $z \in C$. So we can assume that $\|x - y_0\| > 0$. We now construct an increasing binary sequence $(\lambda_n)_{n=0}^\infty$ with $\lambda_0 = 0$, and a sequence $(y_n)_{n=0}^\infty$ of elements of C such that

$$\begin{aligned}\lambda_n = 0 &\implies \|x - y_n\| < (1 - \sigma)\|x - y_{n-1}\|, \\ \lambda_n = 1 &\implies \|x - y_n\| < \|x - z\| + \epsilon \text{ for all } z \in C.\end{aligned}$$

Suppose we have constructed $\lambda_0, \dots, \lambda_n$ and $y_0, \dots, y_n$. If $\lambda_n = 1$, set $\lambda_{n+1} = 1$ and $y_{n+1} = y_n$. If $\lambda_n = 0$, then, applying Lemma 3.5, we have either $\|x - y'\| < (1 - \sigma)\|x - y_n\|$ for some $y' \in C$, or $\|x - y_n\| < \|x - z\| + \epsilon$ for all $z \in C$. In the former case, set $\lambda_{n+1} = 0$ and $y_{n+1} = y'$; in the latter case, set $\lambda_{n+1} = 1$ and $y_{n+1} = y_n$. Since $0 < \sigma < 1$, we have $(1 - \sigma)^n \rightarrow 0$ as $n \rightarrow \infty$. Choose N such that

$$(1 - \sigma)^n < \frac{\epsilon}{\|x - y_0\|}$$

for all $n \geq N$. Now check $\lambda_0, \dots, \lambda_N$. If there exists $n \leq N$ such that $\lambda_n = 1$, then we are done; if $\lambda_n = 0$ for all $n \leq N$, then

$$\|x - y_N\| < (1 - \sigma)^N \|x - y_0\| < \epsilon \leq \|x - z\| + \epsilon$$

for all $z \in C$. □

It is an open question whether we can extend these results to unbounded convex sets.

Corollary 3.7. *Let E be a uniformly smooth normed space and C a bounded convex subset of E . Then C is located if and only if $\sup\{(z, J(x)) : z \in C\}$ exists for all $x \in E$.*

Corollary 3.8. *Let E be a uniformly convex and uniformly smooth Banach space and C a bounded convex subset of E . Then C is located if and only if $\sup\{(z, f) : z \in C\}$ exists for each normable linear functional f on E .*

Let T be a linear mapping from a normed space E into a normed space F , and let S be a mapping from the set $F^\dagger$ of all normable linear functionals on F into the set $E^\dagger$ of all normable linear functionals on E . T and S are called *adjoint* if

$$(Tx, f) = (x, Sf) \quad (x \in E, f \in F^\dagger).$$

The mapping S is uniquely determined in terms of T , is written T^* , and is called the *adjoint of T* .

Theorem 3.9. *Let E be a normed space, and let F be a uniformly convex and uniformly smooth Banach space. Then a bounded linear mapping $T : E \rightarrow F$ has an adjoint if and only if the image $T(B)$ of the unit ball B is located.*

Proof. Let $C := T(B)$. Then C is a bounded, convex and balanced subset of F . Therefore

$$\begin{aligned}
 C \text{ is located} &\iff \sup\{(y, f) : y \in C\} \text{ exists for all } f \in F^\dagger \\
 &\iff \sup\{|(y, f)| : y \in C\} \text{ exists for all } f \in F^\dagger \\
 &\iff \sup\{|(Tx, f)| : x \in B\} \text{ exists for all } f \in F^\dagger \\
 &\iff f \circ T \in E^\dagger \text{ for all } f \in F^\dagger \\
 &\iff T^* \text{ exists.}
 \end{aligned}$$

□

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