PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 12, Pages 3925–3927
S 0002-9939(03)06911-9
Article electronically published on June 30, 2003

EXACTLY k-TO-1 MAPS AND HEREDITARILY INDECOMPOSABLE TREE-LIKE CONTINUA

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(Communicated by Alan Dow)

ABSTRACT. In 1947, W.H. Gottschalk proved that no dendrite is the continuous, exactly k-to-1 image of any continuum if $k \geq 2$. Since that time, no other class of continua has been shown to have this same property. It is shown that no hereditarily indecomposable tree-like continuum is the continuous, exactly k-to-1 image of any continuum if $k \geq 2$.

One of the earliest results concerning exactly k-to-1 maps between continua is W. H. Gottschalk's [2] result that no dendrite is the continuous exactly k-to-1 image of any continuum if $k \geq 2$. Since Gottschalk's result, no other class of continua has been shown to repel exactly k-to-1 functions from continua in the manner that dendrites do. It is proved that no hereditarily indecomposable tree-like continuum is the continuous exactly k-to-1 image of any continuum if $k \geq 2$. This result gives more information towards a resolution of a question posed by Nadler and Ward [10] i.e., which continua are k-to-1 images of continua, where $k \geq 2$? The result also generalizes a result of J. Heath [5] who proved that no hereditarily indecomposable tree-like continuum is a two-to-one image of a continuum. It is known that for each $k > 2$ there exists a k-to-1 map between tree-like continua [4]. For more results concerning k-to-1 functions between continua, the reader is directed to a survey paper of J. Heath [7].

A *space* is a compact metric space, a *continuum* is a nonempty, compact, connected metric space, and a *map* is a continuous function. If X and Y are spaces, then a map f from X into Y is said to be *confluent* if for any continuum L in the image, every component of $f^{-1}(L)$ maps onto L.

Lemma 1. *Suppose that f is a confluent map onto a space Y and n is a positive integer such that $n \leq k$. Let $\mathcal{C}$ denote the set of all continua in Y whose inverse image under f has exactly n components. If $\mathcal{C}$ is non-empty, then $\mathcal{C}$ has a minimal element with respect to inclusion.*

Proof. It will be shown that every chain $\mathcal{L} \subset \mathcal{C}$ has a lower bound in $\mathcal{C}$, namely $\bigcap \mathcal{L}$. It need only be shown that $f^{-1}(\bigcap \mathcal{L})$ has n components. If L is an element of $\mathcal{L}$, the fact that f is confluent implies that for every subcontinuum M of L, each component of $f^{-1}(L)$ contains at least one component of $f^{-1}(M)$. If $M \in \mathcal{L}$, then $f^{-1}(M)$ and $f^{-1}(L)$ have the same number of components, in which case

Received by the editors March 1, 2001 and, in revised form, June 5, 2001.
2000 *Mathematics Subject Classification.* Primary 54C10.
Key words and phrases. k-to-1 map, hereditarily indecomposable continua, tree-like continua.

each component of $f^{-1}(L)$ contains exactly one component of $f^{-1}(M)$. Thus, the components of $f^{-1}(M)$, where M's are in $\mathcal{L}$, form n decreasing chains of continua where elements of different chains are disjoint. The intersection of each of these n chains is a continuum, and $f^{-1}(\bigcap \mathcal{L})$ is the union of these intersections. Therefore, $\mathcal{L}$ belongs to $\mathcal{C}$. It follows that $\mathcal{C}$ has a minimal element. $\square$

If X and Y are spaces, a map f is said to be *k-crisp* if for every proper subcontinuum L of Y, $f^{-1}(L)$ is the union of k disjoint continua and f restricted to each of the k disjoint continua comprising $f^{-1}(L)$ is a homeomorphism onto L. Two elements of X, x and y, are said to be *siblings* if $f(x) = f(y)$.

The following lemma extends a lemma of Griffus [3] to a more general class of spaces.

Lemma 2. *Let X be a space and let Y be a continuum. If $f : X \to Y$ is a k-crisp map, then f is locally one-to-one.*

Proof. First, it is shown that f maps every component H of X onto Y. Suppose that $f(H)$ is a proper subcontinuum of M. As a component of X, $H \subset f^{-1}(M)$ is a component of $f^{-1}(M)$. Since f is k-crisp, H is mapped onto M, which is a contradiction. It follows that X has no more than k components.

According to a lemma of Griffus [3], which is a generalization of a result of Mioduszewski [8], there exist nonempty open sets $U_1, U_2, \ldots, U_k$ in X such that

(1) $\overline{U}_i \cap \overline{U}_j = \emptyset$ for every $i, j \in \{1, 2, \ldots k\}$ such that $i \neq j$.
(2) $\left|f^{-1}(f(x)) \cap U_i\right| = 1$ for every $x \in \bigcup_{j=1}^{k} U_j$ and every $i \in \{1, 2, \ldots, k\}$.
(3) For every $i \in \{1, 2, \ldots, k\}$, f is locally one-to-one at each point of $\overline{U}_i$.

Let U denote the set $\bigcup_{i=1}^{k} U_i$. Suppose that f is not locally one-to-one at the point z. Let C denote the component of X containing z. There exist two disjoint sibling sequences $\{p_i\}$ and $\{\hat{p}_i\}$ in $C \setminus \overline{U}$ each of which converges to z.

For each p_i, let C_i be the component of p_i in $C \setminus U$. Since $C \setminus U$ is a nonempty, proper, closed subset of the continuum C, the Janiszewski lemma [9] implies that the continuum C_i must bump the boundary of $C \cap U$. Let x_i be an element of $C_i \cap \mathrm{Bd}(U)$. The k-crisp property implies that there exists a continuum $\hat{C}_i$ that is disjoint from C_i and that contains $\hat{p}_i$. Since all siblings of x_i are in the boundary of U, there is a sibling of x_i, $\hat{x}_i$ in $\hat{C}_i \cap \mathrm{Bd}(U)$.

There exists a common convergent subsequence of the sequence $\{C_i\}$ and of the sequence $\{\hat{C}_i\}$. Denote the limiting continuum of the common subsequence of $\{C_i\}$ by A and the limiting continuum of the common subsequence of $\{\hat{C}_i\}$ by B. The continuum A must contain the point z, as well as a limit point of the sequence $\{x_i\}$, call it a. The continuum B must also contain the point z, as well as a limit point of the sequence $\{\hat{x}_i\}$, call it b. The points a and b must be distinct as well as siblings. However, this is a contradiction since $f(A \cup B) \neq Y$ and f is k-crisp. Therefore, f is locally one-to-one. $\square$

Theorem 3. *No tree-like continuum is the confluent k-to-1 image of any continuum, for any $k \geq 2$.*

Proof. Suppose that there exists a confluent k-to-1 map $f : X \to Y$ from a continuum X onto a tree-like continuum Y, where $k \geq 2$. Choose n to be the largest integer in the set $\{2, 3, \ldots, k-1\}$ such that there is a continuum $M \subseteq Y$ whose inverse image has n components. By Lemma 1, it may be assumed that M is minimal.

Since the property of being tree-like is hereditary, M is tree-like. The restriction of f to $f^{-1}(M)$ is k-crisp. Indeed, if L is a proper subcontinuum of M, then $f^{-1}(L)$ has at least k components. Since f is confluent, each of them is mapped onto L. Hence $f^{-1}(L)$ has exactly k components, on which f is one-to-one.

By Lemma 2, the restriction of f to $f^{-1}(M)$ is locally one-to-one. A theorem of J. Heath [6] states that every locally one-to-one map from a continuum onto a tree-like continuum is a homeomorphism. Therefore, f restricted to any one of the n components of $f^{-1}(M)$ is one-to-one. Hence, f restricted to $f^{-1}(M)$ is n-to-one, which is a contradiction. □

The final result follows from a result of H. Cook [1], namely that any map from a continuum onto a hereditarily indecomposable continuum is confluent.

Corollary 4. *No hereditarily indecomposable tree-like continuum is the continuous k-to-1 image of a continuum for any $k \geq 2$.*

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