# GLOBAL ANALYTIC REGULARITY FOR NON-LINEAR SECOND ORDER OPERATORS ON THE TORUS 

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(Communicated by David S. Tartakoff)


#### Abstract

Assuming a subelliptic a-priori estimate we prove global analytic regularity for non-linear second order operators on a product of tori, using the method of majorant series.


## 1. Introduction

Hypoellipticity for linear partial differential operators has been largely investigated by many authors. In the non-linear case, on the contrary, there are still few results and many open questions.

Some results about $C^{\infty}$-hypoellipticity for non-linear partial differential equations have been obtained in [X] and G], using the para-differential calculus of Bony [B].

We are interested in analytic hypoellipticity for non-linear second order p.d.e.'s. Local analytic regularity for a model operator given by sums of squares of non-linear vector fields has been proved in [TZ]. Here we prove global analytic regularity on the torus for non-linear second order operators constructed from rigid vector fields, generalizing the result obtained for the linear case in T].

The problem of regularity of solutions on the torus in the linear case has been studied by many other authors in the frameworks of $C^{\infty}$, Gevrey and analytic functions (see, for instance, GPY and the references there).

## 2. Notation and main result

Let $\mathbb{T}^{N}$ be the $N$-dimensional torus and split $\mathbb{T}^{N} \simeq \mathbb{T}^{m} \times \mathbb{T}^{n}$. Let us then consider, for $u \in C^{\infty}\left(\mathbb{T}^{N}\right)$ and for some integer $n^{\prime} \geq n$, the operator

$$
\begin{align*}
P & =P_{u}=P(x, u, D)  \tag{1}\\
& =\sum_{j, k=1}^{n^{\prime}} a_{j k}(u(t, x)) X_{j} X_{k}+\sum_{j=1}^{n^{\prime}} b_{j}(u(t, x)) X_{j}+X_{0}+c(u(t, x))
\end{align*}
$$

[^0]defined for $(t, x) \in \mathbb{T}^{m} \times \mathbb{T}^{n}$, where the real analytic coefficients $a_{j k}(u), b_{j}(u)$ and $c(u)$ are complex valued, but the real analytic rigid vector fields
\[

$$
\begin{equation*}
X_{j}=\sum_{k=1}^{n} d_{j k}(x) \frac{\partial}{\partial x_{k}}+\sum_{k=1}^{m} e_{j k}(x) \frac{\partial}{\partial t_{k}}, \quad j=0, \ldots, n^{\prime} \tag{2}
\end{equation*}
$$

\]

are real valued (rigid means that the coefficients $d_{j k}, e_{j k}$ do not depend on $t$ ). Assume also that, for every $x \in \mathbb{T}^{n}$, the fields

$$
\begin{equation*}
X_{j}^{\prime}=\sum_{k=1}^{n} d_{j k}(x) \frac{\partial}{\partial x_{k}}, \quad j=1, \ldots, n^{\prime} \tag{3}
\end{equation*}
$$

span the tangent space $T_{x}\left(\mathbb{T}^{n}\right)$.
Let us now denote by $A\left(\mathbb{T}^{N}\right)$ the space of real analytic functions on $\mathbb{T}^{N}$, and fix a solution $u \in C^{\infty}\left(\mathbb{T}^{N}\right)$ of the equation $P u=f$ for $f \in A\left(\mathbb{T}^{N}\right)$.

We shall assume in the sequel that the following a-priori estimate is satisfied for some $\delta, C>0$ and for all $v \in C^{\infty}\left(\mathbb{T}^{N}\right)$ :

$$
\begin{equation*}
\sum_{i, j=1}^{n^{\prime}}\left\|X_{i} X_{j} v\right\|_{\mu}+\sum_{j=1}^{n^{\prime}}\left\|X_{j} v\right\|_{\mu}+\|v\|_{\mu+\delta} \leq C\left(\left\|P_{u} v\right\|_{\mu}+\|v\|_{\mu}\right) \tag{4}
\end{equation*}
$$

where $\mu$ is a fixed integer with $\mu>N / 2$, so that the Sobolev space $H^{\mu}\left(\mathbb{T}^{N}\right)$ is an algebra and

$$
\|f g\|_{\mu} \leq \Lambda\|f\|_{\mu} \cdot\|g\|_{\mu} \quad \forall f, g \in H^{\mu}\left(\mathbb{T}^{N}\right)
$$

for a positive $\Lambda$ depending only on $N$.
Before giving the analytic regularity result, we first give an example of an operator of type (1) satisfying the required assumptions, and in particular the a-priori estimate (4).

Example 2.1. Let $(t, x) \in \mathbb{T}^{2}$ and consider the operator

$$
P=\partial_{x}^{2}+\sin ^{2} x\left(1+a^{2}(u(t, x))\right) \partial_{t}^{2}
$$

where $a(u)$ is a real analytic function. This operator is of the form (1) with

$$
\begin{aligned}
& X_{1}=X_{1}^{\prime}=\partial_{x}, X_{2}=\sin x \partial_{t} \\
& a_{11}(u) \equiv 1, a_{12}(u) \equiv a_{21}(u) \equiv 0, a_{22}(u)=1+a^{2}(u)
\end{aligned}
$$

We must prove the a-priori estimate (4). From [RS] it easily follows that

$$
\|v\|_{\mu+\delta}^{2} \leq c\left(\sum_{j=1}^{2}\left\|X_{j} v\right\|_{\mu}^{2}+\|v\|_{\mu}^{2}\right) \quad \forall v \in C^{\infty}\left(\mathbb{T}^{2}\right), \delta=1 / 2
$$

This implies, by standard arguments, the following a-priori estimate for the operator $\tilde{P}=\partial_{x}^{2}+\sin ^{2} x \partial_{t}^{2}$ :

$$
\begin{equation*}
\sum_{i, j=1}^{2}\left\|X_{i} X_{j} v\right\|_{\mu}^{2}+\sum_{j=1}^{2}\left\|X_{j} v\right\|_{\mu+\delta}^{2}+\|v\|_{\mu+2 \delta}^{2} \leq c^{\prime}\left|\langle\tilde{P} v, v\rangle_{\mu}\right|+\|v\|_{\mu}^{2} \tag{5}
\end{equation*}
$$

for some $c^{\prime}>0$ and for all $v \in C^{\infty}\left(\mathbb{T}^{2}\right)$.

Since $P=\tilde{P}+a^{2}(u) X_{2}^{2}$, we have that

$$
\begin{aligned}
\left|\langle\tilde{P} v, v\rangle_{\mu}\right| & \leq\left|\langle P v, v\rangle_{\mu}\right|+\left|\left\langle a^{2}(u) X_{2}^{2} v, v\right\rangle_{\mu}\right| \\
& \leq \frac{1}{2}\|P v\|_{\mu}^{2}+\frac{1}{2}\|v\|_{\mu}^{2}+\varepsilon\left\|a^{2}(u) X_{2}^{2} v\right\|_{\mu}^{2}+\frac{1}{4 \varepsilon}\|v\|_{\mu}^{2} \\
& \leq \frac{1}{2}\|P v\|_{\mu}^{2}+\frac{2 \varepsilon+1}{4 \varepsilon}\|v\|_{\mu}^{2}+\varepsilon K\left\|X_{2}^{2} v\right\|_{\mu}^{2}
\end{aligned}
$$

for some constant $K>0$. Substituting in (5) we obtain the desired estimate (4), for $\varepsilon>0$ small enough.

Let us now state the main result of this paper.
Theorem 2.2. Let $P$ be the operator defined in (11), and assume that the vector fields $\left\{X_{j}\right\}_{j=0, \ldots, n^{\prime}}$ are rigid and that for every fixed $x \in \mathbb{T}^{n}$ the $\left\{X_{j}^{\prime}\right\}_{j=1, \ldots, n^{\prime}}$ span $T_{x}\left(\mathbb{T}^{n}\right)$.

Assume moreover that $u \in C^{\infty}\left(\mathbb{T}^{N}\right)$ is a solution of the equation

$$
P(x, u, D) u=f
$$

for some $f \in A\left(\mathbb{T}^{N}\right)$, and that the a-priori estimate (4) is satisfied. Then also $u \in A\left(\mathbb{T}^{N}\right)$.
Remark 2.3. We can follow $[\mathrm{X}]$ to obtain from the a-priori estimate (4) and the use of para-differential operators a result of $C^{\infty}$-hypoellipticity for the operator (11) with the given assumptions on the $X_{j}$ 's: if $f \in C^{\infty}\left(\mathbb{T}^{N}\right)$ and $u \in C^{\mu+3}\left(\mathbb{T}^{N}\right)$ is a solution of $P u=f$, then $u \in C^{\infty}\left(\mathbb{T}^{N}\right)$.

Before giving the proof of Theorem [2.2] we first need some notation.
Define, for $u \in C^{\infty}\left(\mathbb{T}^{N}\right)$,

$$
\left\|\|u\|_{\mu}=\sum_{i, j=1}^{n^{\prime}}\right\| X_{i} X_{j} u\left\|_{\mu}+\sum_{j=1}^{n^{\prime}}\right\| X_{j} u\left\|_{\mu}+\right\| u \|_{\mu+\delta}
$$

and consider the sequence $m_{q}=c q!/(q+1)^{2}$, where the constant $c$ is such that (see (AM])

$$
\begin{equation*}
\sum_{0 \leq \beta \leq \alpha}\binom{\alpha}{\beta} m_{|\beta|} m_{|\alpha-\beta|} \leq m_{|\alpha|} \tag{6}
\end{equation*}
$$

Then set $M_{q}=\varepsilon^{1-q} m_{q}$ for $\varepsilon>0$ and $q \geq 1$. The relation (6) implies that

$$
\begin{equation*}
\sum_{0<\beta<\alpha}\binom{\alpha}{\beta} M_{|\beta|} M_{|\alpha-\beta|} \leq \varepsilon M_{|\alpha|} \tag{7}
\end{equation*}
$$

and hence, if we consider the formal power series

$$
\begin{equation*}
\theta(Y)=\sum_{\alpha>0} \frac{M_{|\alpha|}}{\alpha!} Y^{\alpha} \tag{8}
\end{equation*}
$$

for $Y=(t, x) \in \mathbb{R}^{N}$, we obtain that

$$
\theta^{q}(Y) \ll \varepsilon^{q-1} \theta(Y) \quad \forall q \geq 1, Y \in \mathbb{R}^{N}
$$

meaning that each coefficient of the formal power series on the left is less than or equal to the corresponding coefficient of the formal power series on the right-hand side.

It follows that, if we choose $A, R>0$ satisfying for every integer $q \geq 0$
(9) $\sum_{i, j=1}^{n^{\prime}}\left\|a_{i j}^{(q)}\right\|_{H^{\mu}\left(u\left(\mathbb{T}^{N}\right)\right)}+\sum_{j=1}^{n^{\prime}}\left\|b_{j}^{(q)}\right\|_{H^{\mu}\left(u\left(\mathbb{T}^{N}\right)\right)}+\left\|c^{(q)}\right\|_{H^{\mu}\left(u\left(\mathbb{T}^{N}\right)\right)} \leq A R^{q} q$ !
(which is possible because of the analyticity of the coefficients), and we define the formal power series

$$
\begin{equation*}
\phi(w)=\sum_{q=1}^{+\infty} A R^{q} w^{q}, \quad \text { for } w \in \mathbb{R} \tag{10}
\end{equation*}
$$

then, for every $\rho>0$,

$$
\begin{equation*}
\phi(\rho \theta(Y)) \ll \frac{A}{\varepsilon} \theta(Y) \sum_{q=1}^{+\infty}(\rho R \varepsilon)^{q}=\frac{A R \rho}{1-\varepsilon \rho R} \theta(Y) \tag{11}
\end{equation*}
$$

for all $\varepsilon>0$ such that $\varepsilon \rho R<1$.
Proof of Theorem [2.2, From the given assumptions on the vector fields $X_{j}$, it is sufficient to prove the analytic estimate for $\left\|\partial_{t_{k}}^{b} u\right\|_{\mu}$ for $k=1, \ldots, m$ and for every $b \geq 1$.

We fix $k$ and denote, for simplicity, $t=t_{k}$ and $T=\partial_{t}=\partial_{t_{k}}$. Then we define

$$
[u]_{t, r}=\sup _{0<q \leq r} \frac{\| \| T^{q} u \|_{\mu}}{M_{q}}
$$

and prove by induction on $r \geq 1$ that there exist $\varepsilon, M>0$ such that for all $r \geq 1$,

$$
\begin{equation*}
[u]_{t, r} \leq M \tag{12}
\end{equation*}
$$

We claim that we can take

$$
\begin{equation*}
M=\max \left\{1, \frac{4}{c} \max _{1 \leq q \leq 3}\left\|\mid T^{q} u\right\|_{\mu}\right\} \tag{13}
\end{equation*}
$$

whereas $\varepsilon$ will be chosen in the following.
For $p=1,2,3$ we clearly have that $[u]_{t, p} \leq M$ for $\varepsilon$ small enough. Assume that (12) is satisfied for all $3 \leq r<b$ and let us prove it for $r=b$ (the above request $b \geq 3$ will be understood in the following).

By the a-priori estimate (4) we have that

$$
\begin{align*}
\left\|T^{b} u\right\|_{\mu} & =\sum_{j, k=1}^{n^{\prime}}\left\|X_{j} X_{k} T^{b} u\right\|_{\mu}+\sum_{j=1}^{n^{\prime}}\left\|X_{j} T^{b} u\right\|_{\mu}+\left\|T^{b} u\right\|_{\mu+\delta} \\
& \leq C\left(\left\|P T^{b} u\right\|_{\mu}+\left\|T^{b} u\right\|_{\mu}\right) \tag{14}
\end{align*}
$$

For every $\varepsilon_{1}, \delta_{1}>0$ we can find a positive constant $C_{\varepsilon_{1}, \delta_{1}}>0$ such that

$$
\begin{equation*}
\left\|T^{b} u\right\|_{\mu} \leq \varepsilon_{1}\left\|T^{b} u\right\|_{\mu+\delta}+C_{\varepsilon_{1}, \delta_{1}}\left\|T^{b} u\right\|_{\mu-\delta_{1}} \tag{15}
\end{equation*}
$$

Moreover

$$
\begin{equation*}
\left\|P T^{b} u\right\|_{\mu} \leq\left\|\left[P, T^{b}\right] u\right\|_{\mu}+\left\|T^{b} P u\right\|_{\mu} \tag{16}
\end{equation*}
$$

Since $\varepsilon_{1}\left\|T^{b} u\right\|_{\mu+\delta}$ will be absorbed in the left-hand side of (14), $\left\|T^{b} u\right\|_{\mu-\delta_{1}}$ will be estimated by induction and $\left\|T^{b} P u\right\|_{\mu}$ will not give any problems because of the
analyticity of $f=P u$, we first estimate $\left\|\left[P, T^{b}\right] u\right\|_{\mu}$. To this aim we compute
$\left[T^{b}, P\right]=T^{b} P-P T^{b}$
$=\sum_{j, k=1}^{n^{\prime}} \sum_{b^{\prime}=1}^{b}\binom{b}{b^{\prime}} T^{b^{\prime}}\left(a_{j k}(u(t, x))\right) X_{j} X_{k} T^{b-b^{\prime}}$

$$
\begin{equation*}
+\sum_{j=1}^{n^{\prime}} \sum_{b^{\prime}=1}^{b}\binom{b}{b^{\prime}} T^{b^{\prime}}\left(b_{j}(u(t, x))\right) X_{j} T^{b-b^{\prime}}+\sum_{b^{\prime}=1}^{b}\binom{b}{b^{\prime}} T^{b^{\prime}}(c(u(t, x))) T^{b-b^{\prime}} \tag{17}
\end{equation*}
$$

since the $X_{j}$ 's do not depend on the variable $t$.
Let us denote by $a \circ u$ the generic coefficient $a_{j k}(u(t, x))$ or $b_{j}(u(t, x))$ or $c(u(t, x))$, and write $\left(X^{2}\right)$ for the generic term of the form 1 , or $X_{j}$ or $X_{j} X_{k}$. Then we estimate

$$
\begin{align*}
& \left\|\sum_{b^{\prime}=1}^{b}\binom{b}{b^{\prime}} T^{b^{\prime}}(a \circ u)\left(X^{2}\right) T^{b-b^{\prime}} u\right\|_{\mu} \leq \\
& (18) \quad \Lambda T^{b}(a \circ u)\left\|_{\mu} \cdot\right\|\left(X^{2}\right) u \|_{\mu}  \tag{18}\\
& \\
&
\end{align*}
$$

It can be easily proved by induction on $p \geq 1$ that the derivative $T^{p}$ of the composite function $a(u(t, x))$ can be written as

$$
\begin{aligned}
T^{p}(a \circ u) & =\sum_{\substack{r_{i} \in \mathbb{N} \backslash\{0\} \\
r_{1}+\ldots+r_{q}=p}} C_{q, r} a^{(q)}(u) \partial_{t}^{r_{1}} u \cdots \partial_{t}^{r_{q}} u \\
& =a^{\prime}(u) T^{p} u+\sum_{\substack{r_{1}+\ldots+r_{q}=p \\
0<r_{i}<p}} C_{q, r} a^{(q)}(u) \partial_{t}^{r_{1}} u \cdots \partial_{t}^{r_{q}} u
\end{aligned}
$$

for some $C_{q, r}>0$, and therefore

$$
\begin{aligned}
\left\|T^{b}(a \circ u)\right\|_{\mu} \leq & \left\|a^{\prime}(u) T^{b} u\right\|_{\mu}+\sum_{\substack{r_{1}+\ldots+r_{q}=b \\
0<r_{i}<b}} C_{q, r} \Lambda^{q}\left\|a^{(q)}(u)\right\|_{\mu}\left\|\partial_{t}^{r_{1}} u\right\|_{\mu} \cdots\left\|\partial_{t}^{r_{q}} u\right\|_{\mu} \\
\leq & \Lambda\left\|a^{\prime}(u)\right\|_{\mu} \cdot\left\|T^{b} u\right\|_{\mu} \\
& +\sum_{\substack{r_{1}+\ldots+r_{q}=b \\
0<r_{i}<b}} C_{q, r}\left\|a^{(q)}(u)\right\|_{\mu}\left(\Lambda[u]_{t, b-1}\right)^{q} \partial_{t}^{r_{1}} \theta(0) \cdots \partial_{t}^{r_{q}} \theta(0)
\end{aligned}
$$

since $\left\|\partial_{t}^{r_{h}} u\right\|_{\mu} \leq[u]_{t, b-1} M_{r_{h}}=[u]_{t, b-1} \partial_{t}^{r_{h}} \theta(0)$ for $1 \leq r_{h} \leq b-1, h=1, \ldots, q$, where $\theta(Y)$ is the formal power series defined in (8).

With the choice made for $A, R>0$ in (9), we have that $\left\|a^{(q)}(u)\right\|_{\mu} \leq A R^{q} q$ ! and hence, substituting in (19),

$$
\begin{align*}
\left\|T^{b}(a \circ u)\right\|_{\mu} \leq & \Lambda A R\left\|T^{b} u\right\|_{\mu} \\
& +\sum_{\substack{r_{1}+\ldots+r_{q}=b \\
0<r_{i}<b}} C_{q, r} A R^{q} q!\left(\Lambda[u]_{t, b-1}\right)^{q} \partial_{t}^{r_{1}} \theta(0) \cdots \partial_{t}^{r_{q}} \theta(0) \tag{20}
\end{align*}
$$

Let us now remark that, for $\phi$ given by (10) and $\rho>0$,

$$
\begin{aligned}
T^{b}(\phi(\rho \theta)) & =\sum_{\substack{r_{1}+\ldots+r_{q}=b \\
r_{i}>0}} C_{q, r} \phi^{(q)}(\rho \theta) \rho^{q} \partial_{t}^{r_{1}} \theta \cdots \partial_{t}^{r_{q}} \theta \\
& =\phi^{\prime}(\rho \theta) \rho \partial_{t}^{b} \theta+\sum_{\substack{r_{1}+\ldots+r_{q}=b \\
0<r_{i}<b}} C_{q, r} \phi^{(q)}(\rho \theta) \rho^{q} \partial_{t}^{r_{1}} \theta \cdots \partial_{t}^{r_{q}} \theta
\end{aligned}
$$

and therefore

$$
\left.T^{b}(\phi(\rho \theta(Y)))\right|_{Y=0}=A R \rho M_{b}+\sum_{\substack{r_{1}+\ldots+r_{q}=b \\ 0<r_{i}<b}} C_{q, r} A R^{q} q!\rho^{q} \partial_{t}^{r_{1}} \theta(0) \cdots \partial_{t}^{r_{q}} \theta(0)
$$

since $\partial^{\alpha} \theta(0)=M_{|\alpha|}$ and $\phi^{(q)}(0)=A R^{q} q$..
Substituting in (20) with $\rho=\Lambda[u]_{t, b-1}$,

$$
\begin{align*}
\left\|T^{b}(a \circ u)\right\|_{\mu} \leq & \Lambda A R\left\|T^{b} u\right\|_{\mu}+\left.T^{b}\left(\phi\left(\Lambda[u]_{t, b-1} \theta(Y)\right)\right)\right|_{Y=0} \\
& -\Lambda A R[u]_{t, b-1} M_{b} \tag{21}
\end{align*}
$$

From (11) we deduce that

$$
T^{b}(\phi(\rho \theta(Y))) \ll \frac{A R \rho}{1-\varepsilon \rho R} T^{b} \theta(Y) \quad \text { if } \varepsilon \rho R<1
$$

and hence

$$
\left.T^{b}\left(\phi\left(\Lambda[u]_{t, b-1} \theta(Y)\right)\right)\right|_{Y=0} \leq \frac{A R \Lambda[u]_{t, b-1}}{1-\varepsilon R \Lambda[u]_{t, b-1}} M_{b}
$$

if $\varepsilon R \Lambda[u]_{t, b-1}<1$.
By the inductive assumption we can take $\varepsilon=\varepsilon_{o} /(M R \Lambda)$, with $0<\varepsilon_{o}<1$ to be chosen in the following, so that

$$
\varepsilon R \Lambda[u]_{t, b-1} \leq \frac{\varepsilon_{o}}{M R \Lambda} R \Lambda M=\varepsilon_{o}<1
$$

We thus obtain from (21) and (15) that

$$
\begin{align*}
\left\|T^{b}(a \circ u)\right\|_{\mu} \leq & \Lambda A R\left(\varepsilon_{1}\left\|T^{b} u\right\|_{\mu+\delta}+C_{\varepsilon_{1}, \delta_{1}}\left\|T^{b} u\right\|_{\mu-\delta_{1}}\right)+\frac{\Lambda A R}{1-\varepsilon_{o}}[u]_{t, b-1} M_{b} \\
& -\Lambda A R[u]_{t, b-1} M_{b} \\
(22) & \varepsilon_{1} \Lambda A R\left\|T^{b} u\right\|_{\mu+\delta}+\Lambda A R C_{\varepsilon_{1}, \delta_{1}}\left\|T^{b} u\right\|_{\mu-\delta_{1}}+\frac{\varepsilon_{o}}{1-\varepsilon_{o}} \Lambda A R[u]_{t, b-1} M_{b} \tag{22}
\end{align*}
$$

This estimate will be substituted in (18). In a similar way we obtain the following estimates for $1 \leq b^{\prime} \leq b-1$ :

$$
\begin{align*}
\left\|T^{b^{\prime}}(a \circ u)\right\|_{\mu} & \leq \sum_{\substack{r_{1}+\ldots+r_{q}=b^{\prime} \leq b-1 \\
r_{i}>0}} C_{q, r} A R^{q} q!\left(\Lambda[u]_{t, b-1}\right)^{q} \partial_{t}^{r_{1}} \theta(0) \cdots \partial_{t}^{r_{q}} \theta(0) \\
& =\left.T^{b^{\prime}}\left(\phi\left(\Lambda[u]_{t, b-1} \theta(Y)\right)\right)\right|_{Y=0} \leq \frac{\Lambda A R}{1-\varepsilon_{o}}[u]_{t, b-1} M_{b^{\prime}} . \tag{23}
\end{align*}
$$

Substituting (22) and (23) in (18),

$$
\begin{align*}
& \left\|\sum_{b^{\prime}=1}^{b}\binom{b}{b^{\prime}} T^{b^{\prime}}(a \circ u)\left(X^{2}\right) T^{b-b^{\prime}} u\right\|_{\mu} \leq \Lambda\left\|\left(X^{2}\right) u\right\|_{\mu}\left\{\varepsilon_{1} \Lambda A R\left\|T^{b} u\right\|_{\mu+\delta}\right. \\
& \left.\quad+\Lambda A R C_{\varepsilon_{1}, \delta_{1}}\left\|T^{b} u\right\|_{\mu-\delta_{1}}+\frac{\varepsilon_{o}}{1-\varepsilon_{o}} \Lambda A R[u]_{t, b-1} M_{b}\right\} \\
& \quad+\Lambda \sum_{b^{\prime}=1}^{b-1}\binom{b}{b^{\prime}} \frac{\Lambda A R}{1-\varepsilon_{o}}[u]_{t, b-1} M_{b^{\prime}}\left\|\left(X^{2}\right) T^{b-b^{\prime}} u\right\|_{\mu} \tag{24}
\end{align*}
$$

Let us set $m=\| \| u \|_{\mu}$ (so that $\left\|\left(X^{2}\right) u\right\|_{\mu} \leq m$ ), and estimate the norms $\left\|T^{b} u\right\|_{\mu-\delta_{1}}$ and $\left\|\left(X^{2}\right) T^{b-b^{\prime}} u\right\|_{\mu}$ :

$$
\begin{aligned}
& \left\|T^{b} u\right\|_{\mu-\delta_{1}} \leq\left\|T^{b-1} u\right\|_{\mu+\delta} \leq[u]_{t, b-1} M_{b-1} \quad \text { if } \delta_{1} \geq 1-\delta, \delta_{1}>0 \\
& \left\|\left(X^{2}\right) T^{b-b^{\prime}} u\right\|_{\mu} \leq\left\|T^{b-b^{\prime}} u\right\|_{\mu} \leq[u]_{t, b-1} M_{b-b^{\prime}} \quad \text { since } b^{\prime} \geq 1
\end{aligned}
$$

Then, from (24) and the inductive assumption,

$$
\begin{aligned}
&\left\|\sum_{b^{\prime}=1}^{b}\binom{b}{b^{\prime}} T^{b^{\prime}}(a \circ u)\left(X^{2}\right) T^{b-b^{\prime}} u\right\|_{\mu} \leq \varepsilon_{1} m \Lambda^{2} A R\left\|T^{b} u\right\|_{\mu+\delta}+m \Lambda^{2} A R C_{\varepsilon_{1}, \delta_{1}} M M_{b-1} \\
& \quad+\frac{\varepsilon_{o}}{1-\varepsilon_{o}} m \Lambda^{2} A R M M_{b}+\frac{\Lambda^{2} A R}{1-\varepsilon_{o}} M^{2} \sum_{b^{\prime}=1}^{b-1}\binom{b}{b^{\prime}} M_{b^{\prime}} M_{b-b^{\prime}} \\
& \leq \varepsilon_{1} m \Lambda^{2} A R\left\|T^{b} u\right\|_{\mu+\delta}+m \Lambda^{2} A R C_{\varepsilon_{1}, \delta_{1}} M \varepsilon M_{b} \\
& \quad+\frac{\varepsilon_{o}}{1-\varepsilon_{o}} m \Lambda^{2} A R M M_{b}+\frac{\Lambda^{2} A R}{1-\varepsilon_{o}} M^{2} \varepsilon M_{b}
\end{aligned}
$$

because of (7) and of the estimate $M_{b-1} \leq \varepsilon M_{b}$ for $b \geq 3$. (Here is the only reason why we start with $r=3$ in the induction.)

By the choice of $\varepsilon=\varepsilon_{o} /(M R \Lambda)$ we have

$$
\begin{align*}
& \left\|\sum_{b^{\prime}=1}^{b}\binom{b}{b^{\prime}} T^{b^{\prime}}(a \circ u)\left(X^{2}\right) T^{b-b^{\prime}} u\right\|_{\mu} \leq \varepsilon_{1} m \Lambda^{2} A R\left\|T^{b} u\right\|_{\mu+\delta} \\
& \quad+\varepsilon_{o}\left(m \Lambda A C_{\varepsilon_{1}, \delta_{1}}+\frac{m \Lambda^{2} A R M}{1-\varepsilon_{o}}+\frac{\Lambda A M}{1-\varepsilon_{o}}\right) M_{b} \tag{25}
\end{align*}
$$

From (17) and (25) we finally obtain the desired estimate for $\left\|\left[P, T^{b}\right] u\right\|_{\mu}$ :

$$
\left\|\left[P, T^{b}\right] u\right\|_{\mu} \leq \varepsilon_{1} m \Lambda^{2} A R\left(n^{\prime 2}+n^{\prime}+1\right)\left\|T^{b} u\right\|_{\mu+\delta}
$$

$$
\begin{equation*}
+\varepsilon_{o}\left(m \Lambda A C_{\varepsilon_{1}, \delta_{1}}+\frac{m \Lambda^{2} A R M}{1-\varepsilon_{o}}+\frac{\Lambda A M}{1-\varepsilon_{o}}\right)\left(n^{\prime 2}+n^{\prime}+1\right) M_{b} \tag{26}
\end{equation*}
$$

Then, from (14), (15), (16) and (26),

$$
\begin{aligned}
\left\|T^{b} u\right\|_{\mu} \leq & C\left[\varepsilon_{1} m \Lambda^{2} A R\left(n^{\prime 2}+n^{\prime}+1\right)\left\|T^{b} u\right\|_{\mu+\delta}\right. \\
& +\varepsilon_{o}\left(m \Lambda A C_{\varepsilon_{1}, \delta_{1}}+\frac{m \Lambda^{2} A R M}{1-\varepsilon_{o}}+\frac{\Lambda A M}{1-\varepsilon_{o}}\right)\left(n^{\prime 2}+n^{\prime}+1\right) M_{b} \\
& \left.+B_{f}^{b+1} \frac{b!}{(b+1)^{2}}+\varepsilon_{1}\left\|T^{b} u\right\|_{\mu+\delta}+C_{\varepsilon_{1}, \delta_{1}}[u]_{t, b-1} M_{b-1}\right] \\
\leq & \varepsilon_{1}\left[C m \Lambda^{2} A R\left(n^{\prime 2}+n^{\prime}+1\right)+C\right] \cdot\left\|T^{b} u\right\|_{\mu+\delta} \\
& +\varepsilon_{o} C M\left(m \Lambda A C_{\varepsilon_{1}, \delta_{1}}+\frac{m \Lambda^{2} A R}{1-\varepsilon_{o}}+\frac{\Lambda A}{1-\varepsilon_{o}}\right)\left({n^{\prime 2}}^{2}+n^{\prime}+1\right) M_{b} \\
& +B_{f}^{b+1} \frac{b!}{(b+1)^{2}}+C C_{\varepsilon_{1}, \delta_{1}} M \frac{\varepsilon_{o}}{M R \Lambda} M_{b}
\end{aligned}
$$

where $B_{f}>0$ is given by the analyticity of $f$.
We now choose $0<\varepsilon_{1}<1$ with

$$
A_{\varepsilon_{1}}=\varepsilon_{1}\left[C m \Lambda^{2} A R\left(n^{\prime 2}+n^{\prime}+1\right)+C\right]<1
$$

and then $0<\varepsilon_{o}<1$ sufficiently small so that, for $b \geq 3$,

$$
\frac{B_{f}^{b+1}}{A_{\varepsilon_{1}}} \frac{b!}{(b+1)^{2}} \leq \frac{1}{2} M_{b}=\frac{c}{2} \frac{\varepsilon_{o}^{1-b}}{(M R \Lambda)^{1-b}} \frac{b!}{(b+1)^{2}}
$$

and

$$
\varepsilon_{o} \frac{C}{A_{\varepsilon_{1}}}\left[\left(m \Lambda A C_{\varepsilon_{1}, \delta_{1}}+\frac{m \Lambda^{2} A R}{1-\varepsilon_{o}}+\frac{\Lambda A}{1-\varepsilon_{o}}\right)\left(n^{\prime 2}+n^{\prime}+1\right)+\frac{C_{\varepsilon_{1}, \delta_{1}}}{R \Lambda}\right] \leq \frac{1}{2}
$$

With such choices we finally have that $\left\|\mid T^{b} u\right\| \|_{\mu} \leq M M_{b}$ and hence, by the inductive assumption,

$$
[u]_{t, b}=\sup _{0<q \leq b} \frac{\| \| T^{q} u \|_{\mu}}{M_{q}}=\max \left\{\sup _{0<q<b} \frac{\left\|T^{q} u\right\|_{\mu}}{M_{q}}, \frac{\left\|T^{b} u\right\|_{\mu}}{M_{b}}\right\} \leq M
$$

The theorem is therefore proved.

## 3. The case of coefficients also depending on $(t, x) \in \mathbb{T}^{m} \times \mathbb{T}^{n}$

Let us now consider the case in which the operator $P$ has real analytic complex valued coefficients which depend also on $(t, x) \in \mathbb{T}^{m} \times \mathbb{T}^{n} \simeq \mathbb{T}^{N}$ :

$$
\begin{align*}
& P=P(t, x, u, D)  \tag{27}\\
& =\sum_{j, k=1}^{n^{\prime}} a_{j k}(t, x, u(t, x)) X_{j} X_{k}+\sum_{j=1}^{n^{\prime}} b_{j}(t, x, u(t, x)) X_{j}+X_{0}+c(t, x, u(t, x))
\end{align*}
$$

where the real valued rigid vector fields $\left\{X_{j}\right\}_{0 \leq j \leq n^{\prime}}$ are defined as in (2) and satisfy the same assumptions as in $\$ 2$

Then we can prove the analogue of Theorem 2.2.
Theorem 3.1. Let $P$ be the operator defined in (27) and assume that the vector fields $\left\{X_{j}\right\}_{j=0, \ldots, n^{\prime}}$ are rigid and that for every fixed $x \in \mathbb{T}^{n}$ the $\left\{X_{j}^{\prime}\right\}_{j=1, \ldots, n^{\prime}}$ span $T_{x}\left(\mathbb{T}^{n}\right)$.

Assume moreover that $u \in C^{\infty}\left(\mathbb{T}^{N}\right)$ is a solution of the equation

$$
P(t, x, u, D) u=f
$$

for some $f \in A\left(\mathbb{T}^{N}\right)$ and that the a-priori estimate (4) is satisfied. Then also $u \in A\left(\mathbb{T}^{N}\right)$.

Proof. It is analogous to that of Theorem[2.2, and therefore we give here only the sketch of it.

Following the same outline as in the proof of Theorem 2.2 we replace (9) and (10) defining $A, R>1$ and the formal power series $\phi$ by the following formulas:

$$
\begin{aligned}
& \sum_{i, j=1}^{n^{\prime}}\left\|\partial_{t}^{r} \partial_{u}^{q} a_{i j}\right\|_{H^{\mu}\left(\mathbb{T}^{N} \times u\left(\mathbb{T}^{N}\right)\right)}+\sum_{j=1}^{n^{\prime}}\left\|\partial_{t}^{r} \partial_{u}^{q} b_{j}\right\|_{H^{\mu}\left(\mathbb{T}^{N} \times u\left(\mathbb{T}^{N}\right)\right)} \\
& \quad+\left\|\partial_{t}^{r} \partial_{u}^{q} c\right\|_{H^{\mu}\left(\mathbb{T}^{N} \times u\left(\mathbb{T}^{N}\right)\right)} \leq A R^{r+q}(r+q)!\quad \forall r, q \geq 0 \\
& \quad \phi(w)=\sum_{q=1}^{+\infty} A(2 R)^{q} w^{q} \quad \text { for } w \in \mathbb{R}
\end{aligned}
$$

so that we have (11) with $\bar{R}=2 R$ instead of $R$. We then set

$$
M=\max \left\{1, \frac{4}{c} \max _{1 \leq q \leq 3}\| \| T^{q} u \|_{\mu}, 2 A\right\}
$$

instead of the choice made for $M$ in (13).
Following the proof of Theorem [2.2, we must estimate the $H^{\mu}\left(\mathbb{T}^{N}\right)$-norm of $T^{b} a(t, x, u(t, x))$. To this aim we first recall the following formula, which can be easily proved by induction on $p \geq 1$ :

$$
\begin{align*}
& \partial_{t}^{p} a(t, x, u(t, x))=\left(\partial_{t}^{p} a\right)(t, x, u(t, x)) \\
& +\sum_{s=1}^{p}\binom{p}{s} \sum_{\substack{r_{1}+\ldots+r_{q}=s \\
r_{i}>0}} C_{q, r}\left(\partial_{u}^{q} \partial_{t}^{p-s} a\right)(t, x, u(t, x)) \partial_{t}^{r_{1}} u(t, x) \cdots \partial_{t}^{r_{q}} u(t, x) \tag{28}
\end{align*}
$$

for some $C_{q, r}>0$. Then

$$
\begin{align*}
& T^{b} a(t, x, u(t, x))=\left(\partial_{t}^{b} a\right)(t, x, u(t, x))+\left.\partial_{t}^{b} a(\tau, x, u(t, x))\right|_{\tau=t} \\
& \quad+\sum_{s=1}^{b-1}\binom{b}{s} \sum_{\substack{r_{1}+\ldots+r_{q}=s \\
r_{i}>0}} C_{q, r}\left(\partial_{u}^{q} \partial_{t}^{b-s} a\right)(t, x, u(t, x)) \partial_{t}^{r_{1}} u(t, x) \cdots \partial_{t}^{r_{q}} u(t, x) \tag{29}
\end{align*}
$$

for $b \geq 3$. Let us denote by $a \circ u$ the composite function $a(t, x, u(t, x))$. The first term on the right-hand side of (29) is then easily estimated by

$$
\left\|\partial_{t}^{b} a \circ u\right\|_{\mu} \leq A R^{b} b!\leq \frac{M}{2} M_{b},
$$

if $0<\varepsilon \leq \bar{\varepsilon}$ with $\bar{\varepsilon}$ small enough in order that

$$
\begin{equation*}
\bar{R}^{p} p!\leq M_{p}=\varepsilon^{1-p} c \frac{p!}{(p+1)^{2}} \quad \forall p \geq 2 \tag{30}
\end{equation*}
$$

The second term on the right-hand side of (29) is estimated as in (22) with $\bar{R}=2 R$ instead of $R$ and $0<\bar{\varepsilon}_{o}<\min \{1, \bar{\varepsilon} M \bar{R} \Lambda\}$ instead of $\varepsilon_{o}$, taking $\varepsilon=\bar{\varepsilon}_{o} /(M \bar{R} \Lambda)$.

We have to estimate the third term of the right-hand side of (29). To this aim we first recall that $(k+j)!\leq 2^{k+j} k!j$ ! for all $k, j \in \mathbb{N}$, and hence

$$
\left\|\partial_{u}^{q} \partial_{t}^{b-s} a \circ u\right\|_{\mu} \leq A R^{q+b-s}(q+b-s)!\leq A \bar{R}^{q+b-s} q!(b-s)!.
$$

Therefore

$$
\begin{aligned}
& \left\|\sum_{s=1}^{b-1}\binom{b}{s} \sum_{\substack{r_{1}+\ldots+r_{q}=s \\
r_{i}>0}} C_{q, r} \partial_{u}^{q} \partial_{t}^{b-s} a \circ u \partial_{t}^{r_{1}} u \cdots \partial_{t}^{r_{q}} u\right\|_{\mu} \\
\leq & \sum_{s=1}^{b-1}\binom{b}{s} \bar{R}^{b-s}(b-s)!\sum_{\substack{r_{1}+\ldots+r_{q}=s \\
r_{i}>0}} C_{q, r} A \bar{R}^{q} q!\left(\Lambda[u]_{t, b-1}\right)^{q} \partial_{t}^{r_{1}} \theta(0) \cdots \partial_{t}^{r_{q}} \theta(0) \\
\leq & \left.\bar{R} b T^{b-1} \phi\left(\Lambda[u]_{t, b-1} \theta(Y)\right)\right|_{Y=0}+\left.\sum_{s=1}^{b-2}\binom{b}{s} M_{b-s} T^{s} \phi\left(\Lambda[u]_{t, b-1} \theta(Y)\right)\right|_{Y=0} \\
\leq & \frac{A \bar{R} \Lambda}{1-\bar{\varepsilon}_{o}}[u]_{t, b-1}\left[\bar{R} b M_{b-1}+\sum_{s=1}^{b-2}\binom{b}{s} M_{b-s} M_{s}\right] \\
\leq & \frac{3 A \bar{R}^{2} \Lambda}{1-\bar{\varepsilon}_{o}}[u]_{t, b-1} \varepsilon M_{b}
\end{aligned}
$$

because of (30), of

$$
\left.T^{p} \phi\left(\Lambda[u]_{t, b-1} \theta(Y)\right)\right|_{Y=0} \leq \frac{A \bar{R} \Lambda}{1-\bar{\varepsilon}_{o}}[u]_{t, b-1} M_{p} \quad \forall p \geq 1
$$

of (17) and of $b M_{b-1} \leq 2 \varepsilon M_{b}$ for $b \geq 3$.
The same arguments can be hold to estimate $\left\|T^{b^{\prime}} a \circ u\right\|_{\mu}$ for $1 \leq b^{\prime} \leq b-1$. We can thus proceed as in the proof of Theorem 2.2 to obtain the desired estimate (12).

## Acknowledgements

The authors are grateful to Professor P. Popivanov for his suggestions about the subject of this paper and for his helpful remarks.

## References

[AM] S. Alinhac - G. Metivier, Propagation de l'analyticité des solutions de systèmes hyperboliques non-linéaires, Invent. Math., 75 (1984), pp. 189-204 MR 86f:35010
[B] J.M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Scient. Éc. Norm. Sup., $4^{e}$ série (1981), pp. 209-246 MR 84h:35177
[GPY] T. Gramchev - P. Popivanov - M. Yoshino, Global properties in spaces of generalized functions on the torus for second order differential operators with variable coefficients, Rend. Sem. Mat. Univ. Pol. Torino, vol. 51, n. 2 (1993) MR 95k:35047
[G] P. Guan, Regularity of a class of quasilinear degenerate elliptic equations, Advances in Mathematics, 132 (1997), pp. 24-45 MR 99a:35068
[RS] L.P. Rothschild - E.M. Stein, Hypoelliptic differential operators and nilpotent groups, Acta Math., 137 (1976), pp. 247-320 MR 55:9171
[T] D.S. Tartakoff, Global (and local) analyticity for second order operators constructed from rigid vector fields on products of tori, Trans. of A.M.S., Vol. 348, n. 7 (1996), pp. 2577-2583 MR 96i:35018
[TZ] D.S. Tartakoff - L. Zanghirati, Analyticity of Solutions for Sums of Squares of Non-linear Vector Fields, to appear in Proc. A.M.S.
[X] C.J. Xu, Regularity of solutions of second order non-elliptic quasilinear partial differential equations, C.R. Acad. Sci. Paris, Sér. I, n. 8, t. 300 (1985), pp. 235-237 MR 86m:35073

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[^0]:    Received by the editors July 4, 2002.
    2000 Mathematics Subject Classification. Primary 35B65, 35B45; Secondary 35H10, 35H20.
    Key words and phrases. Analytic regularity, non-linear, sums of squares of vector fields, torus.

