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ZEROES OF COMPLETE POLYNOMIAL VECTOR FIELDS

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(Communicated by Carmen C. Chicone)

Dedicated to my father

ABSTRACT. We prove that a complete polynomial vector field on $\mathbb{C}^2$ has at most one zero, and analyze the possible cases of those with exactly one which is not of Poincaré-Dulac type. We also obtain the possible nonzero first jet singularities of the foliation $\mathcal{F}_X$ at infinity and the nongenericity of completeness. Connections with the Jacobian Conjecture are established.

INTRODUCTION AND RESULTS

Let $X = P(z_1, z_2)\frac{\partial}{\partial z_1} + Q(z_1, z_2)\frac{\partial}{\partial z_2}$ be a polynomial vector field on $\mathbb{C}^2$ of degree $m = \max\{degP, degQ\} \geq 2$ with isolated zeroes. It is known, [9], that X extends as a rational vector field in $\mathbb{CP}^2$ having a pole along the line at infinity, L_∞. Removing the pole, we obtain a foliation $\mathcal{F}_X$ of degree d, where $d = m$ if L_∞ is invariant and $d = m - 1$ if it is not. We denote by $Sing(\mathcal{F}_X)$ the singular set of $\mathcal{F}_X$.

Recall that a holomorphic vector field X in a complex manifold M is said to be complete if, for every $p \in M$, the differential equation defined by X can be solved for every complex time t.

In this paper we study complete polynomial vector fields X on $\mathbb{C}^2$ through some properties of the leaves of $\mathcal{F}_X$. In section 1, we analyze the trajectories of X at infinity and we give in Theorem 1.1 the possible nonzero first jet of $\mathcal{F}_X$ at its singular points in L_∞, thus proving Corollary 1.1: foliations induced by complete polynomial vector fields of degree m give a nowhere dense set in the space of degree m foliations, $\mathcal{F}(m, 2)$, providing a polynomial version of Buzzard-Fornaess's result, [5]. We also apply our results to the problem of exploding orbits of complex polynomial Hamiltonians, obtaining a simple geometric proof of Fornaess and Grellier's result, [8], in that case.

In section 2, we further study the isolated zeroes of X. A natural question (posed in [1] and [15]) is if there exist complete holomorphic vector fields on $\mathbb{C}^2$ with more than one isolated zero. The answer, given in Theorem 2.1, is no for polynomial ones. Our result relies on the study of proper orbits due to Brunella in [4]. We also classify the complete polynomial vector fields with rational first integral and, using

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Andersen's result in [2], those with one zero p which is not of Poincaré-Dulac type, when it is nondicritical and at least two of the separatrices through it are algebraic at infinity. When p is dicritical with no rational first integral, or nondicritical with just one separatrix algebraic at infinity, the induced foliation $\mathcal{F}_X$ is, as in Brunella's result [4], P-complete where P can be written in a simple form due to [17] and [16] (Proposition 2.1 and Theorem 2.2).

In section 3, we state the Jacobian Conjecture in terms of completeness of certain vector fields, and characterize the complete commutative bases of $\mathbb{C}$-derivations of the polynomial ring.

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1. Separatrices at infinity and completeness

A germ Σ of an analytic irreducible curve is said to be a trajectory of X at $p \in L_\infty$ if $p \in \Sigma$ and $\Sigma \setminus \{p\}$ is invariant by X. In this case one can extend $\Sigma \setminus \{p\}$ by analytic continuation to obtain the complex orbit L of X. If $\gamma : \mathbb{D} \to \Sigma$ is the (minimal) Puiseaux's parametrization of a neighborhood U_p of p in Σ, $\mathcal{L} = L \cup \{p\}$ can be endowed with an abstract Riemann surface structure as follows: for any $q \in L$, by the existence of local solutions for X, we can take the parametrization γ_q of an open neighborhood $U_q \subset L$, and define the local chart as $z_q = {\gamma_q}^{-1} : U_q \to \mathbb{C}$. Otherwise, γ defines the local chart around p in $\mathcal{L}$ as $z_p = \gamma^{-1} : \Sigma \to \mathbb{D}$.

Lemma 1.1. *Let X be a polynomial vector field in $\mathbb{C}^2$, and let Σ be a trajectory of X at $p \in L_\infty$. Then, if X is complete on $\mathcal{L} \setminus \{p\}$, it extends to p as a zero of order 1 or 2.*

Proof. As $\mathcal{L} \setminus \{p\}$ is uniformized by $\mathbb{C}$, and it is contained in the Stein manifold $\mathbb{C}^2$, then $\mathcal{L} \setminus \{p\}$ is (analytically) isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$. If $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}$, it follows that $\mathcal{L} \simeq \mathbb{CP}^1$ and X extends to p as zero of order 2, by Riemann-Roch. On the other hand, if $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}^*$, then X extends to p as zero of order 1. We refer to [10] for the study of complete vector fields on Riemann surfaces. □

Corollary 1.1. *If X is a complete polynomial vector field on $\mathbb{C}^2$, then L_∞ is invariant by $\mathcal{F}_X$.*

Remark 1.1. If $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}$, by Chow's Theorem $\mathcal{L} \setminus \{p\}$ is contained in a rational curve.

Remark 1.2. Lemma 1.1 is valid for polynomial vector fields on $\mathbb{C}^n$, $n \geq 2$.

Let $p \in Sing(\mathcal{F}_X) \cap L_\infty$, and let $\Sigma \neq L_\infty$ be a separatrix of $\mathcal{F}_X$ through p, parametrized by $\gamma : \mathbb{D} \to \Sigma$. Without loss of generality assume that $p = (0:1:0)$. Then if $\gamma(t) = (y_1(t), y_2(t))$, with $(y_1, y_2) = (\varphi_1 \circ \varphi_0^{-1})(z_1, z_2) = (\frac{1}{z_1}, \frac{z_2}{z_1})$ the usual change of charts in $\mathbb{CP}^2$, we denote by σ the order of $y_1(t)$ at $t = 0$, which is the order of contact of Σ with L_∞ at p. Since $\Sigma \setminus \{p\}$ is invariant by $\mathcal{F}_X$, $\gamma^* X$ is a holomorphic vector field on $\mathbb{D}^*$ whose order at 0 is called the multiplicity of $\mathcal{F}_X$ with respect to Σ. We will denote it by $ind_p(\mathcal{F}_X, \Sigma)$. From now on, if no other conditions are explicitly given, X will be a complete polynomial vector field on $\mathbb{C}^2$ of degree $m \geq 2$ with isolated zeroes.

Lemma 1.2. $ind_p(\mathcal{F}_X, \Sigma) - \sigma(m-1) = 1$ *or* 2.

Proof. We obtain the extension of $X_{|\mathcal{L}\setminus\{p\}}$ to p as $\gamma^*(\varphi_1\circ\varphi_0^{-1})_*X = f(t)\frac{\partial}{\partial t}$. Thus $(y_1(t))^{m-1}f(t)$ equals

$$-\sum_{i=0}^{m}\frac{(y_1(t))^{m+1-i}}{y_1'(t)}\cdot P_i(1,y_2(t)),\ \text{or}\ \sum_{i=0}^{m}\frac{(y_1(t))^{m-i}}{y_2'(t)}\cdot G_i(y_2(t)), \tag{1}$$

where P_i and Q_i denote the homogeneous components of degree i of P and Q respectively, and $G_i(y_2) = Q_i(1,y_2) - y_2P_i(1,y_2)$. As L_∞ is invariant by Corollary 1.1, $y_1^{m-1}(\varphi_1\circ\varphi_0^{-1})_*X$ represents $\mathcal{F}_X$ in U_1. Thus $ord_0 f(t) = ind_p(\mathcal{F}_X,\Sigma) - \sigma(m-1)$, and the result follows from Lemma 1.1. □

1.1. **Foliations with nonzero first jet singularities at infinity.** We say that $\mathcal{F}_X$ has nonzero first jet at a singularity p if the linear part at p of a vector field Y which represents $\mathcal{F}_X$ in a neighbourhood of p is not zero. Let λ and μ be the eigenvalues of DY_p and suppose that λ and μ are not both zero. Then, we say that p is a saddle-node point if $\lambda\mu = 0$. If $\lambda/\mu\in\mathbb{Q}^+$, the singularity is either dicritical or of Poincaré-Dulac type: after a local analytic change of coordinates Y is given by $x\frac{\partial}{\partial x} + (ny + x^n)\frac{\partial}{\partial y}$, with $n\in\mathbb{N}^+$ [3]. We will suppose that $p = (0,\alpha)\in Sing\,(\mathcal{F}_X)\cap L_\infty$. Let us rewrite the jacobian $D(y_1^{m-1}(\varphi_1\circ\varphi_0^{-1})_*X)_p$ as

$$J_p = \begin{pmatrix} -P_m(1,\alpha) & 0 \\ G_{m-1}(\alpha) & G_m'(\alpha)\end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ \nu & \mu \end{pmatrix}. \tag{2}$$

Theorem 1.1. *Let $p\in Sing\,(\mathcal{F}_X)\cap L_\infty$ be a point at which $\mathcal{F}_X$ has nonzero first jet. Let us suppose that λ and μ are not both zero. Then,*

(i) *either p is a saddle-node point and L_∞ defines the strong direction, that is, $\lambda = 0$, $\mu\neq 0$;*
(ii) *or p is of Poincaré-Dulac type.*

Proof. We study the following cases:

1) If $detJ_p = 0$, then $\lambda = 0$. To see this, we use Corollary 1.1, and observe that if $\lambda\neq 0$, L_∞ is a smooth separatrix tangent to the weak direction $\mu = 0$ and there is just one more smooth separatrix Σ, tangent to the strong direction. Σ is transversal to L_∞ at p, so $ind_p(\mathcal{F}_X,\Sigma) = 1 < m$, contradicting Lemma 1.2. Then (i) holds, and there is at most one more separatrix $\Sigma\neq L_\infty$, [11, pp. 521–522].

2) If $detJ_p\neq 0$, then $\lambda/\mu\in\mathbb{Q}^+$, as otherwise there are exactly two transversal smooth separatrices through p [11, pp. 518–521], and we get a contradiction as before. Moreover, p is nondicritical. If not, take a separatrix $\Sigma\neq L_\infty$, then $ind_p(\mathcal{F}_X,\Sigma) = 1 < 1+\sigma(m-1)$, again a contradiction by Lemma 1.2. Thus p is of Poincaré-Dulac type, [3]. □

Remark 1.3. Note that if $\Sigma\neq L_\infty$ is a separatrix through p, then p is a saddle-node point and L_∞ defines the strong direction. Example ([7]): $X = z_1\frac{\partial}{\partial z_1} - z_2(1+z_1)\frac{\partial}{\partial z_2}$, with $p = (0:1:0)\in L_\infty$.

Corollary 1.2. *For each $m\geq 2$, the set of degree m foliations defined by complete polynomial vector fields is a nowhere dense set in $\mathcal{F}(m,2)$.*

Application: exploding orbits of polynomial Hamiltonians. Given $H\in\mathbb{C}[z_1,z_2]_m$, the space of polynomials of degree $\leq m$, we get a polynomial Hamiltonian, X_H. The (complex) orbit of a point $p\in\mathbb{C}^2$ is said to explode if it is unbounded on some $\mathbb{D}^*\subset\mathbb{C}$.

Proposition 1.1. *The existence of a dense set of points in $\mathbb{C}^2$ whose complex orbit explodes is a generic property in $\mathbb{C}[z_1, z_2]_m$, $m \geq 3$.*

Proof. Consider the Zariski open of $\mathbb{C}[z_1, z_2]_m$ defined by $W_m = \{H \mid H_m = 0 \text{ defines } m \text{ distinct points in } \mathbb{CP}^1\}$. For any $H \in W_m$ and $p = (0, \alpha) \in Sing\mathcal{F}_{X_H} \cap L_\infty$, as $\partial H_m / \partial z_2(1, \alpha) \neq 0$, (2) is not 0. Since $\mathcal{F}_{X_H}$ is given by the pencil defined by H, and L_∞ is invariant, p can be taken to be dicritical. For each separatrix $\Sigma \neq L_\infty$ through p, $X_{|\mathcal{L}\setminus\{p\}}$ extends to p as a pole of order $k \geq 1$, Theorem 1.1. Thus the norm of X is unbounded on $\mathbb{D}^*$ and L explodes. □

2. On the number of zeroes of a complete polynomial vector field

Proposition 2.1. *Suppose that X has a rational first integral. Then, there exists a polynomial automorphism $\varphi \in Aut[\mathbb{C}^2]$ such that*

(i) *If X is not singular, $\varphi^* X = \frac{\partial}{\partial z_1}$.*
(ii) *If X is singular, $\varphi^* X = m z_1 \frac{\partial}{\partial z_1} + n z_2 \frac{\partial}{\partial z_2}$ where $m, n \in \mathbb{Z}^*$.*

Proof. Let $H = F/G$ be a rational first integral of X. By Stein's factorization, we may assume that the generic fiber of H is connected, i.e., H is a primitive rational first integral. Since X is complete there exists a subset $E \subset \mathbb{C}^2$ of zero transverse logarithmic capacity, which is invariant by the flow of X, and such that the orbits of X on $\mathbb{C}^2 \setminus E$ are all isomorphic either to $\mathbb{C}$ or to $\mathbb{C}^*$ (see [18], [19]). We say that the generic orbit of X is $\mathbb{C}$ or $\mathbb{C}^*$, and also that H is of type $\mathbb{C}$ or $\mathbb{C}^*$.

• Assume that the generic orbit of X is $\mathbb{C}$. Suppose that $\{H = 0\} \simeq \mathbb{C}$, so that according to Abhyankar-Moh-Suzuki's Theorem [16], there exists $\varphi \in Aut[\mathbb{C}^2]$ with $H \circ \varphi(z_1, z_2) = z_2$. Therefore $\varphi^* X = \frac{\partial}{\partial z_1}$.

• If the generic orbit of X is $\mathbb{C}^*$, following an improvement of a theorem of Saito [17], after a polynomial automorphism Φ, we have that $H \circ \Phi(z_1, z_2) = h \circ Q(z_1, z_2)$, where h is a rational function of degree one and either $Q = (z_1^m (z_1^l z_2 + p(z_1))^n)$, $m, n \in \mathbb{Z}^*$, $l \in \mathbb{N}^+$, $p(z_1)$ is a polynomial of degree $\leq l - 1$ with $p(0) \neq 0$, or $Q(z_1, z_2) = z_1^m z_2^n$. In the first case, removing the one-dimensional singular locus of dQ, $i_{\Phi^* X}(dz_1 \wedge dz_2)$ equals

$$\frac{\lambda (z_1^{-2m})^a ((z_1^l z_2 + p(z_1))^{-2n})^b}{z_1^{m-1} (z_1^l z_2 + p(z_1))^{n-1} h'(Q)} \, dQ, \text{ where } \begin{cases} a = 0 & \text{if } m > 0, \\ a = 1 & \text{if } m < 0, \\ b = 0 & \text{if } n > 0, \\ b = 1 & \text{if } n < 0, \end{cases}$$

and $\lambda \in \mathbb{C}^*$. Thus $\Phi^* X$ equals

$$A z_1^{l+1} \frac{\partial}{\partial z_1} + (B z_1^l z_2 + C p(z_1) + D z_1 p'(z_1)) \frac{\partial}{\partial z_2}, \quad A \in \mathbb{C}^* \text{ and } B, C, D \in \mathbb{C}.$$

Let us consider the trajectory $L = \{z_1^l z_2 + p(z_1) = 0\}$, and let Σ be the branch of $\bar{L}$ at $(0 : 1 : 0)$, parametrized by $\gamma(t) = (t, -\tilde{p}(1, t))$, where $\tilde{p}(x, z)$ is the homogenization of p. Then, $ind_p(\mathcal{F}_{\Phi^* X}, \Sigma) = 1 < 1 + l$, and by Lemma 1.2 $X_{|L}$ is not complete.

If $Q = z_1^m z_2^n$, taking $\varphi = \frac{1}{\sqrt{\lambda}} \Phi$, then $\varphi^* X = m z_1 \frac{\partial}{\partial z_1} + n z_2 \frac{\partial}{\partial z_2}$. □

Proposition 2.2. *Let p be a nondicritical zero of X (polynomial but not necessarily complete). If Γ is an irreducible algebraic invariant curve through p such that $X_{|\Gamma\setminus\{p\}}$ is complete, then there exists $\Phi \in Aut[\mathbb{C}^2]$ such that $\Phi(\Gamma)$ is a line.*

Proof. As $X_{|\Gamma\setminus\{p\}}$ is complete and $\mathbb{C}^2$ is Stein, $\Gamma\setminus\{p\}\simeq\mathbb{C}^*$. Consider the unique branch of Γ at p and its parametrization $\gamma:\mathbb{D}\to\Gamma$. The extension of γ^*X to 0 has a zero of order 1. Then DX_p is not zero, and we denote by λ and μ its eigenvalues.

• If $\lambda=\mu=0$, after a linear change of coordinates

$$DX_p=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Suppose that Γ is singular at p. There exists $\psi\in Aut[\mathbb{C}^2]$ such that $\psi(\Gamma)=\{{z_1}^k-a{z_2}^l=0,(k,l)=1,a\in\mathbb{C}^*\}$, [21]. Then $\gamma(t)=(\varepsilon t^l,t^k)$, with $\varepsilon^k=a$, and $D(\psi_*X)_p=D\psi_p\cdot DX_p\cdot D\psi_p^{-1}$, so we have that $\gamma^*(\psi_*X)=\Delta(t)\frac{\partial}{\partial t}$, where $\Delta(t)$ equals

$$\frac{\alpha b(d\varepsilon t^l-bt^k)+P(\varepsilon t^l,t^k)}{\varepsilon l t^{l-1}}=\frac{\alpha d(d\varepsilon t^l-bt^k)+Q(\varepsilon t^l,t^k)}{kt^{k-1}},\tag{3}$$

where $a,b,c,d\in\mathbb{C}$, $\alpha=(ad-bc)^{-1}\in\mathbb{C}^*$, and $P,Q\in\mathbb{C}[z_1,z_2]$ have order ≥ 2 at p. If $bd\neq 0$ (the case $bd=0$ is similar), as $\gamma^*(\psi_*X)$ has a zero of order 1, the orders of the numerators in (3) are l and k, respectively. It should be $k>l$; otherwise, the term $-\alpha b^2t^k$ is cancelled with one of the terms of $P(\varepsilon t^l,t^k)$, and thus $k=jl$ with $j\geq 2$. But $k>l$ implies that $\alpha d^2\varepsilon t^l$ is cancelled with one of the terms of $Q(\varepsilon t^l,t^k)$, and hence $l=jk$ with $j\geq 2$, a contradiction.

• If $\lambda/\mu\in\mathbb{Q}^+$, as p is nondicritical, p is of Poincaré-Dulac type [3], and hence Γ is smooth at p.

• If $\lambda/\mu\notin\mathbb{Q}^+$, or $\lambda\neq 0$ and $\mu=0$, according to [11, pp. 518–522] Γ is smooth at p.

By [16], there exists $\Phi\in Aut[\mathbb{C}^2]$ such that $\Phi(\Gamma)$ is a line. □

Let Σ be a separatrix through a zero p of X. Consider the orbit L defined extending $\Sigma\setminus\{p\}$. As $\mathbb{C}^2$ is Stein, $L\simeq\mathbb{C}^*$. Thus L has two planar isolated ends; one defined by $\Sigma\setminus\{p\}$ and the other by $L\setminus\Sigma$. If the end defined by $L\setminus\Sigma$ is algebraic (transcendental), one says that Σ is *algebraic (transcendental) at infinity* (see definitions in [4]).

Proposition 2.3. *Either L is defined by the (unique) local branch at p of an algebraic curve $\Gamma\subset\mathbb{C}^2$, such that $\Gamma\setminus\{p\}\simeq L$, or $L\setminus\Sigma$ defines a planar isolated end which is properly imbedded and transcendental.*

Proof. Take $x\in L$ and let $j:\mathbb{C}\to\mathbb{C}^2$ be the map $j(t)=\varphi(t,x)$, where φ is the flow of X. We know that its analytic closure $\overline{L}\subset\mathbb{C}^2$ is of pure dimension 1, [19]. Then L is properly embedded in $\mathbb{C}^2$ (j is proper). If $L\setminus\Sigma$ is not transcendental, then L defines a separatrix through the point $r=lim(L\setminus\Sigma)\in Sing(\mathcal{F}_X)\cap L_\infty$. Therefore $\overline{L}\cup\{r\}\simeq\mathbb{CP}^1$ is an algebraic curve by Chow's Theorem. □

Theorem 2.1. *X has at most one zero in $\mathbb{C}^2$.*

Proof. Suppose that $p_1\neq p_2$ are zeroes of X. By [6], there exists a separatrix Σ_i through p_i, $i=1,2$. First assume that each Σ_i is algebraic at infinity through a nondicritical p_i. Let $\Phi\in Aut[\mathbb{C}^2]$, given in Proposition 2.2, such that $\Phi(\Gamma_1)$ is a line $L_{\Phi(p_1)}$ through $\Phi(p_1)$. Let $\overline{C_2}$ be the closure of $C_2:=\Phi(\Gamma_2)$ in $\mathbb{CP}^2$. Thus $\overline{L_{\Phi(p_1)}}\cap\overline{C_2}=\{r\}\in L_\infty$; otherwise if $\alpha:Z_2\to\overline{C_2}$ is the resolution of $\overline{C_2}$, α^*X extends to Z_2 with at least three zeroes, which is a contradiction. Analogously, $L_\infty\cap\overline{C_2}=\{r\}$. As $\overline{L_{\Phi(p_1)}}$ and L_∞ just intersect $\overline{C_2}$ at r, $\overline{C_2}$ has to be a line as it cannot have two branches at r. Suppose that $L_{\Phi(p_1)}=\{z_1=a\}$ and $L_{\Phi(p_2)}:=$

$C_2 = \{z_1 = b\}$. The orbit of $(z_1^0, z_2^0) \in \mathbb{C}^2$ with $a \neq z_1^0 \neq b$ is defined by the image of the entire map $\varphi_{(z_1^0,z_2^0)}(t) = \varphi(t, z_1^0, z_2^0) = (z_1(t), z_2(t))$, where φ is the flow of $\Phi_* X$. Since $z_1(\mathbb{C}) \subset \mathbb{C} \setminus \{a, b\}$, by Picard's Theorem $z_1(t) \equiv k \in \mathbb{C}$, and thus $\varphi_{(z_1^0,z_2^0)}(\mathbb{C})$ is contained in a line parallel to both $L_{\Phi(p_1)}$ and $L_{\Phi(p_2)}$, and hence $\Phi_* X = \frac{\partial}{\partial z_2}$, a contradiction.

Observe that if p_i is dicritical, Σ_i can be taken to be transcendental at infinity. Otherwise Darboux's Theorem and Proposition 2.1 imply that X has at most one zero. Thus it only remains to analyze the case when Σ_i is transcendental at infinity. Now, we take from [4] the notion of P-completeness, that will be used in what follows. Let $P : \mathbb{C}^2 \to \mathbb{C}$ be a nonconstant polynomial. $\mathcal{F}_X$ is P-complete if there exists a finite set $Q \subset \mathbb{C}$ such that, for all $t \notin Q$, $P^{-1}(t)$ is transverse to $\mathcal{F}_X$ and there is a neighbourhood U_t of t in $\mathbb{C}$ such that $P_{|P^{-1}(U_t)}$ is a fibration and $\mathcal{F}_{X|P^{-1}(U_t)}$ defines a local trivialization on it. Thus, if one Σ_i is transcendental at infinity, it follows from [4] that there is a nonconstant (primitive) polynomial $P : \mathbb{C}^2 \to \mathbb{C}$ of type $\mathbb{C}$ or $\mathbb{C}^*$ such that $\mathcal{F}_X$ is P-complete. The set of points where $\mathcal{F}_X$ is not transverse to P is an algebraic curve $S \subset P^{-1}(Q)$, so $p_i \in S$. If P is of type $\mathbb{C}$, since by [16] there is $\varphi \in Aut[\mathbb{C}^2]$ such that $P \circ \varphi(z_1, z_2) = z_1$, one sees as above that $p_i \in \{z_1 = \lambda\}$, for $i = 1, 2$, again a contradiction.

On the other hand, if P is of type $\mathbb{C}^*$ by [17], as noted in the proof of Proposition 2.1, after a polynomial automorphism φ, $P \circ \varphi$ can be easily written and since $(P \circ \varphi)^{-1}(\lambda) \simeq \mathbb{C}^*$ for all $\lambda \neq 0$, and X is complete on each component of S, one has that $p_i \in (P \circ \varphi)^{-1}(0)$. Therefore $S \cap (P \circ \varphi)^{-1}(0) = \{z_1 = 0\}$ or $\{z_1 z_2 = 0\}$, but in both cases one has two zeroes on an invariant line of X, a contradiction. $\square$

Theorem 2.2. *Let p be a zero of X which is not of Poincaré-Dulac type, and let $\mathcal{S}$ be the set of separatrices through p that are algebraic at infinity. Up to polynomial automorphism,*

(1) *when p is dicritical and all the separatrices through it belong to $\mathcal{S}$: $X = m z_1 \frac{\partial}{\partial z_1} + n z_2 \frac{\partial}{\partial z_2}$ where $m, n \in \mathbb{Z}^*$, and $mn < 0$;*
(2) *when p is nondicritical, but $\sharp\mathcal{S} \geq 2$: $X = z_1(\lambda + q f(z_1^p z_2^q)) \frac{\partial}{\partial z_1} + z_2(\mu + p f(z_1^p z_2^q)) \frac{\partial}{\partial z_2}$, where $f \in \mathbb{C}[z]$, $p, q \in \mathbb{N}$ and $\lambda\mu \in \mathbb{C}^*$.*

If there is at least one separatrix $\Sigma \notin \mathcal{S}$, then $\sharp\mathcal{S} \geq 1$, and either

(3) *$X = \lambda z_1 \frac{\partial}{\partial z_1} + (a(z_1) + b(z_1) z_2) \frac{\partial}{\partial z_2}$, with $a, b \in \mathbb{C}[z_1]$ and $b(0)\lambda \in \mathbb{C}^*$, or*
(4) *$\mathcal{F}_X$ is P-complete for a polynomial $P = (z_1^m (z_1^l z_2 + p(z_1))^n)$, where $m, n, l \in \mathbb{N}^+$, $p \in \mathbb{C}[z_1]$ of degree $\leq l - 1$ with $p(0) \neq 0$, or $P = z_1^m z_2^n$.*

Proof. By Theorem 2.1, p is the unique zero of X in $\mathbb{C}^2$. Suppose that $p = (0, 0)$. Since the restriction of X to any open neighbourhood of p is semicomplete, [13], one has that $\lambda\mu \neq 0$, [14].

We can distinguish two cases. Suppose that p is dicritical. Then, if there is a rational first integral, one has by Proposition 2.1 that X can be written as in (1). If there is no rational first integral, there is a separatrix through p, Σ, which is transcendental at infinity, and according to [4], the foliation $\mathcal{F}_X$ is P-complete, with P of type $\mathbb{C}$ or $\mathbb{C}^*$. As noted in the proof of Theorem 2.1, if P is of type $\mathbb{C}$, then $P = z_1$ and it follows that X is as in (3), while if it is of type $\mathbb{C}^*$, then P can be written so that it reads as in (4).

Assume now that p is nondicritical. As p is not of Poincaré-Dulac type, then there are at least two separatrices through p, [3] and [11]. If at least two of them are algebraic at infinity, according to Propositions 2.2 and 2.3 they are defined by the smooth algebraic curves $\Gamma_1 = \{P_1 = 0\}$ and $\Gamma_2 = \{P_2 = 0\}$. Consider the simply connected algebraic curve $\Gamma_1 \cup \Gamma_2 = \{P_1 P_2 = 0\}$. After a polynomial automorphism Φ, $\Phi(\Gamma_1 \cup \Gamma_2) = \{z_1^k z_2^l = 0\}$, [21]. Moreover, since $\Phi_* X$ is complete on $\mathbb{C}^2 \setminus (\{z_1 = 0\} \cup \{z_2 = 0\}) \simeq (\mathbb{C}^*)^2$, the classification of such vector fields, [2], shows that $\Phi_* X$ takes the form (2) of the statement. The nonexistence of two separatrices through p algebraic at infinity implies the existence of at least one Σ which is transcendental at infinity, and by [4]; X can be expressed as it is pointed out in (3) or (4).

3. Completeness and the Jacobian Conjecture

Jacobian Conjecture. *If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that $\det(JF) \in \mathbb{C}^*$, then F is invertible, that is, F has an inverse which is also a polynomial map.*

In fact, from a theorem due to Bialynicki-Birula and Rosenlicht, [20], if F is injective it is surjective, and the inverse is a polynomial map. Thus the Jacobian Conjecture is equivalent to: "if $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that $\det(JF) \in \mathbb{C}^*$, then F is injective". The conjecture is true for $n = 1$, and is an open problem for $n \geq 2$.

Following Nousiainen and Sweedler, [20], we can associate to $F = (F_1, \dots, F_n)$ n polynomial vector fields on $\mathbb{C}^n$, $\frac{\partial}{\partial F_1}, \dots \frac{\partial}{\partial F_n}$, defined by

$$\left(\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}\right) := \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)(JF)^{-1},$$

with the following properties:

(1) They are $\mathbb{C}$-linearly independent on $\mathbb{C}^n$.
(2) $\mathcal{L}_{\frac{\partial}{\partial F_i}} F_j = DF_j\left(\frac{\partial}{\partial F_i}\right) = \delta_{ij}$ and $\left[\frac{\partial}{\partial F_i}, \frac{\partial}{\partial F_j}\right] = 0$, with $1 \leq i, j \leq n$.

Therefore, we obtain for each $i = 1, \dots, n$ a nonsingular algebraic foliation by curves in $\mathbb{C}^n$ defined by the vector field $\frac{\partial}{\partial F_i}$ whose leaves are given by the intersection of the level sets of F_j, $j \neq i$.

Theorem 3.1. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $\det(JF) \in \mathbb{C}^*$. Then F is injective if and only if the polynomial vector fields $\frac{\partial}{\partial F_i}$, $i = 1, \dots, n$, are complete.*

Proof. Suppose that F is injective. If $F_i(z_1, \dots, z_n) = w_i$, $i = 1, \dots, n$, as the vector fields $\frac{\partial}{\partial w_i}$ are complete, $\frac{\partial}{\partial F_i} = F^* \frac{\partial}{\partial w_i}$ are also complete.

Conversely, if $\frac{\partial}{\partial F_i}$, $i = 1, \dots, n$, are complete, and there are two different points $p, q \in \mathbb{C}^n$ such that $F(p) = F(q) = \alpha = (\alpha_1, \dots, \alpha_n)$, there are n leaves $L_i = \bigcap_{j \neq i} \{F_j = \alpha_j\}$, one of each foliation induced by $\frac{\partial}{\partial F_i}$, having at least two different points p and q of intersection.

But each leaf L_i, $i = 1, \dots, n$, is equipped with a holomorphic 1–form $DF_i|_{L_i}$ such that $DF_i|_{L_i}(\frac{\partial}{\partial F_i}) \equiv 1$. Following [13], since $\frac{\partial}{\partial F_i}$ is complete the 1–form $DF_i|_{L_i}$ is defined by the "différentielle du temps", which is locally given by its flow.

Let us fix $i_0 \in \{1, \dots, n\}$, and an injective smooth path $c_{i_0} : [0, 1] \to L_{i_0}$ from p to q. The integral of $DF_{i_0}|_{L_{i_0}}$ along c_{i_0} has to be nonzero [13], but $\int_{c_{i_0}} DF_{i_0}|_{L_{i_0}} = F_{i_0}(q) - F_{i_0}(p) = 0$. Thus F is injective. $\square$

After our work was completed, we noticed that Theorem 3.1 was proved by Meisters and Olech in [12] for the real case in another context. We denote by $\mathcal{D}$ the $\mathbb{C}[z_1,\dots,z_n]$–module of all $\mathbb{C}$–derivations of $\mathbb{C}[z_1,\dots,z_n]$. It is well known that $\mathcal{D}$ is free and of rank n. A basis is said to be commutative when $[X_i, X_j] = 0$, $0 \le i, j \le n$. If each X_i is complete, we will say that it is complete.

Proposition 3.1. *A commutative basis $(X_1,\dots,X_n)$ of $\mathcal{D}$ is complete if and only if there exists a polynomial automorphism F of $\mathbb{C}^n$ such that $F_* X_i = \frac{\partial}{\partial z_i}$, $i = 1,\dots,n$.*

Proof. Suppose first that $(X_1,\dots,X_n)$ is a complete commutative basis. Then by a result of A. Nowicki [20], there exists a polynomial map $F = (F_1,\dots,F_n)$ with $\det(JF) \in \mathbb{C}^*$ such that $X_i = \frac{\partial}{\partial F_i}$. Thus by Theorem 3.1, F is a polynomial automorphism such that $X_i = (F^{-1})_* \frac{\partial}{\partial z_i}$.

Now, suppose that there is a polynomial automorphism F of $\mathbb{C}^n$ such that $F_* X_i = \frac{\partial}{\partial z_i}$. Then, the X_i are complete and moreover $[X_i, X_j] = (F^{-1})_*[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}] = 0$, thus proving the converse. $\square$

Corollary 3.1. *The Jacobian Conjecture holds if and only if every commutative basis of $\mathcal{D}$ is complete.*

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