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A REPRODUCING KERNEL SPACE MODEL FOR $\mathbf{N}_\kappa$-FUNCTIONS

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Abstract. A new model for generalized Nevanlinna functions $Q \in \mathbf{N}_\kappa$ will be presented. It involves Bezoutians and companion operators associated with certain polynomials determined by the generalized zeros and poles of Q. The model is obtained by coupling two operator models and expressed by means of abstract boundary mappings and the corresponding Weyl functions.

1. Introduction

A function Q, locally meromorphic on $\mathbb{C} \setminus \mathbb{R}$, belongs to the class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{Z}_+$) of generalized Nevanlinna functions if on its domain of holomorphy $\rho(Q)$ it admits the symmetry property $Q(\bar{z}) = \overline{Q(z)}$, and is such that the Nevanlinna kernel

$$\mathsf{N}_Q(z,w) = \frac{Q(z) - Q(\bar{w})}{z - \bar{w}},\ z \neq \bar{w}, \quad \mathsf{N}_Q(z,\bar{z}) = Q'(z), \tag{1.1}$$

$z, w \in \rho(Q)$, has κ negative squares; cf. [16]. As is known [17], every generalized Nevanlinna function $Q \in \mathbf{N}_\kappa$ holomorphic at $i \in \rho(Q)$ admits the representation

$$Q(z) = s + z[v,v] + (z^2+1)[(A-z)^{-1}v, v], \quad s = \bar{s}, \tag{1.2}$$

with a selfadjoint operator (or a linear relation) A acting in a Hilbert space $(\mathfrak{H}, [\cdot,\cdot])$ and a generating vector $v \in \mathfrak{H}$, and then Q is called the Q-function of a symmetric restriction S of A. Models for Pontryagin space selfadjoint operators has been constructed in [16] and [15]. In particular, the model in [15], which is an extension of the one given in [14] for the case $\kappa = 1$, was based on the use of certain distributions to define a Pontryagin space Π_κ, as a finite dimensional extension of a Hilbert space $L_2(\sigma)$, and a selfadjoint multiplication operator in Π_κ.

In the present paper a new and explicit model for the functions $Q \in \mathbf{N}_\kappa$ will be given. This model uses a recent factorization result from [10] which states that every function $Q \in \mathbf{N}_\kappa$ admits a representation

$$Q = r^\sharp Q_0 r, \quad r^\sharp(z) = \overline{r(\bar{z})}, \tag{1.3}$$

where $r = \frac{p}{q}$ is a rational function and Q_0 is a Nevanlinna function ($Q_0 \in \mathbf{N}_0$). The model is constructed as a certain coupling of a Hilbert space selfadjoint operator

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A_0 generated by Q_0 in (1.3) and a selfadjoint operator A_R acting on a finite-dimensional Pontryagin space $\mathfrak{H}_R$. The inner product in $\mathfrak{H}_R$ is defined with the aid of the Bezoutian $B_{p,q}$ associated with the polynomials p, q, and the operator A_R is the orthogonal sum $C_{p^\sharp} \oplus C_p$ of the companion matrices determined by p, $p^\sharp$.

The main tools of the construction are the technique of the reproducing kernel Pontryagin spaces (see e.g. [1], [21]) and the boundary value approach developed in [13], [8], [2], [3]. In particular, some results obtained in [3] for orthogonal couplings of symmetric operators, which are described in terms of abstract boundary conditions, play an important role in the proof of Theorem 3.3. The models constructed in [5], [6] for giving realizations for singular perturbations of selfadjoint operators (cf. e.g. [11]) can be seen as special cases of the present model.

The use of abstract boundary mappings makes it easy to apply the present model, for instance, to spectral problems for differential operators with rationally λ-depending boundary conditions. These problems were shown to be adequate to the eigenvalue problems for some Pontryagin space selfadjoint operators; see [20], [12]. In [7] (see also [6]) a special case of the present model where r is a polynomial has been applied to give realizations for singular perturbations of selfadjoint (differential) operators.

2. Preliminaries

Let $\mathfrak{H}$ be a separable Pontryagin space, let S be a not necessarily densely defined closed symmetric relation in $\mathfrak{H}$ with equal defect numbers $d = d_+(S) = d_-(S)$ $(< \infty)$, and let S^* be the adjoint linear relation of S. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and Γ_j, $j = 0, 1$, are linear mappings from S^* to $\mathcal{H}$, is said to be a boundary triplet for S^*, if the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top : \widehat{f} \to \{\, \Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\,\}$ from S^* into $\mathcal{H} \oplus \mathcal{H}$ is surjective and the abstract Green's identity

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}} \tag{2.1}$$

holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S^*$; cf. [13], [8], [2]. The mapping Γ from S^* onto $\mathcal{H} \oplus \mathcal{H}$ establishes a one-to-one correspondence between the set Ext_S of all closed extensions of S and the set $\widetilde{\mathcal{C}}(\mathcal{H})$ of all closed linear relations in $\mathcal{H}$ via

$$\widetilde{A}_\theta := \Gamma^{-1}\theta = \{\, \widehat{f} \in S^* : \Gamma \widehat{f} \in \theta \,\}, \quad \theta \in \widetilde{\mathcal{C}}(\mathcal{H}). \tag{2.2}$$

It follows from (2.1) that

$$\widetilde{A}_\theta^* = \widetilde{A}_{\theta^*}, \quad \text{for every } \theta \in \widetilde{\mathcal{C}}(\mathcal{H}). \tag{2.3}$$

As usual A_0 and A_1 stand for the selfadjoint extensions $A_j = \ker \Gamma_j$, $j = 0, 1$. Let $\mathfrak{N}_\lambda(S^*) = \ker(S^* - \lambda)$ be a defect subspace of S and let $\widehat{\mathfrak{N}}_\lambda := \{\, \{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathfrak{N}_\lambda(S^*)\,\}$. The γ-field and the Weyl function associated with Π are defined by

$$\gamma(\lambda) = p_1(\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda)^{-1} (\in [\mathcal{H}, \mathfrak{N}_\lambda]), \quad Q(\lambda) = \Gamma_1(\Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda)^{-1} \, (\in [\mathcal{H}]), \tag{2.4}$$

$\lambda \in \rho(A_0) \neq \emptyset$ (see [8], [2]). In (2.4) p_1 denotes the orthogonal projection onto the first component of $\mathcal{H} \oplus \mathcal{H}$. The operator-valued functions γ and Q are holomorphic on $\rho(A_0)$. The set of all points of regular type of S will be denoted by $\widehat{\rho}(S)$. A closed symmetric relation S is said to be simple if

$$\mathfrak{H} = \overline{\mathrm{span}}\,\{\, \gamma(z) : z \in \widehat{\rho}(S) \supset \mathbb{C} \setminus \mathbb{R} \,\}. \tag{2.5}$$

In this case, the number κ of negative squares of the kernel N_Q in (1.1) coincides with the dimension of a maximal negative subspace of $\mathfrak{H}$. The class of Weyl functions Q of S coincides with the class of Q-functions of S in the sense of [16].

Associated with the kernel N_Q is a reproducing kernel Pontryagin space $\mathfrak{H}(Q)$ of analytic vector functions on $\rho(Q)$ (cf. [1], [21]) generated by the vector functions $z \to \mathsf{N}_Q(z,w)h$, $w \in \rho(Q)$, $h \in \mathcal{H}$, and the inner product

$$[\mathsf{N}_Q(\cdot,w)h, \mathsf{N}_Q(\cdot,z)k] = k^*\mathsf{N}_Q(z,w)h, \quad h,k \in \mathcal{H}, \quad z,w \in \rho(Q). \tag{2.6}$$

The characterizing property of $\mathfrak{H}(Q)$ is the equality

$$[F(\cdot), \mathsf{N}_Q(\cdot,w)k] = k^*F(w), \quad f \in \mathfrak{H}(Q), \quad k \in \mathcal{H}, \quad w \in \rho(Q). \tag{2.7}$$

The multiplication operator $S(Q)$ in $\mathfrak{H}(Q)$ defined by

$$S(Q) = \{\, \{f, f'\} \in \mathfrak{H}(Q)^2 : \, f'(w) = wf(w),\, w \in \rho(Q)\,\} \tag{2.8}$$

is symmetric. The next proposition specifies its adjoint $S(Q)^*$ and associates a boundary triplet to $S(Q)^*$, such that Q is the corresponding Weyl function.

Proposition 2.1 ([9], [4])**.** *Let $Q \in \mathbf{N}_\kappa$ be a Weyl function in $\mathcal{H} = \mathbb{C}^d$ of a closed symmetric operator S in a Pontryagin space and let $\mathfrak{H}(Q)$ be the reproducing kernel Pontryagin space associated with the kernel* (1.1)*. Then $S(Q)$ in* (2.8) *is a closed simple symmetric operator in $\mathfrak{H}(Q)$ (which is unitarily equivalent to S if S is simple) and, moreover:*

(i) *the adjoint $S(Q)^*$ is given by*

$$S(Q)^* = \{\, \widehat{F} = \{F, F'\} \in \mathfrak{H}(Q)^2 : \, F'(w) - wF(w) = \xi_1 - Q(w)\xi_0,\, \xi_0, \xi_1 \in \mathcal{H}\,\};$$

(ii) *the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $S(Q)^*$ is given by*

$$\mathcal{H} = \mathbb{C}^d, \quad \Gamma_0\widehat{F} = \xi_0, \quad \Gamma_1\widehat{F} = \xi_1;$$

(iii) *the corresponding Weyl function coincides with Q and the γ-field is given by*

$$\gamma_Q(z) = \mathsf{N}_Q(\cdot, \bar{z}), \quad z \in \rho(Q). \tag{2.9}$$

3. An operator model associated with the class $\mathbf{N}_\kappa$

Let p and q be scalar polynomials with complex coefficients of the form ($p_n \neq 0$)

$$p(\lambda) = p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0, \quad q(\lambda) = q_n\lambda^n + q_{n-1}\lambda^{n-1} + \cdots + q_0. \tag{3.1}$$

Then the Bezoutian $\mathrm{Bez}\,(p,q)$ and the companion matrix C_p are defined by

$$\frac{p(\ell)q(\lambda) - q(\ell)p(\lambda)}{\ell - \lambda} = L\mathrm{Bez}\,(p,q)\Lambda^\top, \quad C_p = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ -\widetilde{p}_0 & -\widetilde{p}_1 & \dots & \dots & -\widetilde{p}_{n-1} \end{pmatrix},$$

with $\Lambda_{(n)} = (I_\mathcal{H}, \lambda I_\mathcal{H}, \dots, \lambda^{n-1}I_\mathcal{H})$, $L_{(n)} = (I_\mathcal{H}, \ell I_\mathcal{H}, \dots, \ell^{n-1}I_\mathcal{H})$, and $\widetilde{p}_i = p_n^{-1}p_i$, $i = 0, 1, \dots, n-1$. The following facts are needed in the sequel (cf. e.g. [19]):

(i) $B_p = \mathrm{Bez}\,(p,1) = [p_{i+j+1}]_{i,j=0}^{n-1}$ ($p_j = 0$ if $j > n$);

(ii) $\mathrm{Bez}\,(p,\lambda^k) = B_pC_p^k$;

(iii) $\mathrm{Bez}\,(p,q) = \sum_{j=0}^n q_j\mathrm{Bez}\,(p,\lambda^j) = B_pq(C_p)$ (the Barnett factorization);

(iv) $B_pC_p = C_p^\top B_p$;
(v) $\sigma(C_p) = \sigma(p)$ (the set of zeros of p).

In particular, (iii) shows that $\operatorname{Bez}(p,q)$ is invertible if and only if $q(C_p)$ is invertible, which holds precisely when $\sigma(C_p)$ does not contain zeros of q, or equivalently, that p and q are relatively prime, i.e., $\sigma(p)\cap\sigma(q)=\emptyset$. The items (iii) and (iv) show that $\operatorname{Bez}(p,q)^\top = \operatorname{Bez}(p,q)$ and $\operatorname{Bez}(p,q)C_p = C_p^\top \operatorname{Bez}(p,q)$. The statement in (v) can be augmented with the corresponding root vectors. For each $\lambda_j\in\sigma(C_p)$ with multiplicity κ_j one associates the set of Vandermonde vectors,

$$V_k(\lambda_j) = \frac{1}{k!}\frac{d^k}{d\lambda^k}\Lambda|_{\lambda=\lambda_j}, \quad k=0,1,\dots,\kappa_j, \tag{3.2}$$

which form a full chain of the root subspace $\ker(C_p-\lambda_j)^{\kappa_j}$. In what follows $(e_j)_{j=1}^n$ stands for the standard basis in $\mathbb{C}^n$. It is easy to check that

$$\begin{aligned} w : = \operatorname{Bez}(p,q)e_n &= \sum_{j=0}^{n-1} p_nq_jB_pC_p^je_n + q_nB_pC_p^ne_n \\ &= (p_nq_0-q_np_0, p_nq_1-q_np_1,\dots,p_nq_{n-1}-q_np_{n-1})^\top. \end{aligned} \tag{3.3}$$

Now introduce the rational functions r, $r^\sharp$, and R by

$$r=\frac{p}{q}, \quad r^\sharp = \frac{p^\sharp}{q^\sharp}, \quad R=\begin{pmatrix} 0 & r\\ r^\sharp & 0\end{pmatrix}, \tag{3.4}$$

where $p^\sharp(z)=\overline{p(\bar z)}$ and $q^\sharp(z)=\overline{q(\bar z)}$. Then R is a matrix function which belongs to the class $\mathbf{N}_\kappa$ with $\kappa=\max\{\deg p,\deg q\}$. The next result gives an explicit form for the reproducing kernel space $\mathfrak{H}(R)$ associated with R.

Proposition 3.1. *Let the polynomials p and q in* (3.1) *be relatively prime and assume that $\deg p\ge\deg q$ and that p is monic. Let w and R be given by* (3.3) *and* (3.4)*, respectively. Then:*

(i) *The reproducing kernel Pontryagin space $\mathfrak{H}(R)$ is isometrically isomorphic to the space $\mathfrak{H}_R=\mathbb{C}^n\oplus\mathbb{C}^n$ equipped with the inner product*

$$[\cdot,\cdot]_{\mathfrak{H}_R} = (\mathcal{B}\,\cdot,\cdot), \quad \mathcal{B}=\begin{pmatrix} 0 & B_{p,q}\\ B_{p,q}^* & 0\end{pmatrix}, \quad B_{p,q}=\operatorname{Bez}(p,q). \tag{3.5}$$

(ii) *The restriction S_R of $\mathcal{C}_p = C_{p^\sharp}\oplus C_p$ to the domain*

$$\operatorname{dom} S_R = \{\, F=f\oplus\widetilde f\in\mathfrak{H}_R:\ \overline{w}^*\widetilde f = w^*f=0\,\} \tag{3.6}$$

is a simple symmetric operator in $\mathfrak{H}_R$, which is unitarily equivalent to $S(R)$.

(iii) *The adjoint S_R^* of S_R takes the form*

$$S_R^* = \left\{\, \widehat F = \{F,\mathcal{C}_pF + (\widetilde c e_n, ce_n)^\top\}:\ F=f\oplus\widetilde f\in\mathfrak{H}_R, \quad c,\widetilde c\in\mathbb{C}\,\right\}. \tag{3.7}$$

(iv) *A boundary triplet $\Pi_R=\{\mathbb{C}^2,\Gamma_0^R,\Gamma_1^R\}$ for S_R^* can be defined by*

$$\Gamma_0^R\widehat F = \begin{pmatrix} \bar q_n\widetilde c + w^*f\\ q_nc+\overline{w}^*\widetilde f\end{pmatrix}, \quad \Gamma_1^R\widehat F=\begin{pmatrix} c\\ \widetilde c\end{pmatrix}, \quad \widehat F\in S_R^*.$$

(v) *The corresponding γ-field is given by $\gamma_R(\lambda)=1/q^\sharp(\lambda)\Lambda^\top\oplus 1/q(\lambda)\Lambda^\top$ and the Weyl function M_R coincides with R in* (3.4)*.*

Proof. The kernel $\mathsf{N}_r(\ell,\lambda)$ can be expressed in terms of $B_{p,q}$ as follows:

$$\mathsf{N}_r(\ell,\lambda)=\frac{1}{q(\ell)}LB_{p,q}\Lambda^*\frac{1}{q(\bar\lambda)}. \tag{3.8}$$

This leads to the following factorization for the kernel $\mathsf{N}_R(\ell,\lambda)$:

$$\mathsf{N}_R(\ell,\lambda)=\begin{pmatrix}\frac{1}{q(\ell)}L & 0\\ 0 & \frac{1}{q^\sharp(\ell)}L\end{pmatrix}\begin{pmatrix}0 & B_{p,q}\\ B_{p,q}^* & 0\end{pmatrix}\begin{pmatrix}\frac{1}{q(\lambda)}\Lambda & 0\\ 0 & \frac{1}{q^\sharp(\lambda)}\Lambda\end{pmatrix}^*. \tag{3.9}$$

Thus, R is a matrix Nevanlinna function with n negative (and n positive) squares.

(i) It follows from the factorization (3.9) that $\mathfrak{H}(R)$ consists of vector functions

$$F(\ell)=\begin{pmatrix}\frac{1}{q(\ell)}L & 0\\ 0 & \frac{1}{q^\sharp(\ell)}L\end{pmatrix}\mathcal{B}F,\quad F=\begin{pmatrix}f\\ \widetilde{f}\end{pmatrix}\in\mathfrak{H}_R. \tag{3.10}$$

Now the identities $[F(\cdot),G(\cdot)]_{\mathfrak{H}(R)}=G^*\mathcal{B}F=[F,G]_{\mathfrak{H}_R}$ show that the mapping $F\to F(\cdot)$ determines an isometric isomorphism between $\mathfrak{H}_R$ and $\mathfrak{H}(R)$.

(ii) By the properties given above $(\mathcal{B}\mathcal{C}_p)^*=\mathcal{B}\mathcal{C}_p$, i.e., $\mathcal{C}_p$ is selfadjoint in $\mathfrak{H}_R$. It is clear from (3.6) that S_R is a closed symmetric operator in $\mathfrak{H}_R$ with defect numbers $(2,2)$. For every $K=k\oplus\widetilde{k}\in\mathfrak{H}_R$ the following equalities are easily checked:

$$\left\{\begin{array}{lcl}\lambda\Lambda B_p\widetilde{k} &=& \Lambda B_pC_p\widetilde{k}+p(\lambda)\widetilde{k}_1,\\ \lambda\Lambda B_p^*k &=& \Lambda B_p^*C_{p^\sharp}k+p^\sharp(\lambda)k_1.\end{array}\right. \tag{3.11}$$

Moreover,

$$(q(C_p)\widetilde{f})_1=q_n(-p_0\widetilde{f}_1-\cdots-p_{n-1}\widetilde{f}_n)+q_{n-1}\widetilde{f}_n+\cdots+q_0\widetilde{f}_1=\overline{w}^*\widetilde{f}$$

and similarly $(q^\sharp(C_{p^\sharp})f)_1=w^*f$. Therefore, applying the identities (3.11) with $\widetilde{k}=q(C_p)\widetilde{f}$, $k=q^\sharp(C_{p^\sharp})f$ and taking into account the Barnett factorization, one arrives at the following decomposition for the function $\lambda F(\lambda)$:

$$\lambda F(\lambda)=\begin{pmatrix}\frac{1}{q(\lambda)}\Lambda & 0\\ 0 & \frac{1}{q^\sharp(\lambda)}\Lambda\end{pmatrix}\mathcal{B}\mathcal{C}_p\begin{pmatrix}f\\ \widetilde{f}\end{pmatrix}+R(\lambda)\begin{pmatrix}w^*f\\ \overline{w}^*\widetilde{f}\end{pmatrix}. \tag{3.12}$$

This implies the unitary equivalence of the operators S_R and $S(R)$.

(iii) Denote by T the set on the right-hand side of (3.7) and assume that $\{F,G\}\in T$. Then the selfadjointness of $\mathcal{C}_p$ in $\mathfrak{H}_R$ shows that for all $\{H,K\}\in S_R$,

$$[G,H]_{\mathfrak{H}_R}-[F,K]_{\mathfrak{H}_R}=\left[\begin{pmatrix}\widetilde{c}e_n\\ ce_n\end{pmatrix},\begin{pmatrix}k\\ \widetilde{k}\end{pmatrix}\right]_{\mathfrak{H}_R}=\left(\mathcal{B}\begin{pmatrix}\widetilde{c}e_n\\ ce_n\end{pmatrix},\begin{pmatrix}k\\ \widetilde{k}\end{pmatrix}\right)=\begin{pmatrix}ck^*w\\ \widetilde{c}\widetilde{k}^*\overline{w}\end{pmatrix}=0;$$

cf. (3.3). Thus, $T\subset S_R^*$ and the equality (3.7) follows by a dimension argument.

(iv) & (v) Since $p_n=1$ one can write $\Lambda w=q(\lambda)-q_np(\lambda)$ and this leads to

$$\frac{1}{q(\lambda)}\Lambda w=1-q_nr(\lambda),\quad \frac{1}{q^\sharp(\lambda)}\Lambda\overline{w}=1-\bar{q}_nr^\sharp(\lambda).$$

Therefore, with $\widehat{F} = \{F, G\} \in S_R^*$ one obtains from (3.3), (3.7), and (3.12)

$$
\begin{aligned}
G(\lambda) - \lambda F(\lambda) &= \begin{pmatrix} \frac{1}{q(\lambda)}\Lambda & 0 \\ 0 & \frac{1}{q^\sharp(\lambda)}\Lambda \end{pmatrix} \mathcal{B}(G - \mathcal{C}_p F) - R(\lambda) \begin{pmatrix} w^* f \\ \overline{w}^* \widetilde{f} \end{pmatrix} \\
&= \begin{pmatrix} \frac{c}{q(\lambda)}\Lambda w \\ \frac{\widetilde{c}}{q^\sharp(\lambda)}\Lambda \overline{w} \end{pmatrix} - R(\lambda) \begin{pmatrix} w^* f \\ \overline{w}^* \widetilde{f} \end{pmatrix} = \begin{pmatrix} c \\ \widetilde{c} \end{pmatrix} - R(\lambda) \begin{pmatrix} \bar{q}_n \widetilde{c} + w^* f \\ q_n c + \overline{w}^* \widetilde{f} \end{pmatrix}.
\end{aligned}
$$

Now, it follows from Proposition 2.1 that Γ_0^R and Γ_1^R as defined in (iv) can be taken to be the boundary operators for S_R^* and that R is the corresponding Weyl function. The form of the γ-field follows from (2.9), (3.9), and (3.10). □

Corollary 3.2. *Let the assumptions be as in Proposition* 3.1. *Then:*

(i) *The selfadjoint extension $A_1^R = \ker \Gamma_1^R$ of S_R is an operator which coincides with $\mathcal{C}_p = C_{p^\sharp} \oplus C_p$.*

(ii) *The selfadjoint extension $A_0^R = \ker \Gamma_0^R$ of S_R is an operator if and only if $\deg p = \deg q$, in which case it coincides with $\mathcal{C}_q = C_{q^\sharp} \oplus C_q$.*

Proof. The statement (i) is obtained by taking $c = \widetilde{c} = 0$ in part (iii) of Proposition 3.1. To see (ii) first assume that $\deg q < \deg p$. Then $q_n = 0$ and it follows from the formulas for S_R^* and Γ_0^R in Proposition 3.1 that $\operatorname{mul} A_0^R$ is nontrivial. Next assume that $\deg q = \deg p$. Then $q_n \neq 0$ and $\widehat{F} \in A_0^R$ implies that

$$
c = -\frac{1}{q_n}\, \overline{w}^* \widetilde{f}, \quad \widetilde{c} = -\frac{1}{\bar{q}_n}\, w^* f. \tag{3.13}
$$

Observe that $w/q_n = (C_p^\top - C_q^\top) e_n$. Substituting this and (3.13) into (3.7) yields

$$
\begin{aligned}
&C_{p^\sharp} f - e_n \frac{1}{\bar{q}_n} w^* f = C_{p^\sharp} f - e_n e_n^\top (C_{p^\sharp} - C_{q^\sharp}) f = C_{p^\sharp} f - (C_{p^\sharp} - C_{q^\sharp}) f = C_{q^\sharp} f, \\
&C_p \widetilde{f} - e_n \frac{1}{q_n} \overline{w}^* \widetilde{f} = C_p \widetilde{f} - e_n e_n^\top (C_p - C_q) \widetilde{f} = C_p \widetilde{f} - (C_p - C_q) \widetilde{f} = C_q \widetilde{f},
\end{aligned}
$$

which proves (ii). □

The construction of the model for $\mathbf{N}_\kappa$-functions is based on the following theorem. An underlying idea here arises from some results on intermediate extensions that were proved in [3]; cf. also [5] for the case of Pontryagin spaces. Given two symmetric operators S_1 and S_2 with the Weyl functions Q_1 and $Q = (Q_{ij})_{i,j=1}^2$, Q decomposed according to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, one produces two intermediate extensions whose Weyl functions are of the form

$$
\widetilde{Q}(\lambda) = Q_1(\lambda)\pi_1 + Q(\lambda) = \begin{pmatrix} Q_1(\lambda) + Q_{11}(\lambda) & Q_{12}(\lambda) \\ Q_{21}(\lambda) & Q_{22}(\lambda) \end{pmatrix}, \tag{3.14}
$$

$$
\widetilde{Q}_2(\lambda) = (\pi_2 Q(\lambda)^{-1} \restriction \mathcal{H}_2)^{-1} = Q_{22}(\lambda) - Q_{21}(\lambda) Q_{11}^{-1}(\lambda) Q_{12}(\lambda), \tag{3.15}
$$

where π_j stands for the orthogonal projection onto $\mathcal{H}_j$, $j = 1, 2$, and then combines these two transforms by applying (3.15) to (3.14). All of this can be shortly expressed by using abstract boundary conditions. The procedure is applied here to a scalar function $Q_1 = -M_0^{-1}$ and the matrix function $Q = R$ in Proposition 3.1.

Theorem 3.3. *Let the polynomials p and q in* (3.1) *be relatively prime, let r and R be given by* (3.4)*, and let S_R be the symmetric operator in $\mathfrak{H}_R$ as defined in Proposition 3.1 with the boundary triplet $\Pi_R = \{\mathbb{C}^2, \Gamma_0^R, \Gamma_1^R\}$. Let S_0 be a symmetric operator in the Hilbert space $\mathfrak{H}_0 = \mathfrak{H}(M_0)$ corresponding to the Weyl function M_0 of the boundary triplet $\Pi_0 = \{\mathbb{C}, \Gamma_0^0, \Gamma_1^0\}$ of S_0^*, and let w be given by* (3.3)*. Denote $\widehat{F} = \{F, F'\}$ with $F = f \oplus \widetilde{f} \in \mathfrak{H}_R$ and let $\widehat{f}_0 = \{f_0, f_0'\} \in S_0^*$. Then:*

(i) *The linear relation*

$$S = \{\, \{f_0 \oplus F, f_0' \oplus (\mathcal{C}_p F + (0, \Gamma_0^0 \widehat{f}_0 e_n)^\top)\} : \overline{w}^* \widetilde{f} = -q_n \Gamma_0^0 \widehat{f}_0,\ w^* f = \Gamma_1^0 \widehat{f}_0 \,\}$$

is closed and symmetric in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_R$.

(ii) *The adjoint S^* of S is given by*

$$S^* = \{\, \{f_0 \oplus F, f_0' \oplus (\mathcal{C}_p F + (\widetilde{c} e_n, \Gamma_0^0 \widehat{f}_0 e_n)^\top)\} : \bar{q}_n \widetilde{c} + w^* f = \Gamma_1^0 \widehat{f}_0,\ \widetilde{c} \in \mathbb{C} \,\}.$$

(iii) *A boundary triplet $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for S^* is determined by*

$$\Gamma_0(\widehat{f}_0 \oplus \widehat{F}) = q_n \Gamma_0^0 \widehat{f}_0 + \overline{w}^* \widetilde{f}, \quad \Gamma_1(\widehat{f}_0 \oplus \widehat{F}) = \widetilde{c}, \quad \widehat{f}_0 \oplus \widehat{F} \in S^*.$$

(iv) *The corresponding γ-field and the Weyl function are given by*

$$\gamma(\lambda) = \gamma_0(\lambda) r(\lambda) \oplus \left(\frac{r(\lambda) M_0(\lambda)}{q^\sharp(\lambda)} \Lambda^\top \dotplus \frac{1}{q(\lambda)} \Lambda^\top \right), \tag{3.16}$$

$$M(\lambda) = r^\sharp(\lambda) M_0(\lambda) r(\lambda). \tag{3.17}$$

Proof. Step 1. Define an intermediate extension $\widetilde{S}$ of $S_0 \oplus S_R$ by

$$\widetilde{S} = \{\, \widehat{f}_0 \oplus \widehat{F} \in S_0^* \oplus S_R^* : \Gamma_1^0 \widehat{f}_0 = \Gamma_0^R \widehat{F} = (\Gamma_0^0 \widehat{f}_0) e_1 - \Gamma_1^R \widehat{F} = 0 \,\}, \tag{3.18}$$

where $e_1 = (1,0)^\top \in \mathbb{C}^2$. Then $\widetilde{S}$ is closed and symmetric. Using (2.3) it can be seen that the adjoint $\widetilde{S}^*$ of $\widetilde{S}$ is of the form (cf. also [5, Proposition 2.3])

$$\widetilde{S}^* = \{\, \widehat{f}_0 \oplus \widehat{F} \in S_0^* \oplus S_R^* : \Gamma_1^0 \widehat{f}_0 - \pi_1 \Gamma_0^R \widehat{F} = 0 \,\}, \tag{3.19}$$

where $\pi_1 = e_1 e_1^*$. Moreover, one can take $\widetilde{\Pi} = \{\mathcal{H}_0 \oplus \mathcal{H}_0, \Gamma_0^R, (-\Gamma_0^0) e_1 + \Gamma_1^R\}$ to be a boundary triplet for $\widetilde{S}^*$. The corresponding Weyl function is given by

$$\widetilde{M} = \begin{pmatrix} -M_0^{-1} & r \\ r^\sharp & 0 \end{pmatrix}. \tag{3.20}$$

Step 2. Next define a closed symmetric extension S of $\widetilde{S}$ in (3.18) via

$$\begin{aligned} S &= \{\, \widehat{f}_0 \oplus \widehat{F} \in \widetilde{S}^* : \pi_2 \Gamma_0^R \widehat{F} = (\Gamma_0^0 \widehat{f}_0) e_1 - \Gamma_1^R \widehat{F} = 0 \,\} \\ &= \{\, \widehat{f}_0 \oplus \widehat{F} \in S_0^* \oplus S_R^* : \Gamma_1^0 \widehat{f}_0 = \bar{q}_n \widetilde{c} + w^* f,\ \Gamma_0^0 \widehat{f}_0 = c,\ q_n c + \overline{w}^* \widetilde{f} = \widetilde{c} = 0 \,\}, \end{aligned}$$

where $\pi_2 = I - \pi_1$. In view of (3.7) S can be rewritten as in (i). Moreover,

$$\begin{aligned} S^* &= \{\, \widehat{f}_0 \oplus \widehat{F} \in \widetilde{S}^* : \Gamma_0^0 \widehat{f}_0 - \pi_1 \Gamma_1^R \widehat{F} = 0 \,\} \\ &= \{\, \widehat{f}_0 \oplus \widehat{F} \in S_0^* \oplus S_R^* : \Gamma_1^0 \widehat{f}_0 = \bar{q}_n \widetilde{c} + w^* f,\ \Gamma_0^0 \widehat{f}_0 = c \,\}, \end{aligned} \tag{3.21}$$

which leads to (ii). A boundary triplet for S^* is obtained by letting

$$\Gamma_0 \widehat{f} = \pi_2 \Gamma_0^R \widehat{F} = q_n \Gamma_0^0 \widehat{f}_0 + \overline{w}^* \widetilde{f}, \quad \Gamma_1 \widehat{f} = \pi_2((-\Gamma_0^0 \widehat{f}_0) e_1 + \Gamma_1^R \widehat{F}) = \pi_2 \Gamma_1^R \widehat{F} = \widetilde{c},$$

which gives (iii) (cf. [5, Proposition 2.2]). Finally, to prove (iv) observe that the defect subspace $\mathfrak{N}_\lambda(S^*)$ consists of vectors

$$f_0 \oplus F = \gamma_0(\lambda)\xi \oplus \begin{pmatrix} \dfrac{1}{q^\sharp(\lambda)}\Lambda^\top & 0 \\ 0 & \dfrac{1}{q(\lambda)}\Lambda^\top \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad \xi, h_1, h_2 \in \mathbb{C},$$

such that $\widehat{f}_{0,\lambda} = \{f_0, \lambda f_0\} \in S_0^*$ and $\widehat{F}_\lambda = \{F, \lambda F\} \in S_R^*$ satisfy the equalities

$$\Gamma_1^0 \widehat{f}_{0,\lambda} - \pi_1 \Gamma_0^R \widehat{F}_\lambda = \Gamma_0^0 \widehat{f}_{0,\lambda} - \pi_1 \Gamma_1^R \widehat{F}_\lambda = 0; \tag{3.22}$$

cf. (3.19), (3.21). One can rewrite (3.22) in the form

$$\xi - r(\lambda)h_2 = M_0(\lambda)\xi - h_1 = 0.$$

Therefore, $\mathfrak{N}_\lambda(S^*)$ is spanned by the vectors

$$\gamma(\lambda)h_2 = \gamma_0(\lambda)r(\lambda)h_2 \oplus \begin{pmatrix} \dfrac{1}{q^\sharp(\lambda)}\Lambda^\top & 0 \\ 0 & \dfrac{1}{q(\lambda)}\Lambda^\top \end{pmatrix} \begin{pmatrix} M_0(\lambda)r(\lambda)h_2 \\ h_2 \end{pmatrix}, \quad h_2 \in \mathbb{C},$$

and this gives (3.16). Similarly, (3.17) follows from $\Gamma_1(\widehat{f}_{0,\lambda} \oplus \widehat{F}_\lambda) = \pi_2 R(\lambda) h = r^\sharp(\lambda)M_0(\lambda)r(\lambda)h_2$. Notice that the same expression for M is also obtained by applying the transform (3.15) to (3.20). □

To explain the importance of Theorem 3.3 let M be a generalized Nevanlinna function in $\mathbf{N}_\kappa$. Let α_j be all the poles in $\mathbb{C}^+$ and the generalized poles of nonpositive type in $\mathbb{R}$ of M with multiplicities κ_j, $j = 1, \dots, t$, and let β_i be all the zeros in $\mathbb{C}_+$ and the generalized zeros of nonpositive type in $\mathbb{R}$ of M with multiplicities π_i, $i = 1, \dots, s$; see [18]. Define the polynomials p and q as follows:

$$p(z) = \prod_{j=1}^{s} (z - \beta_j)^{\pi_j}, \quad q(z) = \prod_{i=1}^{t} (z - \alpha_i)^{\kappa_i}. \tag{3.23}$$

The factorization result in [10] when applied to $M \not\equiv 0$ shows that there exists a (unique) Nevanlinna function $M_0 \in \mathbf{N}_0$, i.e. $\kappa = 0$, such that (3.17) holds with $r = p/q$. The converse is also true. If p and q are relatively prime polynomials, and if $M_0 \in \mathbf{N}_0$, $M_0 \not\equiv 0$, then M in (3.17) belongs to $\mathbf{N}_\kappa$, with $\kappa = \max\{\deg p, \deg q\}$. Moreover, the zeros of p and q coincide, counting multiplicities, with the finite (generalized) poles and zeros of nonpositive type of M, and $\deg q - \deg p = \kappa_\infty - \pi_\infty$; cf. [4, Proposition 3.2]. Now Theorem 3.3 applied to the polynomials p and q gives a model for M as a coupling of the finite-dimensional model for R in Proposition 3.1 with $r = p/q$, and a Hilbert space model for M_0. In fact, the approach for constructing the model via Theorem 3.3 allows one to apply the same method immediately to factorized matrix Nevanlinna functions for which the corresponding Bezoutian is invertible.

4. Simplicity of the model operator

The symmetric linear relation S in Theorem 3.3 need not be an operator. In fact, from the form of S it is seen that if $\deg p = \deg q$, then $\operatorname{mul} S = \operatorname{mul} S_0 \oplus \{0\} \oplus \{0\}$, while if $\deg p > \deg q$, then

$$\operatorname{mul} S = \left\{ f_0' \oplus \begin{pmatrix} 0 \\ \Gamma_0^0 \widehat{f}_0 e_n \end{pmatrix} : \widehat{f}_0 = \{0, f_0'\} \in A_1 = \ker \Gamma_1^0 \right\}. \tag{4.1}$$

Hence, if $\deg p = \deg q$, then S is an operator if and only if S_0 is an operator and if $\deg p > \deg q$, then S is an operator if and only if A_1 is an operator. A simple symmetric relation is necessarily an operator. The next theorem characterizes the simplicity of S in spectral theoretical terms.

Theorem 4.1. *Let the symmetric operator S_0 in $\mathfrak{H}_0$ be simple and let S be as defined in Theorem 3.3 with relatively prime polynomials p and q as in* (3.1)*. Then:*

(i) *If $\deg p = \deg q$, then S is simple if and only if*

$$\sigma_p(A_0) \cap \sigma(p) = \emptyset \text{ and } \sigma_p(A_1) \cap \sigma(q) = \emptyset. \tag{4.2}$$

(ii) *If $\deg p > \deg q$, then S is simple if and only if $A_1 = \ker \Gamma_1^0$ is an operator and the conditions* (4.2) *are satisfied.*

(iii) *If $\deg p < \deg q$, then S is simple if and only if $A_0 = \ker \Gamma_0^0$ is an operator and the conditions* (4.2) *are satisfied.*

Moreover, the γ-fields γ and γ_1 of $H_0 = \ker \Gamma_0$ and $H_1 = \ker \Gamma_1$ have the expansions

$$q(\lambda)\gamma(\lambda) = \sum_{k=0}^{\pi_i - 1} (\lambda - \beta_i)^k \Gamma_{k,i} + (\lambda - \beta_i)^{\pi_i - 1} o(1),\ \lambda \widehat{\to} \beta_i \in \sigma(p) \backslash \sigma_p(A_0), \tag{4.3}$$

$$p^\sharp(\lambda)\gamma_1(\lambda) = \sum_{k=0}^{\kappa_j - 1} (\lambda - \bar\alpha_j)^k \widetilde{\Gamma}_{k,j} + (\lambda - \bar\alpha_j)^{\kappa_j - 1} o(1),\ \bar\lambda \widehat{\to} \alpha_j \in \sigma(q) \backslash \sigma_p(A_1), \tag{4.4}$$

$i = 1, \dots, s$, $j = 1, \dots, t$, where $\Gamma_{k,i} = 0 \oplus 0 \oplus V_k(\beta_i)$, $\widetilde{\Gamma}_{k,j} = 0 \oplus V_k(\bar\alpha_j) \oplus 0$, and $V_k(\beta_i)$, $V_k(\bar\alpha_j)$ are the corresponding Vandermonde vectors.

Proof. First the expansions (4.3) and (4.4) are derived. For every $\beta_i \in \sigma(p) \backslash \sigma_p(A_0)$ the γ-field γ_0 and the Weyl function M_0 corresponding to A_0 satisfy the relations

$$\lim_{\lambda \widehat{\to} \beta_i} (\lambda - \beta_i)\gamma_0(\lambda) = \lim_{\lambda \widehat{\to} \beta_i} (\lambda - \beta_i) M_0(\lambda) = 0. \tag{4.5}$$

By incorporating the Taylor series for the vector function Λ at $\lambda = \beta_i$ in the expression for the γ-field in (3.16) and taking into account (4.5), one obtains the expansion (4.3) with $\beta_i \in \sigma(p) \backslash \sigma_p(A_0)$ of multiplicity π_i, $i = 1, \dots, s$. The γ-field γ_1 and the Weyl function M_1 corresponding to $H_1 = \ker \Gamma_1$ are given by

$$\gamma_1 = \gamma M^{-1}, \quad M_1 = -M^{-1} = -\frac{qq^\sharp}{pp^\sharp} M_0^{-1}. \tag{4.6}$$

The γ-field $\gamma_0 M_0^{-1}$ and the Weyl function $-M_0^{-1}$ corresponding to A_1 satisfy

$$\lim_{\lambda \widehat{\to} \bar\alpha_j} (\lambda - \bar\alpha_j)\gamma_0(\lambda) M_0(\lambda)^{-1} = \lim_{\lambda \widehat{\to} \bar\alpha_j} (\lambda - \bar\alpha_j) M_0(\lambda)^{-1} = 0 \tag{4.7}$$

for every $\alpha_j \in \sigma(q) \backslash \sigma_p(A_1)$. It follows from (3.16) that for every $\lambda \in \rho(A_1) \backslash \sigma(p)$,

$$p^\sharp(\lambda)\gamma_1(\lambda) = \gamma_0(\lambda) \frac{q^\sharp(\lambda)}{M_0(\lambda)} \oplus \left(\Lambda^\top \dotplus \Lambda^\top \frac{q^\sharp(\lambda)}{p(\lambda) M_0(\lambda)} \right). \tag{4.8}$$

Hence, by incorporating the Taylor series for Λ at $\lambda = \bar\alpha_j$ in (4.8) and using (4.7) one obtains the expansion (4.4) with $\alpha_j \in \sigma(q) \backslash \sigma_p(A_1)$ of multiplicity κ_j, $j = 1, \dots, t$.

Proof of sufficiency. Let $\widetilde{\mathfrak{H}} = \overline{\operatorname{span}}\{\mathfrak{N}_\lambda(S^*) : \lambda \in \rho(A_0)\}$. Then $\widetilde{\mathfrak{H}} \subset \mathfrak{H}$ and the simplicity of S follows by proving $\mathfrak{H} \subset \widetilde{\mathfrak{H}}$. The expansions (4.3) and (4.4) imply

$$\Gamma_{k,i} \in \widetilde{\mathfrak{H}}, \quad k = 1, \dots, \pi_i, \quad i = 1, \dots, s, \tag{4.9}$$

$$\widetilde{\Gamma}_{k,j} \in \widetilde{\mathfrak{H}}, \quad k = 1, \dots, \kappa_j, \quad j = 1, \dots, t. \tag{4.10}$$

If $\deg p = n$, the Vandermonde vectors $V_k(\beta_i)$, $k = 1, \dots, \pi_i$, $i = 1, \dots, s$, span $\mathbb{C}^n$, and (4.9) gives $\{0\}\oplus\{0\}\oplus\mathbb{C}^n \subset \widetilde{\mathfrak{H}}$. Similarly, if $\deg q = n$, the Vandermonde vectors $V_k(\bar{\alpha}_j)$, $k = 1, \dots, \kappa_j$, $j = 1, \dots, t$, span $\mathbb{C}^n$, and (4.10) gives $\{0\} \oplus \mathbb{C}^n \oplus \{0\} \subset \widetilde{\mathfrak{H}}$. To treat the case $\deg p \neq \deg q$ one may assume that $n = \deg p > \deg q = m$. Then by the assumptions A_1 is an operator, which implies that

$$\lim_{\lambda \widehat{\to} \infty} \gamma_0(\lambda) M_0(\lambda)^{-1} = 0, \quad \lim_{\lambda \widehat{\to} \infty} \frac{1}{\lambda} M_0(\lambda)^{-1} = 0. \tag{4.11}$$

Hence, it follows from (4.8) that

$$\lim_{\lambda \widehat{\to} \infty} \frac{p^\sharp(\lambda)\gamma_1(\lambda)}{\lambda^{n-1}} = 0 \oplus e_n \oplus 0.$$

Analogously, for $k = 1, \dots, n-m-1$ one obtains

$$\lim_{\lambda \widehat{\to} \infty} \lambda^k \left(\frac{p^\sharp(\lambda)\gamma_1(\lambda)}{\lambda^{n-1}} - 0 \oplus \sum_{j=0}^{k-1} \frac{e_{n-j}}{\lambda^j} \oplus 0 \right) = 0 \oplus e_{n-k} \oplus 0.$$

This together with (4.10) implies that $\{0\} \oplus \mathbb{C}^n \oplus \{0\} \subset \widetilde{\mathfrak{H}}$, since the first m coordinates of the Vandermonde vectors $V_k(\bar{\alpha}_j)$, $k = 1, \dots, \kappa_j$, form the full chain of the root subspace of the $m \times m$ companion matrix $C_{q^\sharp}$ at $\bar{\alpha}_j \in \sigma(C_{q^\sharp})$, $j = 1, \dots, t$. The simplicity of S_0 and the form of the γ-field in (3.16) finally show that $\mathfrak{H}_0 \oplus \{0\} \oplus \{0\} \subset \widetilde{\mathfrak{H}}$. Therefore, $\mathfrak{H} \subset \widetilde{\mathfrak{H}}$ and S is simple.

Proof of necessity. If $\deg p > \deg q$ and $\operatorname{mul} A_1 \neq \{0\}$, then also $\operatorname{mul} S \neq \{0\}$, which is not possible if S is simple. Similarly, if $\deg p < \deg q$, then the simplicity of S forces that A_0 is an operator. It remains to prove the necessity of the conditions (4.2). First assume that $\beta_i \in \sigma_p(A_0) \cap \sigma(p)$. Let $h_i \neq 0$ be the corresponding eigenvector of A_0 and let $\widehat{h}_i = \{h_i, \beta_i h_i\}$. Then $\beta_i \in \mathbb{R}$ and $\Gamma_0^0 \widehat{h}_i = 0$. Let $\widehat{v}_i = \{v_i, \beta_i v_i\}$, where

$$v_i = \frac{q^\sharp(\beta_i)}{\Gamma_1^0 \widehat{h}_i} h_i \oplus \begin{pmatrix} V_0(\beta_i) \\ 0 \end{pmatrix}.$$

Here $q^\sharp(\beta_i) \neq 0$, since $\sigma(p) \cap \sigma(q) = \emptyset$, and $\Gamma_1^0 \widehat{h}_i \neq 0$ since S_0 is simple. The definition of w in (3.3) leads to

$$w^* V_0(\beta_i) = e_n^\top B_{p,q}^* V_0(\beta_i) = e_n^\top B_p^* q^\sharp(C_{p^\sharp}) V_0(\beta_i) = e_1^\top q^\sharp(\beta_i) V_0(\beta_i) = q^\sharp(\beta_i).$$

Now the description of S in Theorem 3.3 shows that $\widehat{v}_i \in S$. Hence, β_i is an eigenvalue of S, and thus S is not simple.

Finally, assume that $\alpha_j \in \sigma_p(A_1) \cap \sigma(q)$. Let $k_j \neq 0$ be the corresponding eigenvector of A_1 and let $\widehat{k}_j = \{k_j, \alpha_j k_j\}$. Then $\alpha_j \in \mathbb{R}$, $\Gamma_1^0 \widehat{k}_j = 0$, $p(\alpha_j) \neq 0$, and $\Gamma_0^0 \widehat{k}_j \neq 0$. Now let $\widehat{u}_j = \{u_j, \alpha_j u_j\}$, where

$$u_j = \frac{p(\alpha_j)}{\Gamma_0^0 \widehat{k}_j} k_j \oplus \begin{pmatrix} 0 \\ V_0(\alpha_j) \end{pmatrix}. \tag{4.12}$$

If $\deg p = \deg q$, then (3.3) and $\mathrm{Bez}\,(p,q) = -\mathrm{Bez}\,(q,p) = -B_q p(C_q)$ yield

$$\overline{w}^* V_0(\alpha_j) = -e_n^\top B_q p(C_q) V_0(\alpha_j) = -e_1^\top q_n p(\alpha_j) V_0(\alpha_j) = -q_n p(\alpha_j).$$

Hence, in view of Theorem 3.3, $\widehat{u}_j \in S$. If $\deg p > \deg q = m$, then $q_n = 0$ and

$$\begin{aligned}\overline{w}^* V_0(\alpha_j) &= e_n^\top B_{p,q} V_0(\alpha_j) = e_n^\top B_p q(C_p) V_0(\alpha_j) = e_1^\top q(C_p) V_0(\alpha_j)\\ &= q(e_1^\top C_p) V_0(\alpha_j) = \sum_{k=0}^{m} q_k e_{k+1}^\top V_0(\alpha_j) = q(\alpha_j) = 0.\end{aligned}$$

Moreover,

$$C_p V_0(\alpha_j) + \Gamma_0^0 \left(\frac{p(\alpha_j)}{\Gamma_0^0 \widehat{k}_j} k_j \right) e_n = (\alpha_j V_0(\alpha_j) - p(\alpha_j) e_n) + p(\alpha_j) e_n = \alpha_j V_0(\alpha_j).$$

The description of S in Theorem 3.3 again shows that $\widehat{u}_j \in S$. In both cases α_j is an eigenvalue of S, and thus S is not simple. The case $\deg p < \deg q$ is obtained by changing the roles of A_0 and A_1 (cf. Corollary 3.2). □

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