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POWER BOUNDED OPERATORS AND SUPERCYCLIC VECTORS

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ABSTRACT. By the well-known result of Brown, Chevreau and Pearcy, every Hilbert space contraction with spectrum containing the unit circle has a nontrivial closed invariant subspace. Equivalently, there is a nonzero vector which is not cyclic.

We show that each power bounded operator on a Hilbert space with spectral radius equal to one has a nonzero vector which is not supercyclic. Equivalently, the operator has a nontrivial closed invariant homogeneous subset. Moreover, the operator has a nontrivial closed invariant positive cone.

Let T be a bounded linear operator acting on a complex Banach space X. A vector $x \in X$ is called cyclic (supercyclic, hypercyclic) for T if the set $\{p(T)x : p \text{ polynomial}\}$ $\big(\{\lambda T^n x : \lambda \in \mathbb{C}, n = 0, 1, \dots\}$, $\{T^n x : n = 0, 1, \dots\}$, respectively$\big)$ is dense in X.

Clearly, T has a nontrivial closed invariant subspace (subset, homogeneous subset) if and only if there is a nonzero vector in X which is not cyclic (hypercyclic, supercyclic, respectively); a subset $M \subset X$ is called homogeneous if $\mathbb{C}M \subset M$.

By the well-known example of Read [R], there is an operator on ℓ_1 without nontrivial closed invariant subsets. For operators on Hilbert spaces no negative results are known; the best positive result is that each Hilbert space contraction whose spectrum contains the unit circle has a nontrivial closed invariant subspace [BCP].

The main result of this paper is that each Hilbert space contraction (or more generally power bounded operator) whose spectrum contains at least one point from the unit circle has a nontrivial closed invariant homogeneous subset. With more work we also show that the same result holds for Banach space operators of class C_{00}. In both cases there is also a nontrivial closed invariant positive cone.

Note that each power bounded operator has nontrivial closed invariant subsets since the orbit $\{T^n x : n = 0, 1, \dots\}$ is bounded, and hence nondense, for each vector x.

All spaces considered in this paper are complex. For an operator T acting on a Banach space X denote its spectral radius by $r(T)$ and the essential spectrum by $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$.

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We start with the following lemma.

Lemma 1. *Let $K \geq 1$. Then there exist positive numbers $c_i \quad (i \in \mathbb{N})$ such that $\sum_{i=1}^{\infty} c_i^2 = 1$ and $\sum_{i=k+1}^{\infty} c_i^2 > 3Kc_k$ for all $k \geq 1$.*

Proof. Note first that

$$\lim_{k\to\infty} k^{2/3} \sum_{i=k+1}^{\infty} i^{-4/3} = \infty. \tag{1}$$

Indeed, we have $\sum_{i=k+1}^{\infty} i^{-4/3} \geq \int_{k+1}^{\infty} x^{-4/3} \mathrm{d}x = 3(k+1)^{-1/3}$, and so

$$\lim_{k\to\infty} k^{2/3} \sum_{i=k+1}^{\infty} i^{-4/3} \geq \lim_{k\to\infty} \frac{3k^{2/3}}{(k+1)^{1/3}} = \infty.$$

By (1), there exists k_0 such that

$$k^{2/3} \sum_{i=k+1}^{\infty} i^{-4/3} > 3K \Big(\sum_{i=1}^{\infty} i^{-4/3}\Big)^{1/2} \tag{2}$$

for all $k \geq k_0$. For $j \in \mathbb{N}$ set

$$c_j = (j+k_0)^{-2/3} \Big(\sum_{i=k_0+1}^{\infty} i^{-4/3}\Big)^{-1/2}.$$

Then $\sum_{i=1}^{\infty} c_i^2 = 1$. Let $k \geq 1$. Then, by (2),

$$\begin{aligned}\sum_{i=k+1}^{\infty} c_i^2 &= \frac{\sum\limits_{i=k+1}^{\infty} (k_0+i)^{-4/3}}{\sum\limits_{i=k_0+1}^{\infty} i^{-4/3}} > \frac{3K\Big(\sum\limits_{i=1}^{\infty} i^{-4/3}\Big)^{1/2}}{\sum\limits_{i=k_0+1}^{\infty} i^{-4/3}} \cdot (k+k_0)^{-2/3} \\ &\geq \frac{3K(k+k_0)^{-2/3}}{\Big(\sum\limits_{i=k_0+1}^{\infty} i^{-4/3}\Big)^{1/2}} = 3Kc_k.\end{aligned}$$

□

The next result, which is of independent interest, is a generalization of [M2], Corollary 3.4.

Theorem 2. *Let T be an operator acting on a Hilbert space H such that $1 \in \sigma(T)$ and $T^n x \to 0$ for all $x \in H$. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sup a_n < 1$. Then there exists $x \in H$ of norm one such that $\operatorname{Re}\langle T^n x, x\rangle > a_n$ for all $n \geq 1$.*

Proof. By the Banach-Steinhaus theorem, $\sup_n \|T^n\| < \infty$. Let $K = \sup_n \|T^n\|$. Clearly $K \geq 1$ and $r(T) = 1$.

Suppose first that $1 \notin \sigma_e(T)$. Then 1 is an eigenvalue of T and there exists $x \in H$ of norm one such that $Tx = x$. Then $\operatorname{Re}\langle T^n x, x\rangle = 1$ for all n.

Let $1 \in \sigma_e(T)$. Then $1 \in \partial\sigma_e(T)$, and so $T - I$ is not upper semi-Fredholm; see [HW]. Consequently, for all $\varepsilon > 0$ and $M \subset H$ with $\operatorname{codim} M < \infty$ there exists $u \in M$ of norm one such that $\|Tu - u\| < \varepsilon$. Moreover, given $n_0 \in \mathbb{N}$, we can also find $v \in M$ of norm one such that $\|T^j v - v\| < \varepsilon$ for all $j \leq n_0$.

Replacing the numbers a_n by $\sup\{a_i : i \geq n\}$ we can assume without loss of generality that $1 > a_1 \geq a_2 \geq \cdots$. By Lemma 1, there are positive numbers c_i such that $\sum_{i=1}^{\infty} c_i^2 = 1$ and $\sum_{i=k+1}^{\infty} c_i^2 > 3Kc_k$ for all $k \geq 1$.

For $i = 1, 2, \ldots$ let δ_i be a positive number satisfying $\delta_i < \frac{1-a_1}{2^i}$ and $\delta_i < \min\left\{\frac{Kc_k}{i \cdot 2^{i-k+1}} : k = 1, \ldots, i+1\right\}$.

Find $m_0 \in \mathbb{N}$ such that $a_{m_0} < \sum_{i=2}^{\infty} c_i^2 - 3Kc_1$. We construct inductively an increasing sequence $(m_i)_{i=0}^{\infty}$ of positive integers and a sequence $(x_i)_{i=1}^{\infty} \subset H$ in the following way:

Let $k \in \mathbb{N}$ and suppose that $x_i \in H$ and m_i have already been constructed for all $i < k$. Choose $x_k \in X$ of norm one such that

$$x_k \perp T^j x_i \qquad (i < k, 0 \leq j \leq m_{k-1})$$

and

$$\|T^j x_k - x_k\| < \delta_k \qquad (j \leq m_{k-1}).$$

Find $m_k > m_{k-1}$ such that

$$\|T^j x_i\| < \delta_k \qquad (i \leq k, j \geq m_k)$$

and

$$a_{m_k} < \sum_{i=k+2}^{\infty} c_i^2 - 3Kc_{k+1}.$$

Suppose that x_i and m_i have been constructed in the above described way. Set $x = \sum_{i=1}^{\infty} c_i x_i$. Since (x_i) is an orthonormal sequence, we have $\|x\| = \left(\sum_{i=1}^{\infty} c_i^2\right)^{1/2} = 1$.

For $n \leq m_0$ we have

$$\begin{aligned}
\operatorname{Re}\langle T^n x, x\rangle &= \operatorname{Re}\sum_{i=1}^{\infty} c_i \langle T^n x_i, x\rangle = \operatorname{Re}\sum_{i=1}^{\infty} c_i\big(\langle x_i, x\rangle - \langle x_i - T^n x_i, x\rangle\big)\\
&\geq \sum_{i=1}^{\infty} c_i^2 - \sum_{i=1}^{\infty} c_i \|x_i - T^n x_i\| \geq 1 - \sum_{i=1}^{\infty} c_i \delta_i > 1 - \sum_{i=1}^{\infty} \frac{1-a_1}{2^i} = a_1 \geq a_n.
\end{aligned}$$

Let $k \geq 1$ and $m_{k-1} < n \leq m_k$. Then

$$\begin{aligned}
\operatorname{Re}\langle T^n x, x\rangle &= \operatorname{Re}\sum_{i=1}^{k-1} c_i \langle T^n x_i, x\rangle + \operatorname{Re} c_k \langle T^n x_k, x\rangle + \operatorname{Re}\sum_{i=k+1}^{\infty} c_i \langle T^n x_i, x\rangle\\
&\geq -\sum_{i=1}^{k-1} c_i \|T^n x_i\| - Kc_k + \operatorname{Re}\sum_{i=k+1}^{\infty} c_i\Big(\langle x_i, x\rangle - \langle x_i - T^n x_i, x\rangle\Big)\\
&\geq -\sum_{i=1}^{k-1} c_i \delta_{k-1} - Kc_k + \sum_{i=k+1}^{\infty} c_i^2 - \sum_{i=k+1}^{\infty} c_i \|x_i - T^n x_i\|\\
&\geq -(k-1)\delta_{k-1} - Kc_k + \sum_{i=k+1}^{\infty} c_i^2 - \sum_{i=k+1}^{\infty} \delta_i \geq \sum_{i=k+1}^{\infty} c_i^2 - 3Kc_k > a_{m_{k-1}} \geq a_n.
\end{aligned}$$

Thus $\operatorname{Re}\langle T^n x, x\rangle > a_n$ for all $n \geq 1$. □

Theorem 3. *Let T be a power bounded operator on a Hilbert space H with $\dim H \geq 2$. Suppose that $r(T) = 1$. Then there exists a nonzero vector $x \in H$ which is not* *supercyclic for T. Moreover, there exists a nontrivial closed positive cone invariant with respect to T.*

Proof. Without loss of generality we may assume that T is not a scalar multiple of the identity and that $1 \in \sigma(T)$. We also may assume that the point spectrum of T^* is empty since $\overline{(T-\bar\alpha)H} = \ker(T^*-\alpha)^\perp$ is a nontrivial closed invariant subspace for each eigenvalue α of T^*.

We use the standard reduction of the problem; see [NF]. Set $X_1 = \{x \in H : T^n x \to 0\}$ and $X_2 = \{x \in H : T^{*n}x \to 0\}$. Then X_1 is a closed subspace invariant with respect to T and X_2 is a closed subspace invariant with respect to T^*. So $X_2^\perp$ is invariant with respect to T. Thus we may assume that both X_1 and X_2 are trivial (equal either to $\{0\}$ or to H).

If $X_1 = \{0\} = X_2$, then T is of class C_{11} and therefore has plenty of invariant subspaces; see [NF], Theorem I.5.4. Thus we may assume that either $X_1 = H$ (in this case $T^n x \to 0$ for all $x \in H$) or $X_2 = H$ (and so $T^{*n}x \to 0$ for all x). By Theorem 2, in both cases there exists $x \in H$ of norm one such that $\mathrm{Re}\,\langle T^n x, x\rangle \geq 0$ for all $n \geq 0$. Hence $\mathrm{Re}\,\langle tT^n x, x\rangle \geq 0$ for all $t > 0$ and $n \in \mathbb{N}$, and so the set $\{tT^n x : t > 0, n = 0,1,\dots\}$ is not dense in H. By [LM], this implies that x is not supercyclic for T.

Clearly, the set $\{tT^n x : t > 0, n = 0,1,\dots\}$ is a nontrivial closed positive cone invariant with respect to T. ☐

The situation for operators on Banach spaces is more complicated since the geometry in general Banach space is not so regular. We are able to obtain the main result for operators of class C_{00}, i.e., for operators $T : X \to X$ satisfying $T^n x \to 0$ and $T^{*n}x^* \to 0$ for all $x \in X$ and $x^* \in X^*$.

Lemma 4. *Let $K \geq 1$. Then there are positive numbers c_i $(i \geq 1)$ satisfying $\sum_{i=1}^\infty c_i < 1/3$ and $\sum_{i=k+1}^\infty c_i^2 > 3Kc_k^2$ for all $k \geq 1$.*

Proof. For $j = 1,2,\dots$ set $c_j = (j+18K)^{-2}$. For $k \geq 1$ we have

$$\sum_{i=k+1}^\infty c_i^2 = \sum_{i=k+1}^\infty (i+18K)^{-4} \geq \int_{k+1}^\infty (x+18K)^{-4}\mathrm{d}x$$
$$= \frac{1}{3(k+1+18K)^3} > \frac{1}{6(k+18K)^3} > \frac{18K}{6(k+18K)^4} = 3Kc_k^2.$$

Similarly,

$$\sum_{i=1}^\infty c_i \leq \int_0^\infty (x+18K)^{-2}\mathrm{d}x = \frac{1}{18K} < 1/3.$$

☐

Theorem 5. *Let T be an operator on a Banach space X such that $1 \in \sigma(T)$, $T^n x \to 0$ $(x \in X)$ and $T^{*n}x^* \to 0$ $(x^* \in X^*)$. Let (a_i) be a sequence of positive numbers such that $\lim_{i\to\infty} a_i = 0$. Then there are $x \in X$ and $x^* \in X^*$ such that $\mathrm{Re}\,\langle T^n x, x^*\rangle > a_n$ for all $n \geq 1$.*

Proof. Replacing a_n by $\sup\{a_i : i \geq n\}$ we may assume that $a_1 \geq a_2 \geq \cdots$.

By the Banach-Steinhaus theorem, T is power bounded. Let $K = \sup_n \|T^n\|$. Clearly $K \geq 1$ and $r(T) = 1$.

If $1 \notin \sigma_e(T)$, then 1 is an eigenvalue of T. Let $x \in H$ be a corresponding eigenvector and let $x^* \in X^*$ satisfy $\langle x, x^*\rangle = 1$. Then $\mathrm{Re}\,\langle T^n x, x^*\rangle = 1$ for all n.

In the following we suppose that $1 \in \sigma_e(T)$. As in Theorem 2, $T - I$ is not upper semi-Fredholm. Consequently, for all $\varepsilon > 0$, $n_0 \in \mathbb{N}$ and $M \subset H$ with $\operatorname{codim} M < \infty$ there exists $u \in M$ of norm one such that $\|T^j u - u\| < \varepsilon$ for all $j \le n_0$.

By Lemma 4, there are positive numbers c_i such that $\sum_{i=k+1}^{\infty} c_i^2 > 3Kc_k^2$ for all $k \ge 1$ and $\sum_{i=1}^{\infty} c_i \le 1/3$.

For $i = 1, 2, \dots$ set $\delta_i = \min\left\{\frac{c_k^2}{2^{i-k+5}} : k = 1, \dots, i+1\right\}$.

Choose m_0 such that $a_{m_0} < \sum_{i=2}^{\infty} c_i^2 - 3Kc_1^2$.

We construct inductively an increasing sequence (m_i) of positive integers and sequences $(x_i) \subset X$, $(x_i^*) \subset X^*$.

Let $k \ge 1$ and suppose that m_i, x_i and x_i^* have already been constructed for $i \le k-1$. Let $F_k = \bigvee\{T^j x_i : 1 \le i \le k-1, 0 \le j \le m_{k-1}\}$. Clearly, $\dim F_k < \infty$ and, by [M1], Lemma 1, there is a subspace $M_k \subset X$ of finite codimension such that $\left\|f + m\right\| \ge \frac{2}{5}\|m\|$ for all $f \in F_k, m \in M_k$.

Choose $x_k \in M_k \cap \bigcap_{i=1}^{k-1} \ker x_i^*$ such that $\|x_k\| = 1$ and $\|T^j x_k - x_k\| \le \delta_k \quad (j \le m_{k-1})$.

Let x_k^* be the functional on $F_k \vee \{x_k\}$ defined by $\langle f + \alpha x_k, x_k^*\rangle = \alpha$ for all $f \in F_k, \alpha \in \mathbb{C}$. We have $\left|\langle f + \alpha x_k, x_k^*\rangle\right| = |\alpha| \le \frac{5}{2}\left\|f + \alpha x_k\right\|$. Hence $\|x_k^*\| \le 5/2$ and, by the Hahn-Banach theorem, we can extend x_k^* to a functional on X (denoted by the same symbol x_k^*) with the same norm.

Find m_k such that $\|T^j x_i\| < \delta_k$ and $\|T^{*j} x_i^*\| < \delta_k$ for all $j \ge m_k$ and $i = 1, \dots, k$, and $a_{m_k} < \sum_{i=k+1}^{\infty} c_i^2 - 3Kc_k^2$.

Let $(x_i), (x_i^*)$ and (m_i) be the sequences constructed in the above-described way. Note that $\langle x_i, x_j^*\rangle = \delta_{i,j}$ (the Kronecker symbol) for all $i, j \ge 1$. Set $u = \sum_{i=1}^{\infty} c_i x_i$ and $u^* = \sum_{i=1}^{\infty} c_i x_i^*$. Then $\|u\| \le \sum_{i=1}^{\infty} c_i \le 1$ and $\|u^*\| \le \sum_{i=1}^{\infty} \frac{5}{2} c_i \le 1$.

Let $k \ge 1$ and $m_{k-1} < n \le m_k$. Then

$$\begin{aligned}
\operatorname{Re}\langle T^n u, u^*\rangle &= \operatorname{Re}\sum_{i=1}^{k-1} c_i \langle T^n x_i, u^*\rangle + \operatorname{Re}\left\langle c_k x_k, \sum_{i=1}^{k-1} c_i T^{*n} x_i^*\right\rangle + \operatorname{Re}\langle c_k T^n x_k, c_k x_k^*\rangle \\
&\quad + \operatorname{Re}\left\langle c_k T^n x_k, \sum_{i=k+1}^{\infty} c_i x_i^*\right\rangle + \operatorname{Re}\sum_{i=k+1}^{\infty} c_i \langle T^n x_i, u^*\rangle \ge -\sum_{i=1}^{k-1} c_i \|T^n x_i\| \\
&\quad - c_k \sum_{i=1}^{k-1} c_i \|T^{*n} x_i^*\| - \frac{5}{2} K c_k^2 + \operatorname{Re}\sum_{i=k+1}^{\infty} c_i \Big(\langle x_i, u^*\rangle - \langle x_i - T^n x_i, u^*\rangle\Big) \\
&\ge -2\delta_{k-1} - \frac{5}{2} K c_k^2 + \sum_{i=k+1}^{\infty} c_i^2 - \sum_{i=k+1}^{\infty} c_i \delta_i \ge \sum_{i=k+1}^{\infty} c_i^2 - 3Kc_k^2 > a_{m_{k-1}} \ge a_n.
\end{aligned}$$

For $n \le m_0$ we have

$$\begin{aligned}
\operatorname{Re}\langle T^n u, u^*\rangle &= \operatorname{Re}\sum_{i=1}^{\infty} c_i \langle T^n x_i, u^*\rangle \ge \operatorname{Re}\sum_{i=1}^{\infty} c_i \langle x_i, u^*\rangle - \operatorname{Re}\sum_{i=1}^{\infty} c_i \langle x_i - T^n x_i, u^*\rangle \\
&\ge \sum_{i=1}^{\infty} c_i^2 - \sum_{i=1}^{\infty} c_i \delta_i \ge \sum_{i=1}^{\infty} c_i^2 - \sum_{i=1}^{\infty} \frac{c_i^2}{2} > 0.
\end{aligned}$$

Thus for certain positive multiples x of u and x^* of u^* we have $\operatorname{Re}\langle T^n x, x^*\rangle > a_n$ for all $n \ge 1$. $\square$

Corollary 6. *Let T be an operator of class C_{00} acting on a Banach space X such that $r(T) = 1$. Then there exists a nonzero vector $x \in X$ which is not supercyclic for T.*

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