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PRINCIPAL EIGENVALUES FOR INDEFINITE WEIGHT PROBLEMS IN ALL OF $\mathbb{R}^d$

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ABSTRACT. We show the existence of principal eigenvalues of the problem $-\triangle u = \lambda g u$ in $\mathbb{R}^d$ where g is an indefinite weight function. The existence of a continuous family of principal eigenvalues is demonstrated. Also, we prove the existence of a principal eigenvalue for which the principal eigenfunction $u \to 0$ at ∞.

1. INTRODUCTION

In this paper, we consider the following eigenvalue problem with indefinite weight:

$$(P) \qquad \begin{cases} \triangle u + \lambda g u &= 0 \ \text{ in } \mathbb{R}^d \text{ in the distributional sense,} \\ u > 0 & \quad\ \text{on } \mathbb{R}^d \end{cases}$$

for the case $d \geq 3$, where g is a function in $K_d^{loc}(\mathbb{R}^d)$ that changes sign (i.e. g is an indefinite weight). A principal eigenvalue of (P) is a positive constant (λ_0) for which (P) has a positive solution for $\lambda = \lambda_0$.

Recently, a number of authors have investigated the existence of principal eigenvalues for (P).

Brown, Cosner and Fleckinger in [5] showed that if $d \geq 3$ and g is negative and bounded away of from 0 near ∞, then (P) has a principal eigenvalue. Brown and Tertikas in [6] improved the result in [5] if $g^+ = \max\{g, 0\}$ has a compact support. When g is bounded and $g^+ \in L^{\frac{d}{2}}(\mathbb{R}^d)$, the existence of one eigenvalue and infinitely many other eigenvalues was proved by Allegretto in [1]. Zhiren Jin in [10] considered the case when g is locally Hölder continuous on $\mathbb{R}^d$. The author proved that if $d \geq 3$, $g(x_0) > 0$ for some $x_0 \in \mathbb{R}^d$ and if $\int_{\mathbb{R}^d} |g^+(y)|^{\frac{d}{2}}\, dy < \infty$, then there exists a continuous family of principal eigenvalues for the problem (P). Moreover, the author showed that if in addition there exists $p > \frac{d}{2}$ such that $\int_{\mathbb{R}^d} |g(y)|^p \,(1 + |y|^2)^{2p-d} dy < \infty$, then (P) has a principal eigenvalue (λ_0) and a positive eigenfunction $u(x)$ such that $u(x)\, \|x\|^{d-2} \to c_0$ for a nonnegative constant c_0.

In our case, we do not give any assumption on the continuity and boundedness for g and we will give a generalisation of the results cited above. In this paper, we assume that $g^+ \not\equiv 0$, and we shall prove the following main results.

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Theorem 1.1. *Let $g \in K_d^{Loc}(\mathbb{R}^d)$ and let $G(x,y)$ denote the Green function on $\mathbb{R}^d$. If*

$$\left\|g^+\right\|_{\mathbb{R}^d} = \sup_{x\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} G(x,y)g^+(y)dy < \infty,$$

then there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ there exists a positive continuous solution for the problem (P).

Theorem 1.2. *Let g be in the Kato class $K_d(\mathbb{R}^d)$ such that $g^+ \in L^s(\mathbb{R}^d)$ for $0 < s < \frac{d}{2}$. Then, there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ there exists a continuous positive solution for the problem (P).*

Remark 1.1. Note that if g is bounded, then $g \in K_d(\mathbb{R}^d)$.

Theorem 1.3. *Let $g \in K_d^{Loc}(\mathbb{R}^d)$ such that*

$$(G) \qquad \left\{ \begin{array}{l} \|g^+\|_{\mathbb{R}^d} < \infty, \\ \lim_{\|x\|\to\infty}(\int\limits_{\mathbb{R}^d} \frac{g^+(y)}{\|x-y\|^{d-2}dy}) = 0. \end{array} \right.$$

Then, (P) has a principal eigenvalue $\lambda^* > 0$ such that the corresponding eigenfunction u satisfies $\lim_{\|x\|\to\infty} u(x) = 0$.

Remark 1.2. Note that the conditions of Theorem 1.3 are less restrictive than the conditions of Zhiren Jin [10], where the author imposed that g is locally Hölder continuous such that $g^+ \in L^{\frac{d}{2}}$ and

$$(G'') \qquad \int\limits_{\mathbb{R}^d} |g(y)|^p \left(1 + \|y\|^2\right)^{2p-d} dy < \infty$$

for some $p > \frac{d}{2}$. Indeed, any function in L^p $(p > \frac{d}{2})$ is in $K_d(\mathbb{R}^d)$ and we will show in Proposition 3.4 that any function which satisfies the condition (G'') lies in $L^s(\mathbb{R}^d)$ for some $s < \frac{d}{2}$ and hence satisfies the condition (G). Moreover, we show the following general statement:

Theorem 1.4. *Let g be in the Kato class $K_d(\mathbb{R}^d)$ such that $g^+ \in L^q$ for some $q < \frac{d}{2}$. Then the result of Theorem 1.3 holds.*

Theorem 1.5. *Let $g \in K_d^{Loc}(\mathbb{R}^d)$ such that g^+ is a Green tight function in $\mathbb{R}^d$; namely, g^+ is a Borel measurable function in $\mathbb{R}^d$ satisfying that*

$$\textit{The family } \left\{ \frac{g^+(.)}{\|. - y\|^{d-2}} \right\} \textit{ is uniformly integrable}$$

over $\mathbb{R}^d$ with the parameter $y \in \mathbb{R}^d$. Then (P) has a principal eigenvalue λ^* and a positive eigenfunction u such that $\lim_{\|x\|\to\infty} u(x) = 0$.

Moreover, we prove the following statement:

Corollary 1.1. *We suppose that $g^+(x) \leq \frac{1}{\|x\|^\alpha}$ for $\|x\|$ large and some $\alpha > 2$. Then there exists λ^* such that the problem (P) has a positive continuous eigenfunction such that $|u(x)| \leq c\|x\|^{2-d}$ for large $\|x\|$.*

2. Preliminary

Next, we recall from [2] the following definition:

Definition 2.1. A function V is said to be in $K_d^{loc}(\mathbb{R}^d)$ if and only if, for every $R \geq 0$,

$$\lim_{r\to 0}(\sup_{\|x\|\leq R} \int_{\|x-y\|\leq r} G(x,y)|V(y)|dy) = 0.$$

A function V is said to be in $K_d(\mathbb{R}^d)$ if and only if

$$\lim_{r\to 0}(\sup_{x\in\mathbb{R}^d} \int_{\|x-y\|\leq r} G(x,y)|V(y)|dy) = 0.$$

$G(x,y)$ is the Green function associated to the Laplace operator and dy is the Lebesgue measure on $\mathbb{R}^d$.

Definition 2.2. For a bounded domain Ω in $\mathbb{R}^d$ let $G^\Omega(x,y)$ be the Green function defined on $\Omega \times \Omega$. We define the kernel associated to V by

$$K_\Omega^V = \int_\Omega G^\Omega(.,y)V(y)dy$$

and for every measurable function g, we define

$$K_\Omega^V g = K_\Omega(Vg) = \int_\Omega G^\Omega(.,y)V(y)g(y)dy.$$

In this paper, we say that $u = 0$ on $\partial\Omega$ if $u(x_n) \to 0$ for every regular sequence (x_n) in Ω. Particularly, if Ω is regular, then $u(x_n) \to 0$ for every sequence (x_n) converging to $z \in \partial\Omega$.

As in [4], we denote by $S_b^V(\Omega)$ the set of bounded functions u such that $u + K_\Omega^V u$ is a superharmonic function in Ω. If $V = 0$, we will note $S_b^V(\Omega) = S_b(\Omega)$. Next, we recall the following definition (see [8]).

Definition 2.3. We say that the operator $I + K_\Omega^V$ is positive-invertible if the operator $I + K_\Omega^V : B_b(\Omega) \to B_b(\Omega)$ is invertible and for every function $s \in S_b^+(\Omega)$, we have $(I + K_\Omega^V)^{-1} s \geq 0$.

Without loss of generality, set $g = g_1 - g_2$ with $g_2 > 0$, $g_1 > m > 0$.

Since Kg_1 is a strict potential in Ω, then by Theorem 4.1 in [8], for any $\lambda > 0$ there exists a unique principal eigenvalue $\zeta(\lambda,\Omega) > 0$ and a continuous eigenfunction $u_\lambda > 0$ on Ω such that

$$\Xi(\lambda,\Omega) \quad \begin{cases} \triangle u_\lambda - \lambda g_2 u_\lambda + \zeta(\lambda,\Omega) g_1 u_\lambda = 0 \text{ in } \Omega \text{ in the distributional sense,} \\ u_\lambda > 0 \quad \text{in } \Omega, \\ u_\lambda = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Using a result in [3], the function $\lambda \to \zeta(\lambda,\Omega)$ is continuous and for some $0 < \lambda < \mu$, we have $\zeta(\lambda,\Omega) - \lambda > 0 > \zeta(\mu,\Omega) - \mu$, whence $\zeta(\lambda,\Omega) = \lambda$ for some λ. If $\lambda(\Omega) = \inf\{\lambda > 0 : \zeta(\lambda,\Omega) = \lambda\}$, then $\lambda(\Omega)$ is a principal eigenvalue for

$$E(\lambda,\Omega) \quad \begin{cases} \triangle u + \lambda g u = 0 \text{ in } \Omega \text{ in the distributional sense,} \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega. \end{cases}$$

Remark 2.1. The map $\lambda \to \lambda(\Omega)$ is decreasing. Indeed, let $\Omega_1 \subset \Omega$ and set $\alpha = \lambda(\Omega_1)$, $\beta = \lambda(\Omega)$. Then, using Theorem 3.5 in [8], we obtain $\alpha = \zeta(\alpha, \Omega_1) \geq \zeta(\alpha, \Omega)$. Since for λ small we have $\lambda < \zeta(\lambda, \Omega)$, we get the existence of $\omega \leq \alpha$ such that $\omega = \zeta(\omega, \Omega)$ which yields that $\beta \leq \alpha$.

3. Construction of solutions of (P)

Let $g = g^+ - g^-$ where $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \max\{-g(x), 0\}$. We suppose that $\|g^+\|_{\mathbb{R}^d} < \infty$. Note that if G^Ω denotes the Green function on Ω, then

$$K_\Omega g^+ = \int_\Omega G^\Omega(.,y) g^+(y) dy \leq c(d) \left\|g^+\right\|_{\mathbb{R}^d}$$

where $c(d)$ is a constant depending only on the dimension d.

We see that if u_λ is a solution of $E(\lambda, \Omega)$ with $\|u_\lambda\|_\infty = 1$, then

$$\lambda(\Omega) = \frac{u_\lambda + \lambda(\Omega) K_\Omega g^-(u)}{K_\Omega g^+(u)} \geq \frac{u_\lambda}{c(d) \left\|g^+\right\|_{\mathbb{R}^d}}.$$

Hence

$$\lambda(\Omega) \geq \frac{1}{c(d) \left\|g^+\right\|_{\mathbb{R}^d}}.$$

By Remark 2.1, since the map $\Omega \to \lambda(\Omega)$ is decreasing, then

$$\lambda^* = \inf_{\Omega \subset \mathbb{R}^d} \lambda(\Omega) > 0.$$

3.1. **Proof of Theorem 1.1.** Next, let $0 < \mu \leq \lambda^*$. Then for all bounded domains $\Omega \subset \mathbb{R}^d$ we have $\mu < \lambda(\Omega)$.

Next, we claim that $\mu < \zeta(\mu, \Omega)$. Indeed, assume that $\mu > \zeta(\mu, \Omega)$. By using that for λ small we have $\lambda < \zeta(\lambda, \Omega)$, we get the existence of $\lambda \in]0, \mu[$ such that $\lambda = \zeta(\lambda, \Omega)$ which is impossible by the definition of $\lambda(\Omega)$. Hence, by Theorem 3.8 in [8], the operator $(I - \mu K_\Omega^g)$ is positive-invertible and for every $f \in C(\partial\Omega)$ there exists a function u_f satisfying

$$\begin{cases} \triangle u_f + \mu g u_f = 0 & \text{in } \Omega, \\ u_f = f & \text{on } \partial\Omega. \end{cases}$$

Moreover $u_f > 0$ on the set $\{f > 0\}$.

Let B_n be the ball centered at origin with radius $n, n = 1, 2, \ldots$. Then for each $n \in \mathbb{N}^*$, the boundary value problem

$$(P_n) \qquad \begin{cases} \triangle u_n + \mu g u_n = 0 & \text{in } B_n, \\ u_n = n & \text{on } \partial B_n \end{cases}$$

has a solution $u_n > 0$ on B_n. We normalize $u_n(x)$ by setting $u_n(0) = 1$. Then

$$(3.1) \qquad \begin{cases} \triangle u_n + \mu g u_n = 0 & \text{in } B_n, \\ u_n > 0 & \text{on } B_n, \\ u_n(0) = 1. \end{cases}$$

By Harnack's inequality for positive solutions of elliptic equations (see [2]), we see that for any compact set Ω on $\mathbb{R}^d$, there are constants N and M (where M depends only on Ω, μ and g, N depends only on Ω) such that

$$0 < u_n \leq M \text{ on } \Omega \text{ for } n \geq N.$$

Then it is clear that $(u_n)_n$ has a subsequence which converges to a continuous nonnegative function u on any compact subset of $\mathbb{R}^d$.

Therefore

$$\begin{cases} \triangle u + \mu g u = 0 & \text{in } \mathbb{R}^d, \\ u \geq 0 & \text{on } \mathbb{R}^d, \\ u(0) = 1. \end{cases}$$

Now, by an application of the minimum principle in [7], we get that $u > 0$ in every bounded domain of $\mathbb{R}^d$ and hence $u > 0$ on $\mathbb{R}^d$.

3.2. **Proof of Theorem 1.2.** Since $g^+ \in K_d(\mathbb{R}^d)$, then there exists $\delta > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{\|x-y\|\leq\delta} \frac{g^+(y)}{\|x-y\|^{d-2}} dy \leq 1.$$

Hölder's inequality implies

$$\begin{aligned} \int_{\mathbb{R}^d - B(x,\delta)} \frac{g^+(y)}{\|x-y\|^{d-2}} dy &\leq \left\|g^+\right\|_s \Big(\int_{\mathbb{R}^d - B(x,\delta)} \frac{1}{\|x-y\|^{(d-2)s'}} dy\Big)^{\frac{1}{s'}} \\ &= |B_1| \left\|g^+\right\|_s \Big(\int_\delta^\infty r^{d-1-s'(d-2)} ds\Big)^{\frac{1}{s'}} \\ &= |B_1| \left\|g^+\right\|_s \delta^{2-\frac{d}{s}}. \end{aligned}$$

The result now follows from Theorem 1.1.

3.3. **Proof of Theorem 1.3.** For each $k \in \mathbb{N}^*$, let us denote by $\lambda_k = \lambda(B_k)$ and by ω_k the principle eigenfunction to the problem

$$E(\lambda, B_k) \qquad \begin{cases} \triangle u + \lambda_k g u = 0 & \text{in } B_k, \\ u > 0 & \text{on } B_k, \\ u = 0 & \text{on } \partial B_k. \end{cases}$$

The functions ω_k are chosen such that $\|\omega_k\|_\infty = 1$.

Next, let $\overline{\omega} = \lambda_1 \int_{\mathbb{R}^d} G(x,y) g^+(y) dy$. Thus

$$\begin{cases} \triangle\overline{\omega} = -\lambda_1 g^+, \\ \lim_{\|x\|\to\infty} \overline{\omega}(x) = 0. \end{cases}$$

By Remark 2.1, the sequence λ_k is decreasing. Hence on B_k, we have

$$\begin{aligned} \triangle(\omega_k - \overline{\omega}) &= -\lambda_k g \omega_k + \lambda_1 g^+ \\ &= \lambda_1 g^+ (1 - \omega_k) + \lambda_k g^- \omega_k + (\lambda_1 - \lambda_k) g^+ \omega_k \\ &\geq 0. \end{aligned}$$

Since $\omega_k \leq \overline{\omega}$ on ∂B_k, we get $\omega_k \leq \overline{\omega}$ on B_k. Thus, we can choose a subsequence of $(\omega_k)_k$ which converges uniformly on every compact of $\mathbb{R}^d$ to a nonnegative function w such that

$$\triangle\omega + \lambda^* g \omega = 0 \quad \text{in } \mathbb{R}^d$$

and satisfying

$$0 \leq \omega \leq \overline{\omega}.$$

Since g^+ satisfies the condition (G), we get that $\lim_{\|x\|\to\infty} \overline{\omega} = 0$.

Next, we prove the following lemma.

Lemma 3.1. *$\omega > 0$ on $\mathbb{R}^d$.*

Proof. Let $x_k \in B_k$ such that $\omega_k(x_k) = 1$ and suppose that $\|x_k\| \to \infty$. Thus, we get

$$1 = \omega_k(x_k) \leq \overline{\omega}(x_k) \to 0.$$

Consequently, we can assume that there exists $x_0 \in \mathbb{R}^d$ such that $x_0 = \lim_{k\to\infty} x_k$. Let k_0 be such that $x_k \in B_{k_0}$ for all $k \in \mathbb{N}$. Since

$$\omega_k = \lambda_k \int_{B_k} G^{B_k}(\cdot, y) g(y) \omega_k(y) dy$$

we conclude that the family $\{\omega_k : k \in \mathbb{N}\}$ is equicontinuous and hence

$$\omega(x_0) = \lim_{k\to\infty} \omega_k(x_k) = 1.$$

Then, it follows that $\omega > 0$ on $\mathbb{R}^d$. □

3.4. **Proof of Theorem 1.4.** We recall the following theorem (Theorem 1.8 in [12]).

Theorem 3.2. *Let μ be a positive Radon measure on $\mathbb{R}^d$ and let f be nonnegative and μ-measurable. Then*

$$\int_{\mathbb{R}^d} d\mu = \int_0^\infty \mu\{x : f(x) > t\} dt.$$

Thus, we get the following lemma.

Lemma 3.3. *Let μ be a positive Radon measure on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Then*

$$\int_{\|x-y\|\geq\eta} \frac{d\mu(y)}{\|x-y\|^{d-2}} dy \leq (d-2) \int_\eta^\infty r^{1-d} \mu(B(x,r) dr.$$

Proof. Using Fubini's Theorem and Theorem 3.2, we obtain

$$\begin{aligned}
\int_{\|x-y\|\geq\eta} \frac{d\mu(y)}{\|x-y\|^{d-2}} dy &= (d-2) \int_0^\infty r^{1-d} \mu(B(x,r) 1_{\{\|x-y\|\geq\eta\}}) dr \\
&\leq (d-2) \int_\eta^\infty (\int_{\|x-y\|<r} r^{1-d} d\mu(y)) dr \\
&\leq (d-2) \int_\eta^\infty r^{1-d} \mu(B(x,r) dr.
\end{aligned}$$

□

Next, we give the proof of Theorem 1.4.

Since $g^+ \in L^q(\mathbb{R}^d)$, there exists $k \geq 0$ such that for every $x \in \mathbb{R}^d$ and $r \geq 0$, we have

$$\int_{B(x,r)} g^+(y) dy \leq k \left\|g^+\right\|_q r^{d\frac{q-1}{q}}. \tag{3.2}$$

Let $0 < \eta < M$ and $x \in \mathbb{R}^d$ such that $\|x\| > M$. Using the last lemma and (3.2) we get

$$\int_{\mathbb{R}^d} \frac{g^+(y)}{\|x-y\|^{d-2}} dy \leq \int_{\|x-y\|\leq\eta} \frac{g^+(y)}{\|x-y\|^{d-2}} dy + (d-2)\int_{\eta}^{\infty}(\int_{B(x,r)} g^+(y)dy)\frac{dr}{r^{d-1}}$$

$$\leq \int_{\|x-y\|\leq\eta} \frac{g^+(y)}{\|x-y\|^{d-2}} dy + C(\int_{\eta}^{M}\int_{B(x,r)} g^+(y)dy\frac{dr}{r^{d-1}} + \int_{M}^{\infty} r^{d\frac{q-1}{q}}\frac{dr}{r^{d-1}})$$

$$\leq \int_{\|x-y\|\leq\eta} \frac{g^+(y)}{\|x-y\|^{d-2}} dy$$

$$+ C((\eta^{2-\frac{d}{q}} - M^{2-\frac{d}{q}})(\int_{\|y\|\geq\|x\|-M} (g^+)^q(y)dy)^{\frac{1}{q}} + \frac{1}{M^{\frac{d}{q}-2}})$$

for some positive constant C.

Finally, let $\epsilon > 0$. We choose, then, η and M such that $C\frac{1}{M^{\frac{d}{q}-2}} < \frac{\epsilon}{3}$ and

$$\sup_{x\in\mathbb{R}^d} \int_{\|x-y\|\leq\eta} \frac{g^+(y)}{\|x-y\|^{d-2}} dy < \frac{\epsilon}{3}.$$

Let $A \geq 0$ such that

$$C(\eta^{2-\frac{d}{q}} - M^{2-\frac{d}{q}})(\int_{\|y\|\geq A} (g^+(y))^q dy)^{\frac{1}{q}} < \frac{\epsilon}{3}.$$

Thus, for $\|x\| \geq A + M$, we get

$$C(\eta^{2-\frac{d}{q}} - M^{2-\frac{d}{q}})(\int_{\|y\|\geq\|x\|-M} (g^+(y))^q dy)^{\frac{1}{q}} < \frac{\epsilon}{3}.$$

Hence, g^+ satisfies the condition (G).

Next, we prove the following statement

Proposition 3.4. *Let g be a measurable function such that $g^+ \not\equiv 0$ and*

$$(G'') \qquad \int_{\mathbb{R}^d} |g(y)|^p (1+\|y\|^2)^{2p-d} dy < \infty$$

for some $p > \frac{d}{2}$. Then, g is in the Kato-class $K_d(\mathbb{R}^d)$ and there exists $q < \frac{d}{2}$ such that $g^+ \in L^q$.

Proof. By (G''), we get that $g \in L^p$ for $p > \frac{d}{2}$, which implies by a result in [2] that $g \in K_d(\mathbb{R}^d)$. Now, since $p > \frac{d}{2}$, we get that $\frac{dp}{4p-d} < \frac{d}{2}$. Let $\frac{dp}{4p-d} < q < \frac{d}{2}$. Using the fact that $\frac{q}{p} < 1$ and the Hölder inequality we get

$$\int_{\mathbb{R}^d} |g|^q \leq (\int_{\mathbb{R}^d} |g|^p (1+\|y\|^2)^{2p-d} dy)^{\frac{1}{p}} (\int_{\mathbb{R}^d} \frac{dy}{(1+\|y\|^2)^{\frac{(2p-d)q}{p-q}}})^{\frac{p-q}{p}}$$

$$\leq C(\int_0^{\infty} \frac{r^{d-1}dr}{(1+r^2)^{\frac{(2p-d)q}{p-q}}})^{\frac{p-q}{p}}.$$

Using the assumptions on p and q, we get that $g \in L^q$. □

3.5. **Proof of Theorem 1.5.**

Definition 3.5. A Borel function k in $\mathbb{R}^d$ is called Green-bounded if and only if

$$\|k\|_{\mathbb{R}^d} = \sup_{x\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \frac{|k(y)|}{\|x-y\|^{d-2}} dy < \infty.$$

Definition 3.6. A Borel function k in $\mathbb{R}^d$ is called Green-tight if and only if

$$k \in K_d^{Loc}\left(\mathbb{R}^d\right) \qquad \text{and} \qquad \lim_{M\to\infty}\left(\sup_{x\in\mathbb{R}^d} \int\limits_{\|y\|\geq M} \frac{|k(y)|}{\|x-y\|^{d-2}} dy\right) = 0.$$

Remark 3.1. Any Green-tight function is Green-bounded.

Remark 3.2. Note that if there exists $\alpha > 2$ and $c \geq 0$ such that $|k(x)| \leq \frac{c}{\|x\|^{\alpha}}$ for $\|x\|$ large, then k is Green-tight. In fact we prove the following statement.

Proposition 3.7. *Let k be a Borel function in $\mathbb{R}^d (d \geq 3)$. Suppose that k is in K_d^{Loc} and there exists a number L and a positive function $\varphi(r)$ on $[L, \infty[$ with $\int\limits_L^\infty \frac{\varphi(r)}{r} dr < \infty$ such that for all $\|x\| \geq L$*

$$|k(x)| \leq \frac{\varphi(\|x\|)}{\|x\|^2}.$$

Then k is Green-tight.

For the proof see [14].

3.6. **Proof of Corollary 1.1.**

Let $\overline{\omega}$ be the function defined in the proof of Theorem 1.3. Thus, using the results of [11] if $g^+(x) \leq \frac{1}{\|x\|^{\alpha}}$ for $\|x\|$ large and some $\alpha > 2$, then

$$\overline{\omega}(x) = \lambda_1 \int\limits_{B_1} G(x,y) g^+(y) dy \leq C \|x\|^{2-d}$$

for large $\|x\|$ where C is a positive constant. Since the solution ω defined in the previous section is such that $\omega \leq \overline{\omega}$, we get the desired proof.

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