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ON NON-MEASURABILITY OF ℓ_∞/c_0 IN ITS SECOND DUAL

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ABSTRACT. We show that $\ell_\infty/c_0 = C(\mathbb{N}^*)$ with the weak topology is not an intersection of $\aleph_1$ Borel sets in its Čech-Stone extension (and hence in any compactification). Assuming (CH), this implies that $(C(\mathbb{N}^*), \mathrm{weak})$ has no continuous injection onto a Borel set in a compact space, or onto a Lindelöf space. Under (CH), this answers a question of Arhangel'skiĭ.

1. Introduction

All of our spaces are completely regular. We shall identify ℓ_∞ with $C(\beta\mathbb{N})$, the space of continuous functions on the Čech-Stone compactification of the natural numbers $\mathbb{N}$, and ℓ_∞/c_0 with the function space $C(\mathbb{N}^*)$. Talagrand [Ta] demonstrated that ℓ_∞ and ℓ_∞/c_0, canonically embedded in their second duals, are not Borel with respect to the weak* topology; cf. Edgar [Ed]. One checks that ℓ_∞ is an intersection of $2^{\aleph_0}$ Borel sets in $({\ell_\infty}^{**}, \mathrm{weak}^*)$; cf. Remark 3.2. However, assuming the Continuum Hypothesis (CH), the analogous fact is not true for ℓ_∞/c_0. This follows readily from the next result.

Theorem 1.1. *The space* $(\ell_\infty/c_0, \mathrm{weak})$ *is not an intersection of* $\aleph_1$ *Borel sets in its Čech-Stone extension.*

Indeed, the universal properties of the Čech-Stone extension imply that $(\ell_\infty/c_0, \mathrm{weak})$ is not an intersection of $\aleph_1$ Borel sets in any of its compactifications, and hence also in its σ-compact extension $({\ell_\infty}^{**}, \mathrm{weak}^*)$.

The following corollary answers under (CH) (in a rather strong form) a question posed by A.V. Arhangel'skiĭ [Ar1, Problem 34], [Ar2, Problem 5] (cf. Remark 3.3). This question was also mentioned in [C-S] where, as a partial answer, it was shown that $C_p(\mathbb{N}^*)$ had no one-to-one continuous map onto a compact space, a result obtained independently in [AP].

Corollary 1.2. *Assuming (CH), no continuous image of* $(C(\mathbb{N}^*), \mathrm{pointwise})$ *under a map with* σ*-compact fibers is an intersection of* $2^{\aleph_0}$ *Borel sets in its Čech-Stone compactification.*

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2. Proof of Theorem 1.1

We shall prove the assertion in a more general setting. Let us recall that a compact space K is an *F-space* if each continuous $c : U \to [0,1]$, defined on an open σ-compact set U in K, extends continuously over K. A compact set $L \subseteq K$ is a P-set if any countable intersection of neighborhoods of L is a neighborhood of L. The space $\beta\mathbb{N}$, and all of its compact subspaces, are F-spaces, and $\mathbb{N}^*$ contains compact non-open P-sets. From results of Jayne, Namioka and Rogers [JNR] it follows that for any infinite compact F-space K, $(C(K), \text{weak})$ is not Borel in its Čech-Stone extension.

Proposition 2.1. *Let K be a compact F-space containing a compact non-open P-set. Then the function space* $(C(K), \text{weak})$ *is not an intersection of* $\aleph_1$ *Borel sets in its Čech-Stone compactification.*

Proof. By inductively choosing complements of neighborhoods of a compact non-open P-set in K, one can define a strictly increasing sequence

(1) $W_1 \subseteq W_2 \subseteq \ldots \subseteq W_\xi \subseteq \ldots, \xi < \omega_1, \overline{W_\xi} \subseteq W_{\xi+1}, W_\xi \neq W_{\xi+1}$

of open sets in K. Let

(2) $H = \{f \in C(K) : 0 \leq f \leq 1,\ \ f|(K \setminus \bigcup_{\xi<\omega_1} W_\xi) \equiv 0\}$.

We let H_w denote the set H equipped with the topology inherited from $(C(K),$ weak). We also consider a topology τ on H generated by basic neighborhoods

(3) $N(f,\xi) = \{g \in H : f|W_\xi = g|W_\xi\}$.

We shall write H_τ for (H,τ). The identity map

(4) $i : H_\tau \to H_w$

is continuous; cf. [DJP], [Ha], [JNR]. Since K is an F-space, one easily checks that for any sequence $N(f_1,\xi_1) \supseteq N(f_2,\xi_2) \supseteq \ldots, \xi_1 < \xi_2 < \ldots$, there is $f \in H$ such that, for $\eta = \sup\limits_i \xi_i$,

(5) $\bigcap\limits_i N(f_i,\xi_i) \supseteq N(f,\eta)$.

This implies that H_τ is a Baire space and provides the essential property for the proof of the next lemma. □

Lemma 2.2. *The Čech-Stone compactification βH_τ cannot be covered by $\aleph_1$ meager sets.*

Proof of Lemma 2.2. Assume $\beta H_\tau = \bigcup\limits_{\alpha<\omega_1} A_\alpha$, where every A_α is closed and nowhere dense. By (5) one can inductively define $g_\alpha \in H$, $\phi(\alpha) < \omega_1$, with $\phi(\alpha) < \phi(\beta)$, for $\alpha < \beta$, such that $N(g_\alpha,\phi(\alpha)) \supseteq N(g_\beta,\phi(\beta))$ and $\overline{N(g_\alpha,\phi(\alpha))}^\beta \cap A_\alpha = \emptyset$ (the closure taken in βH_τ). But then, $\bigcap\limits_{\alpha<\omega_1} \overline{N(g_\alpha,\phi(\alpha))}^\beta$ is simultaneously non-empty, and disjoint from $\bigcup\limits_{\alpha<\omega_1} A_\alpha$, a contradiction. That concludes the proof of the lemma.

Following an idea of [De] we shall show that H_w is not an intersection of $\aleph_1$ Borel sets in βH_w. Since H_w is closed in $(C(K), \text{weak})$, this will also show that the assertion of the proposition is true.

Aiming for a contradiction to Lemma 2.2, assume that

(6) $H_w = \bigcap\limits_{\alpha<\omega_1} E_\alpha$, where every $E_\alpha \subseteq \beta H_w$, $\alpha < \omega_1$, is Borel,

and let

(7) $i^\beta : \beta H_\tau \to \beta H_w$

be the continuous extension of the identity map given in (4).

Let

(8) $S_\alpha = (i^\beta)^{-1}(E_\alpha)$.

Then S_α is a Borel set in βH_τ containing the Baire space H_τ, and therefore, S_α being open modulo meager sets in βH_τ,

(9) $\beta H_\tau \setminus M_\alpha \subseteq S_\alpha$ for M_α meager in βH_τ.

From (6), (8) and (9) we obtain

(10) $\beta H_\tau \setminus \bigcup_{\alpha<\omega_1} M_\alpha \subseteq (i^\beta)^{-1}(H_w)$.

Now let C_α be the closure in βH_w of the set $\{f \in H : f|(K \setminus W_\alpha) \equiv 0\}$. If there exists $h \in H_w \setminus \bigcup_{\alpha<\omega_1} C_\alpha$, then there is $a > 0$ such that $h^{-1}([a,1]) \cap (K \setminus W_\alpha) \neq \emptyset$, for all $\alpha < \omega_1$. But, K compact gives $h^{-1}([a,1]) \setminus \bigcup_{\alpha<\omega_1} W_\alpha \neq \emptyset$, a contradiction. Hence,

(11) $H_w \subseteq \bigcup_{\alpha<\omega_1} C_\alpha$ and C_α contains no $N(f,\xi)$.

Let

(12) $L_\alpha = (i^\beta)^{-1}(C_\alpha)$.

The second part of (11) shows that $C_\alpha \cap H$ has empty interior in H_τ, and therefore

(13) L_α is nowhere dense in βH_τ.

Then (10), (11) and (12) yield

(14) $\beta H_\tau = \bigcup_{\alpha<\omega_1} (M_\alpha \cup L_\alpha)$,

with M_α, L_α meager in βH_τ. This, however, is impossible by Lemma 2.2. □

3. Proof of Corollary 1.2 and remarks

Corollary 1.2 is an immediate consequence of the following observation.

Lemma 3.1. *Let $u : X \to Y$ be a continuous surjection with σ-compact fibers. If $|Y| = 2^{\aleph_0}$ and Y is is an intersection of $2^{\aleph_0}$ Borel sets in βY, then X is an intersection of $2^{\aleph_0}$ Borel sets in βX.*

Proof. Let $u^\beta : \beta X \to \beta Y$ be the continuous extension and let

(1) $B(Y) = (u^\beta)^{-1}(Y)$,

(2) $B(y) = \beta X \setminus ((u^\beta)^{-1}(y) \setminus X)$.

Then $B(Y)$ is an intersection of $2^{\aleph_0}$ Borel sets in βX. Since every $u^{-1}(y) = (u^\beta)^{-1}(y) \cap X$ is σ-compact and hence F_σ, the sets $B(y) = \beta X \setminus ((u^\beta)^{-1}(y) \setminus u^{-1}(y))$ are also Borel sets in βX. Now, one readily checks that

$$X = B(Y) \cap \bigcap \{B(y) : y \in Y\};$$

hence X is an intersection of $2^{\aleph_0}$ Borel sets in βX. □

Remark 3.2. Similar arguments show that $\ell_\infty = C(\beta\mathbb{N})$ is an intersection of $2^{\aleph_0}$ Borel sets in $(C(\beta\mathbb{N})^{**}, \text{weak}^*)$. Indeed, let Λ be the σ-compact space of all bounded sequences of reals with the pointwise topology. If $\delta_n \in C(\beta\mathbb{N})^*$ is the functional identified with the probability measure supported by $\{n\}$, the map $u : C(\beta\mathbb{N})^{**} \to \Lambda$, defined by $u(\phi)(n) = \langle \delta_n, \phi \rangle$, is a surjection which is continuous with respect to the weak*-topology. (Here, $\langle\, , \rangle$ represents the duality map on $C(\beta\mathbb{N})^* \times C(\beta\mathbb{N})^{**}$.) For $y \in \Lambda$, let y^β be the continuous extension over $\beta\mathbb{N}$, and let

$$B(y) = C(\beta\mathbb{N})^{**} \setminus (u^{-1}(y) \setminus \{y^\beta\}).$$

Then, $B(y)$ is Borel in $(C(\beta\mathbb{N})^{**}, \text{weak}^*)$ and $C(\beta\mathbb{N}) = \bigcap\{B(y) : y \in \Lambda\}$.

Remark 3.3. Corollary 1.2 also shows that, assuming (CH), $(C(\mathbb{N}^*), \text{pointwise})$ cannot be mapped onto a Lindelöf space by any continuous function with σ-compact fibers (this answers the second part of a question of Arhangel'skiĭ in [Ar1, Problem 34]). Assume on the contrary that a Lindelöf space X is an image of $C(\mathbb{N}^*)$ under such a map. Since $(C(\mathbb{N}^*), \text{pointwise})$ is of cardinality $2^{\aleph_0}$, so is X. Therefore, X has a continuous injection into the Tychonoff cube K of weight $2^{\aleph_0}$ and in effect, one can assume that the Lindelöf space X is a subspace of K. But then X is an intersection of $2^{\aleph_0}$ σ-compact subsets of K. Indeed, X being Lindelöf, each point in $K \setminus X$ is contained in a compact G_δ-set in $K \setminus X$, and there are $2^{\aleph_0}$ compact G_δ-sets in K. We arrive at a contradiction with Corollary 1.2.

Remark 3.4. Let κ be a regular uncountable cardinal, let κ_d be the set κ with the discrete topology, and let K_κ be obtained from $\beta\kappa_d$ by identifying to a point the set of uniform ultrafilters $\beta\kappa_d \setminus \bigcup\{\overline{A} : A \subseteq \kappa, |A| < \kappa\}$. Then, K_κ is an F-space with a non-isolated P-point. The reasoning in the proof of Proposition 2.1 shows that $(C(K_\kappa), \text{weak})$ is not an intersection of κ Borel sets in its Čech-Stone compactification. Assuming that κ is strongly inaccessible, one shows also, as in the proof of Corollary 1.2, that any continuous image of $(C(K_\kappa), \text{pointwise})$ under a map with σ-compact fibers, has Lindelöf number κ.

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