PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 12, Pages 3845–3849
S 0002-9939(03)06991-0
Article electronically published on March 25, 2003

ON CHARACTERIZATIONS OF COMMUTATIVITY OF C*-ALGEBRAS

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(Communicated by David R. Larson)

Abstract. We give three kinds of characterizations of the commutativity of C*- algebras. The first is the one from operator monotone property of functions regarded as the nonlinear version of Stinespring theorem, the second one is the characterization of commutativity of local type from expansion formulae of related functions and the third one is of global type from multiple positivity of those nonlinear positive maps induced from functions.

There are several characterizations for the commutativity of C*-algebras. One type is the well-known Stinespring theorem, that is, a C*-algebra A is commutative if and only if every positive linear map from A to another C*-algebra B (or from B to A) becomes completely positive. To be precise, A becomes commutative if and only if every positive linear map to B becomes two-positive (and then automatically completely positive). Note that this is the beginning of the long and fruitful developments of understanding the matricial order structure of (noncommutative) operator algebras (cf. [1]). Another type is based on an operator monotone function such as x^p for $p > 1$ on a C*-algebra, that is, a C*-algebra A is commutative if and only if x^p is operator monotone on A for some $p > 1$ (cf. [2, 3]). Very recently, Wu in [6] gave another characterization for commutativity based on the function $\exp x$. We observe that both x^p and $\exp x$ are monotone increasing functions on the positive axis but not operator monotone on M_2, the matrix algebra of all complex 2×2 matrices.

Thus, regarding the problem of operator monotone functions as the counterpart of a kind of nonlinear positive map we prove our first result (Theorem 1) that includes all known previous results and corresponds the Stinespring theorem mentioned above. We then present a characterization of commutativity of local type (Theorem 2) from expansion formulae of related functions, where just two positive operators are concerned. The third result is the one from multiple positivity of nonlinear positive maps (Theorem 3) as another counterpart of Stinespring theorem relative to nonlinear maps on A.

Throughout this article we assume that a C*-algebra A is unital. By an operator monotone functon on a C*-algebra A we mean that it is monotone on the positive

Received by the editors March 2, 2001 and, in revised form, July 29, 2002.
2000 *Mathematics Subject Classification.* Primary 46L05.
Key words and phrases. C*-algebra, commutativity.
This research was supported by the Excellent Young Teachers Program of MOE, P.R.C. and by the National Natural Science Foundation of China (No.10071047).

axis and induces an operator monotone function on A. We call a function $f(x)$ matrix monotone of order n if it induces a monotone function on the matrix algebra M_n. Then it is well known that $f(x)$ is operator monotone if and only if it is matrix monotone of all orders.

We first illustrate a few generalized examples of monotone functions which are not matrix monotone of order 2 as well as the exponential function together with elementary proofs. We call the following function $f(x)$ a generalized polynomial of order t_n:

$$f(x) = \alpha_0 x^{t_n} + \cdots + \alpha_{n-1}x^{t_1} + \alpha_n,$$

where $t_n > t_{n-1} > \cdots > t_1 > 0$. t_n is its degree.

Proposition 1. *Every generalized polynomial whose order is greater than* 1 *is not matrix monotone of order* 2 *as well as the exponential function.*

Proof. Write $f(x) = \alpha_0 x^{t_n} + \cdots + \alpha_{n-1}x^{t_1} + \alpha_n$. Take a pair $a,\ b$ such that $a \leq b$ in M_2. Then $sa \leq sb$ for every positive number s. Thus if $f(x)$ is monotone we have $f(sa) \leq f(sb)$ and $f(sa)/s^{t_n} \leq f(sb)/s^{t_n}$. Letting s to infinity we have $\alpha_0 a^{t_n} \leq \alpha_0 b^{t_n}$. As is well known, this is absurd whether α_0 is positive or negative. We give here however a simple counterexample for completeness of our arguments. At first, obviously we do not have the case $\alpha_0 < 0$, and $\alpha_0 > 0$. Now consider the matrices $e_0 = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $a_0 = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$ with $e_0 \leq a_0$. Then since e_0 is a projection, ${e_0}^t = e_0$, and ${a_0}^t = \begin{pmatrix} (\frac{3}{2})^t & 0 \\ 0 & (\frac{3}{4})^t \end{pmatrix}$ for every $t > 1$. But a straightfoward calculation shows that $\det|{a_0}^t - {e_0}^t| < 0$ because the function x^t is convex for $t > 1$. Hence we obtain a contradiction.

For the exp function we also use the above examples of e_0 and a_0. Namely in this case, $\exp e_0 = I + (e-1)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e+1) & \frac{1}{2}(e-1) \\ \frac{1}{2}(e-1) & \frac{1}{2}(e+1) \end{pmatrix}$ and $\exp a_0 = \begin{pmatrix} \exp\frac{3}{2} & 0 \\ 0 & \exp\frac{3}{4} \end{pmatrix}$. Put $t = \exp\frac{1}{4}$; then we have

$$\begin{aligned} & \det|\exp a_0 - \exp e_0| \\ = \quad & \exp\frac{9}{4} + e - \frac{1}{2}(\exp\frac{5}{2} + \exp\frac{3}{2} + \exp\frac{7}{4} + \exp\frac{3}{4}) \\ = \quad & \frac{1}{2}(2(\exp\frac{1}{4})^9 + 2(\exp\frac{1}{4})^4 - (\exp\frac{1}{4})^{10} - (\exp\frac{1}{4})^6 - (\exp\frac{1}{4})^7 - (\exp\frac{1}{4})^3) \\ = \quad & -\frac{1}{2}(t^{10} - 2t^9 + t^7 + t^6 - 2t^4 + t^3) \\ = \quad & -\frac{1}{2}t^3(t^7 - 2t^6 + t^4 + t^3 - 2t + 1) \\ = \quad & -\frac{1}{2}t^3(t-1)^4(t+1)(t^2+t+1) < 0 \end{aligned}$$

since $t = \exp\frac{1}{4} > 1$. Thus $\exp a_0$ cannot majorize $\exp e_0$. The proof is completed. □

We are indebted to K. Minemura and T. Ito for the last of the above estimations.

If we make use of the operator monotone property of the log function, we can deduce from monotone assumption for exp the monotone property of the function x^t for any $t > 1$. To discuss on a C*-algebra, this will also considerably shorten the proof in [6], where Wu reduces his whole proof to the monotone property of the function x^2 and uses the old Ogasawara's result [2]. On the matrix algebra M_2 however, we prefer the above elementary approach.

Now we state the monotone function version of the Stinespring theorem in the following theorem.

Theorem 1. *Let A be a C^*-algebra. The following are equivalent:*

(1) *A is commutative;*

(2) *every continuous monotone function on the positive axis becomes operator monotone on A; and*

(3) *there exists a continuous monotone function on the positive axis which is not matrix monotone of order 2 but operator monotone on A.*

Proof. We first recall that a C*-algebra A is commutative if and only if every irreducible representation of A is 1-dimensional. We shall show our proof along this line.

From the above proposition, we see that the assertion (2) implies (3).

(3) $\Longrightarrow$ (1) Let π be an irreducible representation of A on a Hilbert space H. If there is a two dimensional projection $e \in B(H)$, we have by [4, Theorem 4.18] that $\pi(A)e = B(H)e$, hence $e\pi(A)e = eB(H)e = B(eH)$. The latter is naturally isomorphic to the matrix algebra M_2. Put $B = \{a \in A : \pi(a)e = e\pi(a)\}$; then by definition B is a C*-subalgebra of A. Now for every self-adjoint operator $b \in B(H)$, there is a self-adjoint element $a \in A$ such that $\pi(a)e = be$ by the same theorem cited above. If b leaves the subspace eH invariant, then b commutes with e and $\pi(a)$ commutes with e as well. Therefore, the map: $a \to \pi(a)e$ is a $*$-homomorphism from B onto $B(eH)$. Let $c, d \in B(eH)$ be two positive elements such that $c \leq d$. Then by an elementary way, we can find positive elements $a, b \in B$ satisfying $a \leq b$ and $\pi(a) = c$ and $\pi(b) = d$.

Now let f be a monotone function in assertion (3); then we have that $\pi(f(a)) = f(\pi(a)) = f(c)$ and $\pi(f(b)) = f(d)$. Hence, f becomes a monotone function on M_2, a contradiction. Thus every irreducible representation of A has to be one dimensional and A is commutative.

For the implications from (1) to (2), it is enough to notice that when A is commutative a point evaluation φ_x for a point x in the spectrum of A commutes with function operations of the above class as in the case of $*$-homomorphisms. Thus every continuous monotone function on the positive axis becomes a monotone operator function on the C*-algebra A. □

Next we give a characterization of commutativity of local type from the point of view of expansion formulae of relevant functions, where no C*-algebras are around. Namely we have the following.

Theorem 2. *Let a and b be bounded positive linear operators on a Hilbert space. Then the following assertions are equivalent:*

(1) *$ab = ba$;*

(2) *$\exp(a + tb) = \exp a \exp tb$ for every positive number t;*

(3) *$(a+tb)^n = a^n + na^{n-1}tb + \cdots + na(tb)^{n-1} + (tb)^n$ for every positive number t and for every positive integer n, that is, the expansion follows the usual formula; and*

(4) *$(a+tb)^n = a^n + na^{n-1}tb + \cdots + na(tb)^{n-1} + (tb)^n$ for every positive number t and for some positive integers $n \geqq 2$.*

Proof. We note that the implications from (1) to other assertions and from (3) to (4) are trivial.

(2) $\Longrightarrow$ (1) Note first that $\exp a \exp tb = \exp tb \exp a$. Hence $\exp(-tb)\exp a = \exp a \exp(-tb)$. Thus, this commutativity holds for all real number s. Take a positive number λ; then

$$\int_0^\infty \exp(-s(\lambda + b))ds \exp a = \exp a \int_0^\infty \exp(-s(\lambda + b))ds,$$

hence, $(\lambda+b)^{-1}\exp a = \exp a(\lambda+b)^{-1}$. It follows that $\exp a(\lambda+b) = (\lambda+b)\exp a$. Thus, letting $\lambda \to 0$, we have $(\exp a)b = b(\exp a)$. Next, note that for arbitrary positive numbers s and t, $\exp(sa+tb) = (\exp(a+t/sb))^s = (\exp a \exp t/sb)^s = (\exp t/sb \exp a)^s$, and then $\exp(sa+tb) = \exp sa \exp tb = \exp tb \exp sa$. Therefore, by the above argument we see that $(\exp sa)b = b(\exp sa)$ for every positive s, which implies that $ab = ba$.

(4) $\Longrightarrow$ (1) Let $g(t) = (a+tb)^n$. Then we have

$$\begin{aligned} g(t) &= a^n + nta^{n-1}b + \cdots + nt^{n-1}ab^{n-1} + t^n b^n \\ &= a^n + ntba^{n-1} + \cdots + nt^{n-1}b^{n-1}a + t^n b^n \end{aligned}$$

by the assumption and the self-adjointness of $g(t)$ because both a and b are positive. It follows that $ab^{n-1} = b^{n-1}a$, which implies that $ab = ba$. $\square$

Corollary 1. *Let A be a C^*-algebra. The following are equivalent:*

(1) *A is commutative;*

(2) $\exp(a+b) = \exp a \exp b$ *for every pair of positive elements $a, b \in A$;*

(3) $(a+b)^n = a^n + na^{n-1}b + \cdots + nab^{n-1} + b^n$ *for every pair of positive elements $a, b \in A$ and for every positive integer n; and*

(4) $(a+b)^n = a^n + na^{n-1}b + \cdots + nab^{n-1} + b^n$ *for every pair of positive elements $a, b \in A$ and for some positive integers $n \geq 2$.*

We may similarly show that A is commutative if and only if $a^n - b^n = (a-b)(\sum_{i=1}^{n} a^{n-i}b^{i-1})$ for every pair of positive elements $a, b \in A$ and for every positive integer n if and only if $a^n - b^n = (a-b)(\sum_{i=1}^{n} a^{n-i}b^{i-1})$ for every pair of positive elements $a, b \in A$ and for some positive integers $n \geq 2$.

Now as we have mentioned above, it would be interesting to investigate the meaning of degree of positivity even for nonlinear positive mappings; that is, how their degrees are concerned with the structure of C*-algebras. Since we are discussing here a quite limited situation, namely about positive mappings induced from functions, we start from this point.

Recall first that a linear map φ from A to B is two-positive if it induces a positive map from $A \otimes M_2$ to $B \otimes M_2$, that is, if $\tilde{\varphi} : (a_{ij}) \to (\varphi(a_{ij}))$ is positive. Now if we simply apply this definition for a continuous function $f(x)$ we have to be necessarily involved in the meaning of the expression $f(a)$ for a general element a of a C*-algebra A. If we assume f as an entire function we may simply avoid this trouble. Therefore, in order to investigate the multiple positivity of mappings coming from functions in general, the starting point itself has to involve some discussions. Thus, since we are concerned with the characterizations of the commutativity, so far we are restricted here only to those functions of x^n and $\exp x$. We say that they become two-positive functions on a C*-algebra A, if their entry-wise actions on a 2×2 positive matrix in $A \otimes M_2$ also become a positive matrix. Then, we have

Theorem 3. *Let A be a C^*-algebra. The following are equivalent:*
(1) *A is commutative;*
(2) *$\exp x$ is two-positive on A;*
(3) *x^n is two-positive for some integer $n \geq 2$ on A; and*
(4) *x^n is two-positive for every positive integer n on A.*

Proof. (2)$\Longrightarrow$(1) We recall that if a is an invertible positive element in A, then

$$\begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$$

is positive if and only if $b \geq c^* a^{-1} c$. Let $a, b \in A$ such that $a \leq b$; then

$$\begin{pmatrix} a & a \\ a & b \end{pmatrix}$$

is positive. By assumption (2), we have that

$$\begin{pmatrix} \exp a & \exp a \\ \exp a & \exp b \end{pmatrix}$$

is positive, which implies that $\exp b \geq \exp a \exp(-a) \exp a = \exp a$. It follows that $\exp x$ is operator monotone on A. Thus A is commutative by Theorem 1.

Similarly we may prove that (3) implies (1) by inducing that x^n is monotone on A. That (4) implies (3) is clear.

Conversely, let A be commutative; we can easily prove that (4) holds. In fact, for integer n, let $(a_{ij}) \in A \otimes M_2$ be positive. Then $((a_{ij})^n)$ is the Schur product of n-copies of (a_{ij}), and thus is positive since A is commutative. Now we have

$$(\exp a_{ij}) = (\sum_{n=0}^{\infty} \frac{1}{n!} (a_{ij})^n) = \sum_{n=0}^{\infty} \frac{1}{n!} ((a_{ij})^n).$$

Hence (2) holds. The proof is complete. □

We note that if either x^n or $\exp x$ is two-positive, then A is commutative. Thus they become completely positive. That is, A is commutative if and only if either x^n for some integer $n \geq 2$ or $\exp x$ is completely positive on A.

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