PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 12, Pages 3821–3825
S 0002-9939(03)06998-3
Article electronically published on June 18, 2003

A NOTE ON THE ISOPERIMETRIC INEQUALITY

JANI ONNINEN

(Communicated by Juha M. Heinonen)

ABSTRACT. We show that the sharp integral form on the isoperimetric inequality holds for those orientation-preserving mappings $f \in W_{loc}^{\frac{n^2}{n+1}}(\Omega, \mathbb{R}^n)$ whose Jacobians obey the rule of integration by parts.

1. INTRODUCTION

The familiar geometric form of the isoperimetric inequality reads as

$$n^{n-1}\omega_{n-1}|U|^{n-1} \leq |\partial U|^n, \tag{1}$$

where $|U|$ stands for the volume of a domain $U \subset \mathbb{R}^n$ and $|\partial U|$ is its $(n-1)$-dimensional surface area. Now, if $f : B_r \to U$ is a diffeomorphism of a ball $B_r = B(x_0, r) \subset \mathbb{R}^n$ onto U, then $|U| = \left|\int_{B_r} J(x, f)\,dx\right|$ and $|\partial U| \leq \int_{\partial B_r} |D^\sharp f(x)|\,dx$. Here $D^\sharp f(x)$ stands for the cofactor matrix of the differential matrix $Df(x)$. In this way, we obtain what is known as the integral form of the isoperimetric inequality, namely

$$\left|\int_{B_r} J(x,f)\,dx\right| \leq I(n)\left(\int_{\partial B_r} |D^\sharp f(x)|\,dx\right)^{\frac{n}{n-1}} \tag{2}$$

with $I(n) = (n\sqrt[n-1]{\omega_{n-1}})^{-1}$. Above, we used the operator norm of the cofactor matrix, defined by $|D^\sharp f(x)| = \sup\{|D^\sharp f(x)h| : |h| = 1\}$.

Reshetnyak proved in [14] the sharp Hölder-continuity for a mapping of bounded distortion by extending certain ideas of Morrey's [10]. This required him to prove the isoperimetric inequality (2) for a mapping in the Sobolev class $W^{1,n}$ [15] (see also [2, Theorem 4.5.9 (31)]). Reshetnyak's proof is based on integration by parts as are the related proofs given in [11], [12] by Müller et al. One can check using a standard approximation argument that it suffices to prove the isoperimetric inequality (2) for all smooth mappings. The sharp constant $I(n)$ in inequality (2) plays a very crucial role in Reshetnyak's argument (also see [6, Chapter 7.7]). The Sobolev regularity

Received by the editors April 18, 2002 and, in revised form, July 23, 2002.
2000 *Mathematics Subject Classification.* Primary 26D10.
The author was supported in part by the Academy of Finland, project 39788, and by the foundations Magnus Ehrnroothin Säätiö and Vilho, Yrjö ja Kalle Väisälän Rahasto. This research was done when the author was visiting the University of Michigan. He thanks the Department of Mathematics for their hospitality.

$W^{1,n}$ cannot be substantially relaxed. Indeed, the mapping

$$f(x) = \frac{x}{|x|} \log \left(\frac{e}{|x|} \right) \tag{3}$$

belongs to $\bigcap_{p<n} W^{1,p}(B(0,1), \mathbb{R}^n)$ but (2) fails for all $0 < r < 1$.

For example in non-linear elasticity (see [1], [16] and [12]) it is natural to assume that the Jacobians of the mappings in consideration are positive a.e., because a deformation of an elastic body should be orientation preserving. Recently, a generalization of mappings of bounded distortion, the theory of mappings of finite distortion, with subexponentially distortion has emerged, partially motivated by non-linear elasticity. We refer the interested reader to the monograph [6] by Iwaniec and Martin. The assumptions of this theory imply that $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$, $J(x, f) \geq 0$ a.e.,

$$|Df|^n \in L^P_{loc}(\Omega) \tag{4}$$

where

$$\text{the function } t \to P(t^{\frac{n}{n+1}}) \text{ is increasing for large values of } t, \tag{5}$$

$$\int_1^\infty \frac{P(t)}{t^2}\, dt = \infty \tag{6}$$

and P is an Orlicz-function (see [6, Chapter 4.12]). One can improve example (3) and find, for each given function P for which the integral (6) converges, a radial stretching f so that (4) holds and (2) fails ([9]). We proved in [5] that, under the above assumptions, the isoperimetric inequality holds, with some constant, depending only on the dimension n. In this paper, we will give a simple limiting argument to show that, under the above assumptions, the isoperimetric inequality (2) holds with the sharp constant $I(n)$. Actually this is a simple case of our more general theorem.

Let $f \in W^{1,\frac{n^2}{n+1}}_{loc}(\Omega, \mathbb{R}^n)$. We say that the Jacobian $J(\cdot, f)$ of f obeys the rule of integration by parts if the equation

$$\int_\Omega \varphi(x) J(x, f)\, dx = -\int_\Omega f_i(x) J(x, f_1, ..., f_{i-1}, \varphi, f_{i+1}, ..., f_n)\, dx \tag{7}$$

is valid for every test function $\varphi \in C_0^\infty(\Omega)$ and each index $i = 1, ..., n$. Under the assumption $f \in W^{1,\frac{n^2}{n+1}}_{loc}(\Omega, \mathbb{R}^n)$, different choices of indices i yield the same value of the integral; see [3]. It is important to note that the right-hand side is well defined for mappings lying in the Sobolev space $W^{1,\frac{n^2}{n+1}}_{loc}(\Omega, \mathbb{R}^n)$ and so equation (7) implies, when the Jacobian does not change the sign, that

$$J(\cdot, f) \in L^1_{loc}(\Omega). \tag{8}$$

As an example, the Jacobian of an orientation-preserving mapping (i.e. $J(\cdot, f) \geq 0$ a.e.) in the class $W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ so that (4)-(6) hold, obeys the rule of integration by parts ([4], [9], [3] and [6, Theorem 7.2.1]; see also the fundamental paper [7] by Iwaniec and Sbordone).

Theorem 1.1. *Suppose that the Jacobian of $f \in W^{1,\frac{n^2}{n+1}}_{loc}(\Omega, \mathbb{R}^n)$ is non-negative a.e. and the mapping f obeys the rule (7) of integration by parts. Then f satisfies* *the isoperimetric inequality* (2) *for every* $x_0 \in \Omega$ *and almost every radius* $r \in (0, dist(x_0, \partial\Omega))$.

The question of the sharp constant is motivated by the study of sharp modulus of continuity properties for mappings of finite distortion; see the forthcoming papers [8] and [13].

2. Proof of Theorem 1.1

Let $B_R = B(x_0, R) \subset \Omega$ be a ball such that $\overline{B}_R \subset \Omega$. We approximate f in $W^{1,\frac{n^2}{n+1}}(B_R, \mathbb{R}^n)$ by mappings $f^i \in C^\infty(B_R, \mathbb{R}^n)$. Since the functions $|D^\sharp f^i|$ converge to $|D^\sharp f|$ in $L^1(B_R)$ (observe that the cofactor matrix is made up of $n-1$ subdeterminants of the differential matrix and $\frac{n^2}{n+1} \geq n-1$), we find by Fubini's theorem that $|D^\sharp f^i|$ converges to $|D^\sharp f|$ in $L^1(\partial B_r)$ for almost every radius $r \in (0, R)$. Fix $r \in (0, R)$ so that the functions $|D^\sharp f^i|$ converge to $|D^\sharp f|$ in $L^1(\partial B_r)$. Pick $0 < \epsilon < \frac{r}{2}$. We take a convolution approximation u_t^ϵ to the characteristic function $\chi_{B_{r-\epsilon}}$ of the ball $B_{r-\epsilon}$ by using the standard mollifiers Φ_t (see [6, Formula (4.6)]) where t is chosen to be so small that $u_t^\epsilon \in C_0^\infty(B_r)$. Then $0 \leq u_t^\epsilon \leq 1$ and so

$$\int_{B_r} u_t^\epsilon(x) J(x, f^i)\, dx \leq \int_{B_r} J(x, f^i)\, dx \leq I(n) \left(\int_{\partial B_r} |D^\sharp f^i(x)|\, dx \right)^{\frac{n}{n-1}}. \tag{9}$$

Applying Stokes' theorem for the smooth mapping f^i we find that

$$\int_{B_r} u_t^\epsilon(x) J(x, f^i)\, dx = -\int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, ..., f_n^i)\, dx. \tag{10}$$

The telescoping decomposition of the Jacobian (cf. [6, Chapter 8]) leads to the equation

$$\begin{aligned} &\int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2, ..., f_n)\, dx - \int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, ..., f_n^i)\, dx \\ = &\int_{B_r} (f_1(x) - f_1^i(x)) J(x, u_t^\epsilon, f_2, ..., f_n)\, dx \\ &+ \sum_{k=2}^n \int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2^i, ..., f_{k-1}^i, f_k - f_k^i, f_{k+1}, ..., f_n)\, dx. \end{aligned} \tag{11}$$

Combining Hadamard's inequality with Hölder's inequality we find that

$$\begin{aligned} &\left| \int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2, ..., f_n)\, dx - \int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, ..., f_n^i)\, dx \right| \\ \leq &\int_{B_r} |f_1 - f_1^i||\nabla u_t^\epsilon||Df|^{n-1} + \sum_{k=2}^n \int_{B_r} |f_1||\nabla u_t^\epsilon||Df^i|^{k-2}|Df - Df^i||Df|^{n-k} \\ \leq &|\nabla u_t^\epsilon|_{L^\infty(B_r)} \left(\int_{B_r} |f_1 - f_1^i|^{n^2} \right)^{\frac{1}{n^2}} \left(\int_{B_r} |Df|^{\frac{n^2}{n+1}} \right)^{\frac{n^2-1}{n^2}} \\ &+ C(n) |\nabla u_t^\epsilon|_{L^\infty(B_r)} \left(\int_{B_r} |f_1|^{n^2} \right)^{\frac{1}{n^2}} \left(\int_{B_r} (|Df^i| + |Df|)^{\frac{n^2}{n+1}} \right)^{\frac{n^2-n-2}{n^2}} \\ &\left(\int_{B_r} |Df - Df^i|^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n^2}}. \end{aligned} \tag{12}$$

By the Sobolev-Poincaré inequality we see that the right-hand side of inequality (12) tends to zero as i goes to infinity. Combining this with inequality (9) and equation (10) we find that

$$-\int_{B_r} f_1(x)J(x,u_t^\epsilon,f_2,...,f_n)\,dx \le I(n)\left(\int_{\partial B_r}|D^\sharp f(x)|\,dx\right)^{\frac{n}{n-1}}. \tag{13}$$

Applying the assumptions $u_t^\epsilon \in C_0^\infty(B_r)$ and (7) we conclude that

$$\int_{B_r} u_t^\epsilon(x)J(x,f)\,dx \le I(n)\left(\int_{\partial B_r}|D^\sharp f(x)|\,dx\right)^{\frac{n}{n-1}}. \tag{14}$$

Since $u_t^\epsilon(x)J(x,f) \le \chi_{B_r}(x)J(x,f)$ and $J(\cdot,f) \in L^1_{loc}(\Omega)$ by (8), we can use the dominated convergence theorem. First letting $t \to 0$ and then $\epsilon \to 0$, the claim follows.

Acknowledgements

The author wishes to express his thanks to Professor Pekka Koskela for several useful suggestions and for reading the manuscript.

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Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, Fin-40351 Jyväskylä, Finland

E-mail address: `jaonnine@maths.jyu.fi`

Current address: Department of Mathematics, University of Michigan, 525 E. University Ave., Ann Arbor, MI 48109-1109, USA

E-mail address: `jonninen@umich.edu`