PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 12, Pages 3827–3832
S 0002-9939(03)07020-5
Article electronically published on April 24, 2003

ASYMPTOTIC LIMIT FOR CONDENSATE SOLUTIONS IN THE ABELIAN CHERN-SIMONS HIGGS MODEL II

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(Communicated by David S. Tartakoff)

ABSTRACT. In this paper we show that the maximal condensate solutions $(\phi^\epsilon, A^\epsilon)$ in the Abelian Chern-Simons Higgs model converge to (ϕ_*, A_*) in higher norms, where ϕ_* is a harmonic map.

1. INTRODUCTION

In this paper we continue our study discussed in [4] on asymptotic behaviors of the maximal solutions of the following equations on Ω:

$$D_1\phi + iD_2\phi = 0, \tag{1.1}$$

$$F_A + \frac{2}{\epsilon^2}|\phi|^2(|\phi|^2 - 1) = 0. \tag{1.2}$$

Here $\epsilon > 0$, $\phi : \Omega \to \mathbb{C}$ is the complex Higgs field, $A : \Omega \to \mathbb{R}^2$ is the coupled gauge potential, $D_A\phi = \nabla\phi - iA\phi$ is the covariant derivative, and $F_A = \mathbf{curl}A$ is the magnetic field. The domain Ω is a basic lattice cell in $\mathbb{R}^2$ generated by two independent vectors $\mathbf{a}^1$ and $\mathbf{a}^2$, namely,

$$\Omega = \{x \in \mathbb{R}^2 \mid x = s_1\mathbf{a}^1 + s_2\mathbf{a}^2,\ \ 0 < s_1, s_2 < 1\}.$$

The equations (1.1) and (1.2) arise from the self-dual Abelian Chern-Simons Higgs model proposed by Hong-Kim-Pac [5] and Jackiw-Weinberg [6]. It is easy to see that (1.1) and (1.2) are invariant under the gauge transformation

$$(\phi, A) \to (e^{i\chi}\phi, A + \nabla\chi),$$

where $\chi : \Omega \to \mathbb{R}$. In view of gauge invariance we impose the following 't Hooft boundary condition on Ω:

$$\begin{aligned} \exp(i\xi_j(x + \mathbf{a}^j))\phi(x + \mathbf{a}^j) &= \exp(i\xi_j(x))\phi(x), \\ (A_k + \partial_k\xi_j)(x + \mathbf{a}^j) &= (A_k + \partial_k\xi_j)(x), \quad k = 1, 2, \\ x \in \Gamma^1 \cup \Gamma^2 - \Gamma^j, & \qquad j = 1, 2, \end{aligned} \tag{1.3}$$

where ξ_1 and ξ_2 are real-valued smooth functions defined in a neighborhood of $\Gamma^2 \cup \{\mathbf{a}^1 + \Gamma^2\}$, $\Gamma^1 \cup \{\mathbf{a}^2 + \Gamma^1\}$, respectively. Here

$$\Gamma^j = \{x \in \mathbf{R}^2 | x = s\mathbf{a}^j,\ \ 0 < s < 1\}, \quad j = 1, 2.$$

Received by the editors July 24, 2002.
2000 *Mathematics Subject Classification.* Primary 35B40, 81T13.
Key words and phrases. Chern-Simons-Higgs model, self-duality equations, maximal solutions.
This research was supported by the research fund of Hankuk University of Foreign Studies, 2002.

The solutions of (1.1) and (1.2) on Ω under the above boundary conditions are called condensate solutions.

Using the classical Jaffe-Taubes arguments [7], one can show that the equation (1.1) implies that ϕ is holomorphic up to a nonvanishing multiple factor and has exactly N zeros allowing multiplicities. Thus in light of gauge invariance we may assume that ϕ takes the form

$$\phi(x) = \exp\Big(\frac{1}{2}u(x) + i\sum_{j=1}^{k} n_j \arg(x-p_j)\Big), \tag{1.4}$$

where the points $p_1, \cdots, p_k$, called the vortex points, are the distinct zeros of ϕ with multiplicities $n_1, \cdots, n_k$, respectively. Then the equation (1.2) can be reduced to

$$\begin{aligned} &\Delta u = \frac{4}{\epsilon^2}e^u(e^u-1) + 4\pi\sum_{j=1}^{k} n_j\delta_{p_j} \quad \text{on } \Omega, \\ &u : \text{doubly periodic.} \end{aligned} \tag{1.5}$$

Here δ_p denotes the Dirac measure concentrated on the point p. Conversely, once we find a solution u of (1.5), we may recover A from (1.1) by the formula

$$A_1 + iA_2 = -2i\overline{\partial}\ln\phi, \tag{1.6}$$

where $\overline{\partial} = (\partial_1 + i\partial_2)/2$.

The first result about the existence of solutions of (1.1) and (1.2) was given by Caffarelli-Yang [2]. They showed that there is a critical value $\epsilon_c < \sqrt{|\Omega|/4\pi N}$ so that for $0 < \epsilon < \epsilon_c$ the equations $(1.1)_\epsilon$ and $(1.2)_\epsilon$ with the boundary condition (1.3) admits a maximal solution $(\phi^\epsilon, A^\epsilon)$ of the form (1.4) and (1.6). The solution $(\phi^\epsilon, A^\epsilon)$ is maximal in the sense that $|\phi^\epsilon|$ has the largest possible value among all the solutions to $(1.1)_\epsilon$ and $(1.2)_\epsilon$ with the same zeros and multiplicities. In [8] it was proved by Tarantello that there exists another solution, and some asymptotic behaviors of solutions as $\epsilon \to 0$ were studied. See also [3] and references therein for recent progress for related topics.

In [4] we showed that

Theorem 1.1. *Let* $(\phi^\epsilon, A^\epsilon)$ *be the maximal solution of* (1.1) *and* (1.2) *corresponding to* ϵ. *Let* $\Omega' = \Omega\backslash\{p_1, \cdots, p_k\}$. *Then for each* $\alpha \in (0,1)$, *we have*

$$(\phi^\epsilon, A^\epsilon) \to (\phi_*, A_*) \quad \text{in} \quad C^{1,\alpha}_{loc}(\Omega', \mathbb{C}) \times C^{0,\alpha}_{loc}(\Omega', \mathbb{R}^2)$$

as $\epsilon \to 0$, *where* (ϕ_*, A_*) *belongs to* $W^{2,p}_{loc}(\Omega', \mathbb{C}) \times W^{1,p}_{loc}(\Omega', \mathbb{R}^2)$ *for all* $p > 1$ *and satisfies*

$$\begin{aligned} \Delta\phi_* + \phi_*|\nabla\phi_*|^2 &= 0, \\ |\phi_*| &= 1, \\ \mathbf{deg}(\phi_*, p_j) &= n_j, \\ A_* &= -i\overline{\phi_*}\nabla\phi_* \end{aligned} \tag{1.7}$$

on Ω'. *In fact,*

$$\phi_*(z) = \prod_{j=1}^{k} \frac{(z-p_j)^{n_j}}{|z-p_j|^{n_j}} \tag{1.8}$$

on Ω'.

The purpose of the present paper is to obtain the speed of the convergence of $(\phi^\epsilon, A^\epsilon)$ to (ϕ_*, A_*) in higher norms. In fact, we establish

Theorem 1.2. *Let* $(\phi^\epsilon, A^\epsilon)$ *be the maximal solution of* (1.1) *and* (1.2) *corresponding to* ϵ. *Let* $\Omega' = \Omega\backslash\{p_1, \cdots, p_k\}$. *Then for each positive integer* s, *we have*

$$(\phi^\epsilon, A^\epsilon) \to (\phi_*, A_*) \qquad in \quad C^s_{loc}(\Omega', \mathbb{C}) \times C^{s-1}_{loc}(\Omega', \mathbb{R}^2).$$

Furthermore for each $K \subset\subset \Omega'$, *as* $\epsilon \to 0$,

$$\|\phi^\epsilon - \phi_*\|_{C^s(K)} \le C_{K,s}\epsilon^2 \tag{1.9}$$

and

$$\|A^\epsilon - A_*\|_{C^{s-1}(K)} \le C_{K,s}\epsilon^2. \tag{1.10}$$

In the next section we give a proof of Theorem 1.2.

2. Proof of Theorem 1.2

We notice that the equation (1.5) can be regarded as an exponential nonlinearity version of the following Ginzburg-Landau equations:

$$-\Delta v = \frac{1}{\epsilon^2} v(1 - |v|^2) \qquad \text{in } \omega, \tag{2.1}$$

where $v : \omega \subset \mathbb{R}^2 \to \mathbb{C}$. The main strategy of the proof of Theorem 1.2 is to apply the same arguments used in [1] to describe the asymptotic behavior of solutions of (2.1).

Let us denote

$$\Theta(x) = 2\sum_{j=1}^{k} n_j \arg(x - p_j).$$

Then it comes from (1.4), (1.6), (1.7), and (1.8) that

$$\begin{aligned} \phi^\epsilon &= \exp(u_\epsilon/2 + i\Theta/2), \\ 2A^\epsilon &= \mathbf{curl} u^\epsilon + \nabla\Theta, \\ \phi_* &= \exp(i\Theta/2), \\ 2A_* &= \nabla\Theta, \end{aligned}$$

where $\mathbf{curl} u = (\partial_2 u, -\partial_1 u)$.

In order to prove Theorem 1.2, we need some lemmas.

Lemma 2.1 ([1])**.** *Let* U *be a bounded domain in* $\mathbb{R}^n$. *Suppose that*

$$\Delta v = f \qquad in \quad U.$$

Then for each $K \subset\subset U$,

$$\|\nabla v\|^2_{L^\infty(K)} \le C_K \|v\|_{L^\infty(U)} \Big(\|f\|_{L^\infty(U)} + \|v\|_{L^\infty(U)} \Big).$$

Lemma 2.2 ([3])**.** *For every compact subset* K *of* $\Omega\backslash\{p_1, \cdots, p_k\}$,

$$0 \le 1 - |\phi^\epsilon(x)|^2 \le C_K\epsilon^2, \quad \forall x \in K, \tag{2.2}$$

as $\epsilon \to 0$.

Lemma 2.3. *For each* $K \subset\subset \Omega'$ *and nonnegative integer* s, *we have*

$$\|1 - e^{u_\epsilon}\|_{C^s(K)} \le C_{K,s}\epsilon^2 \tag{2.3}$$

as $\epsilon \to 0$.

Proof. The proof is given by induction. The case $s = 0$ follows from (2.2) and we suppose that (2.3) holds up to $s \geq 0$.

Let us choose $x_0 \in \Omega'$ and $R < \inf\{ |x_0 - p_j|/5 \ : \ j = 1, \cdots, k\}$. Set

$$w_\epsilon = \frac{1 - e^{u_\epsilon}}{\epsilon^2}.$$

Then it comes from (1.5) that

$$\Delta u_\epsilon = 4\epsilon^2 w_\epsilon^2 - 4 w_\epsilon \qquad \text{on} \quad B_{5R}(x_0), \tag{2.4}$$

where $B_{5R}(x_0)$ is a ball of radius $5R$ centered at x_0. Since $\|w_\epsilon\|_{C^s(B_{5R})} \leq C$ by induction assumption, it follows from the elliptic estimates that $\|u_\epsilon\|_{W^{s+2,p}(B_{4R})} \leq C$ for all $p > 1$. In particular,

$$\|u_\epsilon\|_{C^{s+1}(B_{4R})} \leq C. \tag{2.5}$$

A short computation yields

$$\Delta w_\epsilon = \frac{4}{\epsilon^2} e^{2u_\epsilon} w_\epsilon - \frac{1}{\epsilon^2} e^{u_\epsilon} |\nabla u_\epsilon|^2 \equiv f_\epsilon \qquad \text{on} \quad B_{4R}. \tag{2.6}$$

Then by Lemma 2.1

$$\begin{aligned} & \|\partial^{s+1} w_\epsilon\|^2_{L^\infty(B_{3R})} \\ \leq \;& C \|\partial^s w_\epsilon\|_{L^\infty(B_{4R})} \Big(\|\partial^s w_\epsilon\|_{L^\infty(B_{4R})} + \|\partial^s f_\epsilon\|_{L^\infty(B_{4R})} \Big) \\ \leq \;& C \Big(1 + \frac{1}{\epsilon^2} \|w_\epsilon\|_{C^s(B_{4R})} \cdot \|u_\epsilon\|_{C^{s+1}(B_{4R})} \Big) \\ \leq \;& C \epsilon^{-2}. \end{aligned}$$

Hence we have

$$\|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{3R})} \leq C \epsilon^{-1}. \tag{2.7}$$

Differentiating (2.4) $(s+1)$ times and applying Lemma 2.1, we deduce from (2.5) and (2.7) that

$$\begin{aligned} & \|\partial^{s+2} u_\epsilon\|^2_{L^\infty(B_{2R})} \\ \leq \;& C \|\partial^{s+1} u_\epsilon\|_{L^\infty(B_{3R})} \Big(\|\partial^{s+1} u_\epsilon\|_{L^\infty(B_{3R})} + \|\Delta \partial^{s+1} u_\epsilon\|_{L^\infty(B_{3R})} \Big) \\ \leq \;& C \Big(1 + \|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{3R})} \Big) \\ \leq \;& C \epsilon^{-1}. \end{aligned}$$

Thus

$$\|\partial^{s+2} u_\epsilon\|_{L^\infty(B_{2R})} \leq C \epsilon^{-1/2}. \tag{2.8}$$

Next differentiating (2.6) $(s+1)$ times and applying Lemma 2.1 again, we find by (2.7) and (2.8) that

$$\begin{aligned} & \|\partial^{s+2} w_\epsilon\|^2_{L^\infty(B_R)} \\ \leq \;& C \|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{2R})} \Big(\|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{2R})} + \|\partial^{s+1} f_\epsilon\|_{L^\infty(B_{2R})} \Big) \\ \leq \;& \frac{C}{\epsilon^2} \|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{2R})} \Big(\|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{2R})} + \|\partial^{s+2} u_\epsilon\|_{L^\infty(B_{2R})} \Big) \\ \leq \;& C \epsilon^{-4}. \end{aligned}$$

Therefore

$$\|\partial^{s+2}w_\epsilon\|_{L^\infty(B_R)} \le C\epsilon^{-2}. \tag{2.9}$$

Now let us rewrite (2.6) as

$$-\epsilon^2\Delta w_\epsilon + 4w_\epsilon = 8\epsilon^2 w_\epsilon^2 - 4\epsilon^4 w_\epsilon^3 - \epsilon^2\nabla u_\epsilon\cdot\nabla w_\epsilon \equiv h_\epsilon \qquad \text{on} \quad B_R. \tag{2.10}$$

Then

$$-\epsilon^2\Delta\partial^{s+1}w_\epsilon + 4\partial^{s+1}w_\epsilon = \partial^{s+1}h_\epsilon \qquad \text{on} \quad B_R.$$

It is seen from (2.5), (2.7), (2.8), and (2.9) that

$$\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)} \le C.$$

Set

$$v = \partial^{s+1}w_\epsilon - \frac{\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)}}{4}.$$

From (2.7) we are led to

$$\begin{aligned} -\epsilon^2\Delta v + 4v &\le 0 && \text{in} \quad B_R, \\ v &\le \frac{C}{\epsilon} - \frac{\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)}}{4} && \text{on} \quad \partial B_R. \end{aligned} \tag{2.11}$$

On the other hand it is easy to check that for $\epsilon < R$, the function

$$V(r) = \frac{C}{\epsilon}\exp\Big(\frac{1}{2\epsilon R}(r^2 - R^2)\Big), \qquad r = |x - x_0|$$

is a supersolution of (2.11). This implies that

$$\partial^{s+1}w_\epsilon(r) \le \frac{\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)}}{4} + \frac{C}{\epsilon}\exp\Big(-\frac{3R}{8\epsilon}\Big) \qquad \text{on } B_{R/2}.$$

Similarly,

$$\partial^{s+1}w_\epsilon(r) \ge -\frac{\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)}}{4} - \frac{C}{\epsilon}\exp\Big(-\frac{3R}{8\epsilon}\Big) \qquad \text{on } B_{R/2}.$$

As a consequence we conclude that

$$\|\partial^{s+1}w_\epsilon\|_{L^\infty(B_{R/2})} \le C,$$

and the proof is completed. □

Corollary 2.4. *For each $K \subset\subset \Omega'$ and nonnegative integer s, we have*

$$\|\nabla u_\epsilon\|_{C^s(K)} \le C_{K,s}\epsilon^2 \tag{2.12}$$

as $\epsilon \to 0$.

Proof. By (2.2) we may assume that $e^{u_\epsilon} \ge 1/2$ on K as $\epsilon \to 0$. Then (2.12) immediately follows from (2.3). □

We are now in a position to prove Theorem 1.2. Given $K \subset\subset \Omega'$, we may suppose by (2.2) that $e^{u_\epsilon} \ge 1/2$ on K. Then we observe from (2.3) that

$$\|1 - e^{u_\epsilon/2}\|_{C^s(K)} \le C_{K,s}\epsilon^2. \tag{2.13}$$

In the sequel (1.9) and (1.10) is verified from (2.12) and (2.13) by the formula

$$\begin{aligned} \phi_* - \phi^\epsilon &= (1 - e^{u_\epsilon/2})e^{i\Theta/2}, \\ A_* - A^\epsilon &= \frac{1}{2}\mathbf{curl}u_\epsilon. \end{aligned}$$

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