PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 12, Pages 3883–3892
S 0002-9939(03)07074-6
Article electronically published on April 24, 2003

ISOMORPHISMS OF SUBALGEBRAS OF NEST ALGEBRAS

FANGYAN LU

(Communicated by David R. Larson)

ABSTRACT. Let $\mathcal{T}$ be a subalgebra of a nest algebra $\mathcal{T}(\mathcal{N})$. If $\mathcal{T}$ contains all rank one operators in $\mathcal{T}(\mathcal{N})$, then $\mathcal{T}$ is said to be large; if the set of rank one operators in $\mathcal{T}$ coincides with that in the Jacobson radical of $\mathcal{T}(\mathcal{N})$, $\mathcal{T}$ is said to be radical-type. In this paper, algebraic isomorphisms of large subalgebras and of radical-type subalgebras are characterized. Let $\mathcal{N}_i$ be a nest of subspaces of a Hilbert space $\mathcal{H}_i$ and $\mathcal{T}_i$ be a subalgebra of the nest algebra $\mathcal{T}(\mathcal{N}_i)$ associated to $\mathcal{N}_i$ $(i = 1, 2)$. Let ϕ be an algebraic isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$. It is proved that ϕ is spatial if one of the following occurs: (1) $\mathcal{T}_i$ $(i = 1, 2)$ is large and contains a masa; (2) $\mathcal{T}_i$ $(i = 1, 2)$ is large and closed; (3) $\mathcal{T}_i$ $(i = 1, 2)$ is a closed radical-type subalgebra and $\mathcal{N}_i$ $(i = 1, 2)$ is quasi-continuous (i.e. the trivial elements of $\mathcal{N}_i$ are limit points); (4) $\mathcal{T}_i$ $(i = 1, 2)$ is large and one of $\mathcal{N}_1$ and $\mathcal{N}_2$ is not quasi-continuous.

1. INTRODUCTION AND PRELIMINARIES

A nest $\mathcal{N}$ is a totally ordered set of closed subspaces of a Hilbert space $\mathcal{H}$ containing (0) and $\mathcal{H}$ which is closed under intersection and closed span. By $B(\mathcal{H})$, we denote the set of all linearly bounded operators on $\mathcal{H}$, and if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, then $B(\mathcal{H}_1, \mathcal{H}_2)$ denotes the set of all linearly bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. The nest algebra denoted by $\mathcal{T}(\mathcal{N})$ associated to $\mathcal{N}$ is the set of operators in $B(\mathcal{H})$ which leave every element in $\mathcal{N}$ invariant. For $E \in \mathcal{N}$, we use E to denote both a subspace and the orthogonal projection to it. So $E^\perp$ denotes the orthogonal complement $\mathcal{H} \ominus E$ and the difference $I - E$. Given a nest $\mathcal{N}$ and $E \in \mathcal{N}$, define $E_- = \bigvee\{N : N < E, N \in \mathcal{N}\}$ and $E_+ = \bigwedge\{N : N > E, N \in \mathcal{N}\}$. $\mathcal{N}$ is continuous if $E_- = E$ for each $E \in \mathcal{N}$; $\mathcal{N}$ is quasi-continuous if $(0)_+ = (0)$ and $\mathcal{H}_- = \mathcal{H}$; $\mathcal{N}$ is maximal if $\dim(E \ominus E_-) \leq 1$ for every $E \in \mathcal{N}$; $\mathcal{N}$ is sub-maximal if $\dim((0)_+) \leq 1$ and $\dim(\mathcal{H} \ominus \mathcal{H}_-) \leq 1$.

Let x and y be non-zero vectors in $\mathcal{H}$. Then rank one operator $x \otimes y$ is defined by $(x \otimes y)z = (z, y)x$ for any $z \in \mathcal{H}$. Let $\mathcal{T}(\mathcal{N})$ be a nest algebra and $\mathcal{R}_\mathcal{N}$ be the Jacobson radical of $\mathcal{T}(\mathcal{N})$. It is well known that $x \otimes y$ belongs to $\mathcal{T}(\mathcal{N})$ if and only if there is E in $\mathcal{N}$ such that $x \in E$ and $y \in E_-^\perp$, and to $\mathcal{R}_\mathcal{N}$ if and only if there is E in $\mathcal{N}$ such that $x \in E$ and $y \in E^\perp$. Thus $\mathcal{R}_\mathcal{N}$ contains all rank one operators in $\mathcal{T}(\mathcal{N})$ only when $\mathcal{N}$ is continuous.

Received by the editors September 24, 2001 and, in revised form, August 8, 2002.
2000 *Mathematics Subject Classification.* Primary 47L75, 47L35.
Key words and phrases. Algebraic isomorphisms, nest algebras, large subalgebras, radical-type algebras, rank one operators, spatially implemented.

Definition 1.1. Let $\mathcal{T}$ be a subalgebra of $\mathcal{T}(\mathcal{N})$. If $\mathcal{T}$ contains all rank one operators in $\mathcal{T}(\mathcal{N})$, then we say that $\mathcal{T}$ is large. If the set of rank one operators in $\mathcal{T}$ coincides with that in the Jacobson radical of $\mathcal{T}(\mathcal{N})$, we say that $\mathcal{T}$ is radical-type.

In the fundamental work [6] on the theory of nest algebras, R. Ringrose established the Isomorphism Theorem of nest algebras which states that every algebraic isomorphism between two nest algebras is necessarily spatial. In the same paper, he also proved that

Theorem 1.2 ([6, Theorem 4.1])**.** *Let $\mathcal{N}_i$ be a nest of subspaces of a Hilbert space $\mathcal{H}_i$ and $\mathcal{T}_i$ be a subalgebra of nest algebra $\mathcal{T}(\mathcal{N}_i)$ associated to $\mathcal{N}_i$ $(i = 1, 2)$. Let ϕ be an algebraic isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$. Suppose that* (1) *$\mathcal{T}_i$ is large;* (2) *there is a maximal abelian *-subalgebra (masa) $\mathcal{A}_i$ of $B(\mathcal{H}_i)$ in $\mathcal{T}_i$;* (3) $\phi(\mathcal{A}_1) = \mathcal{A}_2$*. Then ϕ is spatially implemented, i.e. there is an invertible operator $S \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $\phi(T) = STS^{-1}$ for every $T \in \mathcal{T}_1$.*

As we observed, the Isomorphism Theorem of nest algebras is a corollary of Theorem 1.2, since in this case the isomorphism is the composition of two isomorphisms, one of which is spatial and the other satisfies hypotheses (1)-(3) in Theorem 1.2 [1, Theorem 17.5]. This suggests that condition (3) in Theorem 1.2 may be removed for appropriate subalgebras. In fact, in [3] we proved

Theorem 1.3 ([3, Theorem 3.4])**.** *Let $\mathcal{N}_i$ be a nest of subspaces of a Hilbert space $\mathcal{H}_i$ and $\mathcal{T}_i$ be a subalgebra of nest algebra $\mathcal{T}(\mathcal{N}_i)$ associated to $\mathcal{N}_i$ $(i = 1, 2)$. Let ϕ be an algebraic isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$. Suppose that* (1) *$\mathcal{H}_i \ominus (\mathcal{H}_i)_-$ and $(0_i)_+$ both have dimension ≤ 1;* (2) *there is a maximal abelian *-subalgebra (masa) $\mathcal{A}_i$ of $B(\mathcal{H}_i)$ in $\mathcal{T}_i$;* (3) *the invariant subspace lattice of $\mathcal{T}_i$ is $\mathcal{N}_i$;* (4) *$\mathcal{T}_i$ contains every rank one operator $x \otimes y$ with $x \in E$ and $y \in E^\perp$ for some element E in $\mathcal{N}_i$. Then ϕ is spatially implemented.*

In the present paper, we continue to investigate algebraic isomorphisms of subalgebras of nest algebras. We pay our attention to algebraic isomorphisms of large subalgebras and of radical-type subalgebras. Unlike Theorem 1.2 and Theorem 1.3, we do not require subalgebras to contain masas. If subalgebras contain masas, unlike Theorem 1.2 and like Theorem 1.3, we do not require the algebraic isomorphism to satisfy condition (3) in Theorem 1.2. Let $\mathcal{N}_i$ be a nest of subspaces of a Hilbert space $\mathcal{H}_i$ and $\mathcal{T}_i$ be a subalgebra of nest algebra $\mathcal{T}(\mathcal{N}_i)$ associated to $\mathcal{N}_i$ $(i = 1, 2)$. Let ϕ be an algebraic isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$. We will prove that ϕ is spatial if one of the following occurs: (1) $\mathcal{T}_i$ $(i = 1, 2)$ is large and contains a masa; (2) $\mathcal{T}_i$ $(i = 1, 2)$ is large and closed; (3) $\mathcal{T}_i$ $(i = 1, 2)$ is large and one of $\mathcal{N}_1$ and $\mathcal{N}_2$ is not quasi-continuous; (4) $\mathcal{T}_i$ $(i = 1, 2)$ is a closed radical-type subalgebra and $\mathcal{N}_i$ $(i = 1, 2)$ is quasi-continuous. In particular, every algebraic isomorphism of compact operator ideals of nest algebras is spatial and so is every isomorphism of the Jacobson radicals of nest algebras associated to quasi-continuous nests.

In order to deal with isomorphisms of large subalgebras and of radical-type subalgebras simultaneously, we introduce θ-subalgebras which are modeled on large subalgebras and radical-type subalgebras.

Definition 1.4. We say that a subalgebra $\mathcal{T}$ of $\mathcal{T}(\mathcal{N})$ is a θ-subalgebra if

(1) θ is a homomorphism from the nest $\mathcal{N}$ to itself such that $\theta(E) = E$ or E_- for every $E \in \mathcal{N}$.

(2) A rank one operator $x \otimes y$ belongs to $\mathcal{T}$ if and only if there is $E \in \mathcal{N}$ such that $x \in E$ and $y \in \theta(E)^\perp$.

Thus a large subalgebra is a θ-subalgebra where $\theta : E \to E_-$ and the Jacobson radical is also a θ-subalgebra where θ is the identity on $\mathcal{N}$. We would like to emphasize the following facts. For a θ-subalgebra $\mathcal{T}$ of $\mathcal{T}(\mathcal{N})$: (1) if $x \otimes y$ belongs to $\mathcal{T}$ and E is the smallest element containing x in $\mathcal{N}$, then $y \in \theta(E)^\perp$; (2) if $x \in E$ and $y \in E^\perp$, then $x \otimes y \in \mathcal{T}$.

In subsequent sections, isomorphisms always refer to algebraic isomorphisms.

2. Isomorphisms of θ-subalgebras

In this section, $\mathcal{N}_i$ is always a nest of subspaces of a Hilbert space $\mathcal{H}_i$ and $\mathcal{T}_i$ is a θ_i-subalgebra of the nest algebra $\mathcal{T}(\mathcal{N}_i)$ $(i = 1, 2)$. ϕ is an isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$. We use $\mathcal{N}_i^{\theta_i}$ to denote the set $\{E \in \mathcal{N}_i : E > (0) \text{ and } \theta_i(E)^\perp \neq (0)\}$.

Lemma 2.1. *Suppose that $\bigvee\{\theta_1(E)^\perp : E \in \mathcal{N}_1^{\theta_1}\} = \mathcal{H}_1$ and ϕ preserves rank one operators. Let E be in $\mathcal{N}_1^{\theta_1}$. If y_0 is a non-zero vector in $\theta_1(E)^\perp$, then there is v_0 with the property that for each $x \in E$ there exists a vector u such that $\phi(x \otimes y_0) = u \otimes v_0$.*

Proof. Let x be any non-zero vector in E. Then $x \otimes y_0$ belongs to $\mathcal{T}_1$. Suppose that $\phi(x \otimes y_0) = u \otimes v$; it suffices to prove that v is a multiple of a fixed vector v_0. We distinguish two cases.

Case 1: $\theta_1(E) < E$. Fix a unit vector x_0 in $E \ominus \theta_1(E)$. Then $x \otimes x_0$ and $x_0 \otimes y_0$ belong to $\mathcal{T}_1$. Suppose that $\phi(x \otimes x_0) = f \otimes g$ and $\phi(x_0 \otimes y_0) = u_0 \otimes v_0$. Then we have that $u \otimes v = \phi(x \otimes y_0) = \phi((x \otimes x_0)(x_0 \otimes y_0)) = (u_0, g) f \otimes v_0$. Thus v must be a multiple of v_0.

Case 2: $\theta_1(E) = E$. Then there is F in $\mathcal{N}_1$ such that $(0) < F < E$.

By an argument similar to Case 1, there is a fixed vector v_0 such that $\phi(x_1 \otimes y_0) = u_1 \otimes v_0$ for every $x_1 \in F$. For a non-zero vector $x_2 \in E \ominus F$, suppose that $\phi(x_2 \otimes y_0) = u_2 \otimes z$ and $\phi(x_1 \otimes x_2) = f \otimes g$. Then $\|x_2\|^2 (u_1 \otimes v_0) = \phi((x_1 \otimes x_2)(x_2 \otimes y_0)) = (u_2, g) f \otimes z$. Thus z is a multiple of v_0 and hence $\phi(x_2 \otimes y_0) = u_2' \otimes v_0$. Writing $x = x_1 + x_2 \in F \oplus (E \ominus F)$, we have that $\phi(x \otimes y_0) = \phi((x_1 + x_2) \otimes y_0) = u_1 \otimes v_0 + u_2' \otimes v_0 = u \otimes v_0$.

Lemma 2.2. *Suppose that $\bigvee\{\theta_1(E)^\perp : E \in \mathcal{N}_1^{\theta_1}\} = \mathcal{H}_1$ and ϕ preserves rank one operators. Let E be in $\mathcal{N}_1^{\theta_1}$. If x_0 is a non-zero vector in E, then there is u_0 with the property that for every $y \in \theta_1(E)^\perp$ there exists a vector v such that $\phi(x_0 \otimes y) = u_0 \otimes v$.*

Proof. We distinguish two cases.

Case 1: $\theta_1(E) < E$. Let y_0 be a fixed unit vector in $E \ominus \theta_1(E)$. Then $x_0 \otimes y_0 \in \mathcal{T}_1$. Suppose that $\phi(x_0 \otimes y_0) = u_0 \otimes v_0$. For $y \in \theta_1(E)^\perp$, suppose that $\phi(x_0 \otimes y) = u \otimes v$ and $\phi(y_0 \otimes y) = f \otimes g$. Then we have that $u \otimes v = \phi((x_0 \otimes y_0)(y_0 \otimes y)) = (f, v_0) u_0 \otimes g$. It follows that u must be a multiple of u_0.

Case 2: $\theta_1(E) = E$. Then there is F such that $(0) < F < E$.

If $x_0 \in F$, by an argument similar to Case 1, the assertion holds.

Now assume that $z_0 = (E - F)x_0 \neq 0$. Fix y_0 in $\theta_1(E)^\perp$ and suppose that $\phi(x_0 \otimes y_0) = u_0 \otimes v_0$. For any $y_1 \in \theta_1(E)^\perp$ which is linearly independent of y_0, suppose that $\phi(x_0 \otimes y_1) = u_1 \otimes v_1$. Since $u_0 \otimes v_0 + u_1 \otimes v_1$ $(= \phi(x_0 \otimes (y_0 + y_1)))$ is a rank one operator, one of the pairs $\{u_1, u_0\}$ and $\{v_1, v_0\}$ must be linearly dependent. If u_1 and u_0 are linearly dependent, then we are done. If u_1 and u_0 are linearly independent, then v_1 and v_0 are linearly dependent. In this case, take x_1 in F. By an argument similar to Case 1, we have that $\phi(x_1 \otimes y_i) = f \otimes g_i$ $(i = 0, 1)$. Suppose that $\phi(x_1 \otimes z_0) = h \otimes w$. Then we have that $\|z_0\|^2 f \otimes g_i = \phi((x_1 \otimes z_0)(x_0 \otimes y_i)) = (u_i, w)h \otimes v_i$, $i = 0, 1$, which implies that g_0 and g_1 are linearly dependent, and hence y_0 and y_1 are linearly dependent, which contradicts the hypothesis that y_0 and y_1 are linearly independent.

Let $\mathcal{N}$ be a nest over $\mathcal{H}$. Let x be a non-zero vector in $\mathcal{H}$. Then $E_x = \bigwedge\{N \in \mathcal{N} : x \in N\}$ is the smallest element in $\mathcal{N}$ to which x belongs. We say that such an E_x is the smallest element of x in $\mathcal{N}$ and x is a maximal vector of E_x. For a non-zero element E in $\mathcal{N}$, we can construct a maximal vector x_E of E as follows. If $E \neq E_-$, then any non-zero vector x_E in $E \ominus E_-$ will be. If $E = E_-$, then there is an increasing sequence $\{E_k\} \subseteq \mathcal{N}$ such that $\lim_{k\to\infty} E_k = E$ in the strong operator topology. Let e_k be a unit vector in $E_{k+1} \ominus E_k$. Then $x_E = \sum_{k=1}^{\infty} \frac{1}{2^k} e_k$ is a maximal vector of E.

Theorem 2.3. *Suppose that* $\bigvee\{\theta_i(E)^\perp : E \in \mathcal{N}_i^{\theta_i}\} = \mathcal{H}_i$ $(i = 1, 2)$ *and* ϕ *carries rank one operators to rank one operators. For each* $E \in \mathcal{N}_1^{\theta_1}$, *let* x_E *be a fixed unit maximal vector of* E. *Then*

(1) *There is an order preserving map* $E \to \widehat{E}$ *from* $\mathcal{N}_1^{\theta_1}$ *to* $\mathcal{N}_2^{\theta_2}$ *such that* $\bigvee\{\theta_2(\widehat{E})^\perp : E \in \mathcal{N}_1^{\theta_1}\} = \mathcal{H}_2$.
(2) *For every* $E \in \mathcal{N}_1^{\theta_1}$, *there exists a unit vector* $u_E \in \widehat{E}$ *and a linear bijective map* A_E *from* $\theta_1(E)^\perp$ *onto* $\theta_2(\widehat{E})^\perp$ *such that* $\phi(x_E \otimes y) = u_E \otimes (A_E y)$ *for any* $y \in \theta_1(E)^\perp$.
(3) *There is a linear bijective map* A *from* $\bigcup\{\theta_1(E)^\perp : E \in \mathcal{N}_1^{\theta_1}\}$ *onto* $\bigcup\{\theta_2(\widehat{E})^\perp : E \in \mathcal{N}_1^{\theta_1}\}$ *such that* $AT^* = \phi(T)^* A$ *on* $\bigcup\{\theta_1(E)^\perp : E \in \mathcal{N}_1^{\theta_1}\}$.
(4) *Moreover, if both* ϕ *and* ϕ^{-1} *are bounded, then* A_E *and* A *are also bounded.*

Proof. (1) Let E be in $\mathcal{N}_1^{\theta_1}$. By Lemma 2.2, there is a fixed unit vector u_E with the property that for every $y \in \theta_1(E)^\perp$ there is v such that

$$\phi(x_E \otimes y) = u_E \otimes v. \tag{2.1}$$

Let $\widehat{E}$ be the smallest element of u_E in $\mathcal{N}_2$. Then the map $E \to \widehat{E}$ is well defined.

Since $\mathcal{T}_2$ is a θ_2-subalgebra, the vector v in the right side of equation (2.1) is in $\theta_2(\widehat{E})^\perp$ for every $y \in \theta_1(E)^\perp$, which implies that $\widehat{E}$ is in $\mathcal{N}_2^{\theta_2}$.

Let E_1 and E_2 be in $N_1^{\theta_1}$ such that $E_1 < E_2$. By the choice of x_E's, there is a vector y_1 in $E_1^\perp$ such that $(x_{E_2}, y_1) \neq 0$. Let y_2 be in $\theta_1(E_2)^\perp$. Then $x_{E_1} \otimes y_1$ and $x_{E_2} \otimes y_2$ belong to $\mathcal{T}_1$. Suppose that $\phi(x_{E_i} \otimes y_i) = u_{E_i} \otimes v_i$ $(i = 1, 2)$. Then $(u_{E_2}, v_1)u_{E_1} \otimes v_2 = (x_{E_2}, y_1)\phi(x_{E_1} \otimes y_2) \neq 0$ and hence $(u_{E_2}, v_1) \neq 0$. Since $u_{E_2} \in \widehat{E_2}$ and $v_1 \in \theta_2(\widehat{E_1})^\perp$, we have that $\widehat{E_2}\theta_2(E_1)^\perp \neq 0$. Thus $\theta_2(\widehat{E_1}) < \widehat{E_2}$ and hence $\widehat{E_1} \leq \widehat{E_2}$.

Let F be in $\mathcal{N}_2^{\theta_2}$. Let u_F be a unit maximal vector of F. Applying Lemma 2.2 to ϕ^{-1}, there is a unit vector x_F with the property that for every $v \in \theta_2(F)^\perp$ there is $y(v)$ such that

$$\phi(x_F \otimes y(v)) = u_F \otimes v. \tag{2.2}$$

Let E be the smallest element of x_F in $\mathcal{N}_1$. Then the vector $y(v)$ in the left side of equation (2.2) is in $\theta_1(E)^\perp$. Hence $E \in \mathcal{N}_1^{\theta_1}$ and $x_E \otimes y(v) \in \mathcal{T}_1$. By equation (2.1), we have that

$$\phi(x_E \otimes y(v)) = u_E \otimes v' \tag{2.1$'$}$$

and v' is in $\theta_2(\widehat{E})^\perp$. By Lemma 2.1 and equations (2.2) and (2.1$'$), v and v' must be linearly dependent, so $v \in \theta_2(\widehat{E})^\perp$. That is to say, for each $F \in \mathcal{N}_2^{\theta_2}$ there is $E \in \mathcal{N}_1^{\theta_1}$ such that $\theta_2(F)^\perp \subseteq \theta_2(\widehat{E})^\perp$. Thus

$$\bigvee\{\theta_2(\widehat{E})^\perp : E \in \mathcal{N}_1^{\theta_1}\} = \bigvee\{\theta_2(F)^\perp : F \in \mathcal{N}_2^{\theta_2}\} = \mathcal{H}_2.$$

(2) By (1), for any $y \in \theta_1(E)^\perp$ there is a unique v in $\theta_2(\widehat{E})^\perp$ such that $\phi(x_E \otimes y) = u_E \otimes v$. Thus the map $A_E : y \to v$ is well defined, linear, from $\theta_1(E)^\perp$ into $\theta_2(\widehat{E})^\perp$. Moreover $\phi(x_E \otimes y) = u_E \otimes (A_E y)$ for any $y \in \theta_1(E)^\perp$. Now we prove that A_E is onto.

Let $y_0 \in \theta_1(E)^\perp$. Suppose that $\phi(x_E \otimes y_0) = u_E \otimes v_0$. For any v in $\theta_2(\widehat{E})^\perp$, applying Lemma 2.2 to ϕ^{-1}, there is a vector y such that $x_E \otimes y \in \mathcal{T}_1$ and $\phi(x_E \otimes y) = u_E \otimes v$. By the choice of x_E, we have that $y \in \theta_1(E)^\perp$.

(3) Fix E_0 in $\mathcal{N}_1^{\theta_1}$. Let E be in $\mathcal{N}_1^{\theta_1}$. We want to prove that there is a scalar λ_E such that

$$\lambda_E A_E y = A_{E_0} y \quad \text{on } \theta_1(E_0)^\perp \cap \theta_1(E)^\perp. \tag{2.3}$$

If $\theta_1(E) < \theta_1(E_0)$, then both $x_E \otimes y$ and $x_{E_0} \otimes y$ belong to $\mathcal{T}_1$ for any $y \in \theta_1(E_0)^\perp$. Thus by (2) we have

$$\phi(x_E \otimes y) = u_E \otimes (A_E y), \tag{2.4}$$

$$\phi(x_{E_0} \otimes y) = u_{E_0} \otimes (A_{E_0} y). \tag{2.5}$$

Let y_0 be a fixed vector in $E_0 \ominus \theta_1(E)$ such that $(y_0, x_{E_0}) = 1$ (by the choice of x_E, such y_0 must exist). Then $u_E \otimes A_E y = \phi((x_E \otimes y_0)(x_{E_0} \otimes y)) = (u_{E_0}, A_E y_0) u_E \otimes A_{E_0} y$, and hence (2.3) holds. Likewise if $\theta_1(E) > \theta_1(E_0)$, there is also a scalar λ_E such that equation (2.3) holds. If $\theta_1(E) = \theta_1(E_0) = F$, assume that $E < E_0$. For every $y \in F^\perp$, by (2.5) and Lemma 2.1,

$$\phi(x_E \otimes y) = u \otimes A_{E_0} y. \tag{2.6}$$

Comparing (2.4) and (2.6), we deduce that $A_E y$ is a non-zero multiple of $A_{E_0} y$. Since this holds for every $y \in F^\perp$, it follows easily that A_E is a non-zero multiple of A_{E_0}. That is, equation (2.3) holds.

Similarly, for any E_1 and E_2 in $\mathcal{N}_1^{\theta_1}$, there is a scalar λ such that $\lambda A_{E_1} y = A_{E_2} y$ on $\theta_1(E_1)^\perp \cap \theta_1(E_2)^\perp$. Hence there is a scalar μ such that $\mu \lambda_{E_1} A_{E_1} y = \lambda_{E_2} A_{E_2} y$ on $\theta_1(E_1)^\perp \cap \theta_1(E_2)$. Since $\lambda_{E_1} A_{E_1} y = A_{E_0} y = \lambda_{E_2} A_{E_2} y$, on $\theta_1(E_0)^\perp \cap \theta_1(E_1)^\perp \cap \theta_1(E_2)^\perp$, we have that $\mu = 1$, that is,

$$\lambda_{E_1} A_{E_1} y = \lambda_{E_2} A_{E_2} y \quad \text{on } \theta_1(E_1)^\perp \cap \theta_1(E_2)^\perp. \tag{2.7}$$

Define A from $\bigcup\{\theta_1(E)^\perp : E \in \mathcal{N}_1^{\theta_1}\}$ to $\bigcup\{\theta_2(\widehat{E})^\perp : E \in \mathcal{N}_1^{\theta_1}\}$ by $Ay = \lambda_E A_E y$ for $y \in \theta_1(E)^\perp$. By equation (2.7), A is well defined and bijective. Moreover, we have $\phi(x_E \otimes y) = \bar{\lambda}_E^{-1} u_E \otimes (Ay)$.

Let $T \in \mathcal{T}_1$ and $y \in \theta_1(E)^\perp$, where $E \in \mathcal{N}_1^{\theta_1}$. We have $\bar{\lambda}_E^{-1} u_E \otimes (AT^* y) = \phi(x_E \otimes T^* y) = \bar{\lambda}_E^{-1} u_E \otimes (Ay)\phi(T)$. Hence $AT^* y = \phi(T)^* Ay$. Since y is arbitrary, we have that

$$AT^* = \phi(T)^* A \text{ on } \bigcup\{\theta_1(E)^\perp : E \in \mathcal{N}_1^{\theta_1}\}. \tag{2.8}$$

(4) If ϕ and ϕ^{-1} are bounded, by (2) we have that $\|A_E\| \leq \|\phi\|$ and $\|A_E^{-1}\| \leq \|\phi^{-1}\|$. By equation (2.3), $|\lambda_E| \leq \|A_E^{-1}\| \|A_{E_0}\| \leq \|\phi^{-1}\| \|\phi\|$. Consequently, by the definition of A, A is bounded.

The next goal in this section is to give a sufficient condition such that an isomorphism of θ-subalgebras preserves rank one operators and is bounded. To this end, we introduce the following concept.

Definition 2.4. Let $\mathcal{N}$ be a nest over a Hilbert space $\mathcal{H}$ and θ an order homomorphism from $\mathcal{N}$ into itself. The map θ is said to be dense if $\bigvee\{E \in \mathcal{N} : \theta(E) < \mathcal{H}\} = \mathcal{H}$ and $\bigvee\{\theta(E)^\perp : E \in \mathcal{N} \text{ and } N > (0)\} = \mathcal{H}$.

Remark 2.5. (1) For a nest $\mathcal{N}$, it is well known that $\bigvee\{E \in \mathcal{N} : E_- < \mathcal{H}\} = \mathcal{H}$ and $\bigvee\{E_-^\perp : E \in \mathcal{N} \text{ and } E > (0)\} = \mathcal{H}$. Therefore, if $\theta(E) = E_-$ for every $E \in \mathcal{N}$, then θ is dense.

(2) If $\mathcal{N}$ is quasi-continuous and θ is a map of $\mathcal{N}$ such that $E_- \leq \theta(E) \leq E$, then $(0) \leq \theta(E) \leq E < \mathcal{H}$ for every $(0) < E < \mathcal{H}$. Thus

$$\bigvee\{E \in \mathcal{N} : \theta(E) < \mathcal{H}\} \geq \bigvee\{E \in \mathcal{N} : E < \mathcal{H}\} = \mathcal{H}$$

and

$$\bigwedge\{\theta(E) : E \in \mathcal{N} \text{ and } E > (0)\} \leq \bigwedge\{E \in \mathcal{N} : E > (0)\} = (0).$$

Hence θ is dense.

The following gives a characterization of rank one operators in a θ-subalgebra, which assures that isomorphisms of θ-subalgebras preserve rank one operators under the density assumption. Recall that an element s of an abstract algebra $\mathcal{A}$ is called a single element of $\mathcal{A}$ if $asb = 0$ and $a, b \in \mathcal{A}$ implies that either $as = 0$ or $sb = 0$ [6]. It is easy to see that every rank one operator is a single element of every operator algebra containing it.

Lemma 2.6. *Let $\mathcal{T}$ be a θ-subalgebra of $\mathcal{T}(\mathcal{N})$ and suppose θ is dense. Then every non-zero single element of $\mathcal{T}$ is of rank one.*

Proof. For the case in which $\mathcal{T}$ is a large subalgebra (i.e. $\theta(E) = E_-$ for every $E \in \mathcal{N}$), refer to [6, Lemma 2.3]. Here the proof is simpler. Suppose that T has rank at least two. Since $\bigvee\{E \in \mathcal{N} : \theta(E) < \mathcal{H}\} = \mathcal{H}$, there is E_1 with $\theta(E_1) < \mathcal{H}$ such that TE_1 has rank at least two. Hence, since $\bigvee\{\theta(N)^\perp : N \in \mathcal{N} \text{ and } N > (0)\} = \mathcal{H}$, there is E_2 with $(0) < E_2$ such that $\theta(E_2)^\perp T E_1$ has rank at least two. Thus we can pick vectors x_1 and x_2 in E_1 such that $\theta(E_2)^\perp T x_1$ and $\theta(E_2)^\perp T x_2$ are non-zero and orthogonal. Take non-zero vectors g in $\theta(E_1)^\perp$ and h in E_2. Let $A = h \otimes (\theta(E_2)^\perp T x_1)$ and $B = x_2 \otimes g$. Then both A and B belong to $\mathcal{T}$. It is easy to see that $ATx_1 \neq 0$, $TBg \neq 0$, $ATB = (Tx_2, \theta(E_2)^\perp T x_1) h \otimes g = 0$. Namely, T is not a single element of $\mathcal{T}$.

Here we give an example which shows that if θ is not dense, then Lemma 2.6 may not hold. Let $\mathcal{N}$ be a nest over $\mathcal{H}$ such that $E = (0)_+ > (0)$. Then for each S in $\mathcal{R}_\mathcal{N}$, the Jacobson radical of $\mathcal{T}(\mathcal{N})$, we have that $SE = ESE = 0$. Let T be in $\mathcal{R}_\mathcal{N}$ such that $T = ETE^\perp$. Then $ST = SET = 0$ for every $S \in \mathcal{R}_\mathcal{N}$. However such T is not necessarily of rank one unless E or $E^\perp$ is of dimension one.

Theorem 2.7. *Suppose that ϕ is an isomorphism from a θ_1-subalgebra $\mathcal{T}_1$ onto a θ_2-subalgebra $\mathcal{T}_2$. If both θ_1 and θ_2 are dense, then ϕ carries rank one operators in $\mathcal{T}_1$ to rank one operators in $\mathcal{T}_2$.*

Proof. By Lemma 2.6, it is a simple algebraic exercise. We omit it.

Lemma 2.8. *Suppose that ϕ is an isomorphism from a θ_1-subalgebra $\mathcal{T}_1$ onto a θ_2-subalgebra $\mathcal{T}_2$ and $u_i \otimes v_i$ $(i = 1, 2)$ belong to $\mathcal{T}_2$. If both θ_1 and θ_2 are dense, then $T \to (u_1 \otimes v_1)\phi(T)(u_2 \otimes v_2)$ is continuous on $\mathcal{T}_1$.*

Proof. By Theorem 2.7, there are $x_1 \otimes y_1$ and $x_2 \otimes y_2$ in $\mathcal{T}_1$ such that $\phi(x_i \otimes y_i) = u_i \otimes v_i$. Thus the continuity of $(u_1 \otimes v_1)\phi(T)(u_2 \otimes v_2)$ is immediate from

$$(u_1 \otimes v_1)\phi(T)(u_2 \otimes v_2) = \phi((x_1 \otimes y_1)T(x_2 \otimes y_2)) = (Tx_2, y_1)\phi(x_1 \otimes y_2).$$

Theorem 2.9. *Suppose that ϕ is an isomorphism from a closed θ_1-subalgebra $\mathcal{T}_1$ onto a closed θ_2-subalgebra $\mathcal{T}_2$. If both θ_1 and θ_2 are dense, then ϕ is automatically continuous.*

Proof. By the closed graph theorem, it suffices to prove that ϕ is a closed operator from $\mathcal{T}_1$ into $\mathcal{T}_2$. Let T_n, T be in $\mathcal{T}_1$ and S in $\mathcal{T}_2$ such that $T_n \to T$ and $\phi(T_n) \to S$.

Let F be in $\mathcal{N}_2^{\theta_2}$ and x in F. We want to prove that $\phi(T)x = Sx$. In fact, take a non-zero vector y in $\theta_2(F)^\perp$; then $x \otimes y$ is in $\mathcal{T}_2$. For any $F' \in \mathcal{N}_2^{\theta_2}$, let u be a fixed non-zero vector in F'. Then for every $v \in \theta_2(F')^\perp$, $u \otimes v \in \mathcal{T}_2$. By the continuity of $(u \otimes v)\phi(\cdot)(x \otimes y)$ on $\mathcal{T}$, we obtain that $(u \otimes v)\phi(T)(x \otimes y) = (u \otimes v)S(x \otimes y)$, and hence $(\phi(T)x, v) = (Sx, v)$. Since $\bigvee\{\theta_2(F)^\perp : F \in \mathcal{N}_2^{\theta_2}\} = \mathcal{H}_2$, we have that $\phi(T)x = Sx$. Furthermore, since $\bigvee\{F : F \in \mathcal{N}_2^{\theta_2}\} = \mathcal{H}_2$, we have that $\phi(T) = S$.

3. Isomorphisms of large subalgebras and of radical-type subalgebras

Theorem 3.1. *Let $\mathcal{N}_i$ be a nest of subspaces of a Hilbert space $\mathcal{H}_i$ and $\mathcal{T}_i$ be a closed subalgebra of the nest algebra $\mathcal{T}_i(\mathcal{N}_i)$ associated to $\mathcal{N}_i$ $(i = 1, 2)$. Let ϕ be an isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$. Then*

(1) *if $\mathcal{T}_i$ is large $(i = 1, 2)$, then ϕ is spatial;*
(2) *if $\mathcal{T}_i$ is radical-type and $\mathcal{N}_i$ is quasi-continuous $(i = 1, 2)$, then ϕ is spatial.*

Proof. By Remark 2.5 and Theorem 2.7, ϕ carries rank one operators to rank one operators. Moreover by Theorem 2.9, ϕ is bounded and hence ϕ^{-1} is also bounded. Thus the bijective linear map A provided by Theorem 2.3 is densely defined and has a dense range and is bounded. So it can be extended to be an invertible operator V from $\mathcal{H}_1$ onto $\mathcal{H}_2$. By equation (2.8), we have that for $T \in \mathcal{T}_1$, $VT^* = \phi(T)^*V$ holds on $\mathcal{H}_1$. Therefore, $\phi(T) = V^{*-1}TV^*$.

The following corollaries are obvious.

Corollary 3.2. *An isomorphism between the compact operator ideals of nest algebras is spatially implemented.*

Let $\mathcal{R}_\mathcal{N}$ and $\mathcal{K}_\mathcal{N}$ be the Jacobson radical and the compact operator ideal of the nest algebra $\mathcal{T}(\mathcal{N})$. Then $\mathcal{R}_\mathcal{N}+\mathcal{K}_\mathcal{N}$ is a closed subalgebra of $\mathcal{T}(\mathcal{N})$ and is called the compact perturbation of $\mathcal{R}_\mathcal{N}$ [4].

Corollary 3.3. *An isomorphism between the compact perturbations of the Jacobson radicals of nest algebras is spatially implemented.*

In the following, an (left or right) ideal $\mathcal{I}$ of a nest algebra $\mathcal{T}(\mathcal{N})$ is said to be diagonal-disjoint if $\mathcal{I}\cap(\mathcal{T}(\mathcal{N})\cap\mathcal{T}(\mathcal{N})^*)=\{0\}$.

Lemma 3.4. *Let $\mathcal{I}$ be a diagonal-disjoint (left or right) ideal of a nest algebra $\mathcal{T}(\mathcal{N})$. If $\mathcal{I}$ contains the Jacobson radical of $\mathcal{T}(\mathcal{N})$, then it is radical-type.*

Proof. See Theorem 2.2 and Remark 3.8 in [5].

Corollary 3.5. *Let $\mathcal{I}_i$ be a diagonal-disjoint closed (left or right) ideal of a nest algebra $\mathcal{T}(\mathcal{N}_i)$ such that it contains the Jacobson radical of $\mathcal{T}(\mathcal{N}_i)$ $(i=1,2)$, and let ϕ be an isomorphism from $\mathcal{I}_1$ onto $\mathcal{I}_2$. If $\mathcal{N}_i$ is quasi-continuous, then ϕ is spatial.*

It is well known that the Jacobson radical, the Larson ideal and $J_\mathcal{N}$ are diagonal-disjoint and contain the Jacobson radical. So the above corollary is rich. However, here is an example that shows that if a nest $\mathcal{N}$ is not quasi-continuous, then an automorphism of the Jacobson radical may not be spatial.

Example 3.6. Let $\mathcal{N}=\{(0)<E_1<E_2<E_3<\mathcal{H}\}$ be a nest over a Hilbert space $\mathcal{H}$. Then T in the Jacobson radical $R_\mathcal{N}$ of $\mathcal{T}(\mathcal{N})$ is of the form

$$T=\begin{bmatrix}0 & T_{12} & T_{13} & T_{14}\\ 0 & 0 & T_{23} & T_{24}\\ 0 & 0 & 0 & T_{34}\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ on } \mathcal{H}=E_1\oplus(E_2\ominus E_1)\oplus(E_3\ominus E_2)\oplus E_3^\perp.$$

Let L be an operator in $B(E_1)$ and R be a non-zero operator from $E_3^\perp$ to $E_2\ominus E_1$. For $T\in R_\mathcal{N}$, define

$$\phi(T)=\begin{bmatrix}0 & T_{12} & T_{13} & T_{14}+LT_{12}R\\ 0 & 0 & T_{23} & T_{24}\\ 0 & 0 & 0 & T_{34}\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

It is easy to verify that ϕ is an algebraic automorphism of $R_\mathcal{N}$. But such ϕ is not spatial unless L is a multiple of the identity on E_1. To see this, suppose that there is an invertible operator S such that

$$ST=\phi(T)S \tag{3.1}$$

for every $T\in R_\mathcal{N}$. Suppose that $S=[S_{ij}]_{4\times4}$ on $\mathcal{H}=E_1\oplus(E_2\ominus E_1)\oplus(E_3\ominus E_2)\oplus E_3^\perp$. Rewrite $S=\left[\begin{smallmatrix}S_1 & S_2\\ S_3 & S_4\end{smallmatrix}\right]$ on $\mathcal{H}=E_2\oplus E_2^\perp$. Substituting $T\in R_\mathcal{N}$ of the form $T=\left[\begin{smallmatrix}0 & T_1\\ 0 & T_2\end{smallmatrix}\right]$ on $\mathcal{H}=E_2\oplus E_2^\perp$ (i.e. $T_{12}=0$) to (3.1), noting that $\phi(T)=T$ for this case, we have $\left[\begin{smallmatrix}0 & S_1T_1+S_2T_2\\ 0 & S_3T_1+S_4T_2\end{smallmatrix}\right]=\left[\begin{smallmatrix}T_1S_3 & T_1S_4\\ T_2S_3 & T_2S_4\end{smallmatrix}\right]$. Thus $T_1S_3=0$ for every $T_1\in B(E_2^\perp,E_2)$ and hence $S_3=0$. Let $T_2=0$. Then we have that $S_1T_1=T_1S_4$ for every $T_1\in B(E_2^\perp,E_2)$ and hence there is a scalar λ such that $S_1=\lambda E_2$ and

$S_4 = \lambda E_2^\perp$. Hence $S_2T_2 = 0$, which forces that $S_{13} = 0$ and $S_{23} = 0$. Therefore

$$S = \begin{bmatrix} \lambda & 0 & 0 & S_{14} \\ 0 & \lambda & 0 & S_{24} \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Substituting

$$T = \begin{bmatrix} 0 & T_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in (3.1) we obtain $T_{12}S_{24} + \lambda L T_{12} R = 0$ for all T_{12}. Since $\lambda \neq 0$, L must be a multiple of the identity on E_1.

Now we deal with isomorphisms of non-closed large subalgebras. As we have seen, when large subalgebras are closed, the boundedness of isomorphisms between them can be concluded by the Closed Graph Theorem. However, when large subalgebras are not closed, the Closed Graph Theorem is not applicable. In the following, we consider two special cases: nests are not quasi-continuous and large subalgebras contain masas.

Theorem 3.7. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be large subalgebras of nest algebras $\mathcal{T}(\mathcal{N}_1)$ and $\mathcal{T}(\mathcal{N}_2)$. Let ϕ be an isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$. If one of $\mathcal{N}_1$ and $\mathcal{N}_2$ is not quasi-continuous, then ϕ is spatial.*

Proof. Without loss of generality, we assume that $\mathcal{N}_1$ is not quasi-continuous.

Case 1: $(0)_+ > (0)$. Let $E \to \widehat{E}$ be the map defined in Theorem 2.3 from $\{E : E \in N_1$ and $E_- < \mathcal{H}_1\}$ to $\{F : F \in \mathcal{N}_2$ and $F_- < \mathcal{H}_2\}$. Then $(\widehat{(0)_+})_-^\perp = \bigvee\{(\widehat{E})_-^\perp : E \in \mathcal{N}_1 \text{ and } E > (0)\} = \mathcal{H}_2$. By Theorem 2.3, there is a bijective linear map A from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that

$$\phi(T)^* A = A T^*. \tag{3.2}$$

Now it suffices to prove that A is bounded. Hence it suffices to prove that A is a closed operator.

Let $\{x_n\} \subset \mathcal{H}_1$ such that $x_n \to x$ and $Ax_n \to y$. We want to show that $Ax = y$. For every rank one operator $S \in \mathcal{T}_2$, by (3.2) and the fact that ϕ preserves rank one operators, S^*A is bounded. Thus $S^*Ax_n \to S^*y$ and $S^*Ax_n \to S^*Ax$. Thus we have that $S^*Ax = S^*y$ for every rank one operator $S \in \mathcal{T}(\mathcal{N}_2)$. Hence $S^*Ax = S^*y$ for every finite rank operator $S \in \mathcal{T}(\mathcal{N}_2)$. By the density of finite rank operators in a nest algebra [2], there is a net $\{S_\alpha\}$ of finite rank operators in $\mathcal{T}(\mathcal{N}_2)$ such that S_α weakly converges to the identical operator on $\mathcal{H}_2$. For every vector u in $\mathcal{H}_2$, it follows from $S^*x = S^*y$ that $(Ax, S_\alpha u) = (y, S_\alpha u)$ and hence $(Ax, u) = (y, u)$. Thus we have that $Ax = y$.

Case 2: $(\mathcal{H}_1)_- < \mathcal{H}_1$. Let $\psi(T^*) = \phi(T)^*$ for every $T \in \mathcal{T}_1$. Then ψ is an isomorphism from $\mathcal{T}_1^*$ onto $\mathcal{T}_2^*$, where $\mathcal{T}_i^* = \{T^* : T \in \mathcal{T}_i\}$ $(i = 1, 2)$. It is easy to see that $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ are large subalgebras of $\mathcal{T}(\mathcal{N}_1^\perp)$ and $\mathcal{T}(\mathcal{N}_2^\perp)$ respectively, where $\mathcal{N}_i^\perp = \{N^\perp : N \in \mathcal{N}_i\}$. Note that $(0)_+ > (0)$ in $\mathcal{N}_1^\perp$ when $(\mathcal{H}_1)_- < \mathcal{H}_1$ in $\mathcal{N}_1$. By Case 1, there is a bounded and invertible operator A such that $\psi(T^*) = AT^*A^{-1}$ and hence $\phi(T) = (A^*)^{-1}TA^*$ for every $T \in \mathcal{T}_1$.

Theorem 3.8. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be large subalgebras of nest algebras $\mathcal{T}(\mathcal{N}_1)$ and $\mathcal{T}(\mathcal{N}_2)$. Let ϕ be an isomorphism from $\mathcal{T}_1$ onto $\mathcal{T}_2$. If $\mathcal{T}_i$ contains a masa, then ϕ is spatial.*

Proof. By Theorem 3.7, we only need consider the case when both $\mathcal{N}_1$ and $\mathcal{N}_2$ are quasi-continuous. For this case, the result follows from Theorem 1.3.

There remains the following question on isomorphisms of large subalgebras:

Question 3.9. Let $\mathcal{N}$ be a quasi-continuous nest and $\mathcal{T}$ be the algebra of all finite rank operators in $\mathcal{T}(\mathcal{N})$. Let ϕ be an automorphism of $\mathcal{T}$. Is ϕ bounded?

Acknowledgment

The author would like to thank the referee for his very helpful comments and a careful reading of the paper.

References

1. K. R. Davidson, *Nest Algebras*, Pitman Research Notes in Mathematics Series 191, Longman Scientific and Technical, Burnt Mill Harlow, Essex, UK, 1988. MR **90f:**47062
2. J. A. Erdos, *Operators of finite rank in nest algebras*, J. London Math. Soc., **43**(1968), 391-397. MR **37:**5721
3. F. Lu and S. Lu, *Isomorphisms between sub-strongly reducible maximal triangular algebras*, Proc. Amer. Math. Soc., **128**(2000), 2121-2128. MR **2001f:**47117
4. F. Lu and S. Lu, *Perturbation of radical of nest algebras*, Chinese Science Bulletin, **43**(1998), 556-560. MR **99j:**47063
5. F. Lu and S. Lu, *Finite rank operators in some ideals of nest algebras*, Acta Math. Sinica, **41**(1998), 89-95. MR **2000c:**47129
6. J. R. Ringrose, *On some algebras of operators II*, Proc. London Math. Soc., **16**(1966), 385-402. MR **33:**4703

Department of Mathematics, Suzhou University, Suzhou 215006, People's Republic of China

E-mail address: fylu@pub.sz.jsinfo.net