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INDUCED LOCAL ACTIONS ON TAUT AND STEIN MANIFOLDS

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ABSTRACT. Let $G = (\mathbb{R}, +)$ act by biholomorphisms on a taut manifold X. We show that X can be regarded as a G-invariant domain in a complex manifold X^* on which the universal complexification $(\mathbb{C}, +)$ of G acts. If X is also Stein, an analogous result holds for actions of a larger class of real Lie groups containing, e.g., abelian and certain nilpotent ones. In this case the question of Steinness of X^* is discussed.

INTRODUCTION

Let X be a complex manifold endowed with an action by biholomorphisms of a connected real Lie group G, i.e., X is a complex G-manifold. If the Lie algebra of the universal complexification $G^{\mathbb{C}}$ of G is the complexification of $Lie(G)$, then one obtains an induced local $G^{\mathbb{C}}$-action by integrating the $\mathbb{C}$-linear extension of the infinitesimal generator associated to the G-action. In many cases this can be understood as the restriction of a global $G^{\mathbb{C}}$-action, that is, it is possible to realize X as a G-invariant domain in a complex $G^{\mathbb{C}}$-manifold X^* to which we will refer as a *globalization* of the local $G^{\mathbb{C}}$-action. For instance, by a result of P. Heinzner ([H]) if X is Stein and G compact, then there exists a Stein globalization X^* with the following universal property: every holomorphic G-equivariant map on X to a complex $G^{\mathbb{C}}$-manifold extends $G^{\mathbb{C}}$-equivariantly on X^*.

Furthermore, for X Stein and G with polar complexification $G^{\mathbb{C}}$ and cocompact discrete subgroup Γ such that $G^{\mathbb{C}}/\Gamma$ is Stein, equivalent conditions for the existence of a Stein universal globalization are given in [CIT]. These can be verified to hold in many concrete situations, however it seems not to be known whether in this setting a globalization always exists. Here we first consider $(\mathbb{R}, +)$-actions on taut manifolds and we prove the following:

Let X be a taut $\mathbb{R}$-manifold. Then there exists a universal globalization X^* of the induced local $\mathbb{C}$-action.

Note that one cannot expect X^* to be taut unless the $\mathbb{R}$-action on X is trivial. If X is also Stein, we show that a similar result holds for G in the above-mentioned class of real Lie groups (Corollary 3). In this case it is natural to ask whether such

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a universal globalization is also Stein. For $G = (\mathbb{R}, +)$ it turns out that this is equivalent to a positive answer to the following open question:

Let Y be a complex manifold and assume there exist lower semicontinuous functions $\alpha, \beta : Y \to \mathbb{R}$ such that $\Omega := \{ (\lambda, y) \in \mathbb{C} \times Y \ : \ -\beta(y) < Im\,\lambda < \alpha(y) \}$ is Stein. Is Y then Stein?

We conclude by pointing out particular cases where this holds true.

Existence of globalizations

For basic facts and results on local actions and their globalizations we refer to [P] and more generally to [HI, §§1-3], from which most notations are inherited. However note that here all manifolds are assumed to be Hausdorff (cf. [HI, §3]).

Theorem 1. *Let X be a taut $\mathbb{R}$-manifold. Then there exists a universal globalization X^* of the induced local $\mathbb{C}$-action.*

Proof. Note that every leaf Σ of Palais' foliation with respect to the induced local $\mathbb{C}$-action is a non-compact Riemann surface, since its projection $p|_\Sigma : \Sigma \to \mathbb{C}$ is not constant. In particular Σ is holomorphically separable and [HI, Corollary, p. 438] applies to show univalency of such a local action. Then by [HI, Theorem 2, p. 38] there exists a possibly non-Hausdorff universal globalization X^*. The result will follow by showing that X^* is Hausdorff.

For this suppose that there exist elements x_1 and x_2 in X^* which are not topologically separable. Since $X^* = \mathbb{C} \cdot X$ and X is $\mathbb{R}$-invariant one may assume that $x_1 \in X$ and $x_2 = it \cdot x_0$ with $x_0 \in X$ and $t \in \mathbb{R}^{>0}$. Note that X is Hausdorff, thus $x_2 \notin X$ and consequently the local $\mathbb{C}$-orbit through x_0 has necessarily complex dimension one. Then one can choose a *local slice* $f : \mathbb{B}^{n-1}(1) \to X$ transversal to $\mathbb{C} \cdot x_0$ with $f(0) = x_0$ and a neighborhood $U \subset \mathbb{C}$ of 0 such that $\varphi : U \times \mathbb{B}^{n-1}(1) \to X$ defined by $\varphi(z, s) := z \cdot f(s)$ is a chart of X. Here n is the complex dimension of X and $\mathbb{B}^{n-1}(r) := \{ \ s \in \mathbb{C}^{n-1} \ : \ |s| < r \ \}$ for all $r > 0$. Let us call such a chart an *adapted chart* of X in x_0.

Now $it \cdot \varphi(rU \times \mathbb{B}^{n-1}(r))$ are open neighborhoods of x_2 for all $0 < r < 1$ and we are assuming that x_1 and x_2 are not separable. Therefore there exists a sequence (z_j, s_j) convergent to $(0, 0)$ in $U \times \mathbb{B}^{n-1}(1)$ such that $X \ni it \cdot \varphi(z_j, s_j) \to x_1$. Thus for $y_j := \varphi(z_j, s_j)$ one has $X \ni y_j \to x_0$ and $X \ni it \cdot y_j \to x_1$. Now recall that X is orbit-connected (cf. [CIT, Lemma 1.6]) and $\mathbb{R}$-invariant in X^*. Then by considering an adapted chart of X in x_1 one checks that there exists $\epsilon > 0$ such that $S := \{z \in \mathbb{C} \ : \ -\epsilon < Im\,z < t + \epsilon\} \subset \Omega(y_j)$ for all $j > 0$, where by definition $\Omega(x) := \{ z \in \mathbb{C} \ : \ z \cdot x \in X \}$ for all $x \in X$.

Define a sequence of holomorphic functions $h_j : S \to X$ by $h_j(z) := z \cdot y_j$, let $a_0,\ b_0 \in \mathbb{R}^{>0}$ be given by $\Omega(x_0) = \{z \in \mathbb{C} \ : \ -b_0 < Im\,z < a_0\}$ and note that $it \cdot x_0 \notin X$, hence $a_0 \le t$. Moreover $h_j(0) \to x_0$ while $ia_0 \cdot x_0 \notin X$ and $is \cdot x_0 \in X$, for s smaller than a_0 and close to it, imply that $h_j(a_0) \to \infty$. Since X is taut, this gives a contradiction and concludes the proof. □

Remark 2. Since X is $\mathbb{R}$-invariant and orbit-connected in X^*, there exist lower semicontinuous positive functions $a, b : X \to \mathbb{R}^{>0}$ such that

$$\Omega(x) = \{ \ z \in \mathbb{C} \ : \ -b(x) < Im\,z < a(x) \ \}$$

for all x in X, where $\Omega(x) := \{ z \in \mathbb{C} \ : \ z \cdot x \in X \}$. An analogous argument as in the above proof applies to show that on a taut manifold, a and b are continuous (if X is Stein one knows that $-a$ and $-b$ are plurisubharmonic [F]).

Let G be a real Lie group with polar complexification $G^{\mathbb{C}}$, i.e., the G-equivariant map $G \times \mathfrak{g} \to G^{\mathbb{C}}$ given by $(g, \xi) \to g \exp i\xi$ is a real analytic diffeomorphism. Furthermore assume that G admits a discrete cocompact subgroup Γ such that $G^{\mathbb{C}}/\Gamma$ is Stein. For instance all abelian and compact real Lie groups are of this kind or more generally products of the form $K \times N$, with K compact and N simply connected and nilpotent with rational structure constants (see [Ma], [GH]). Since $G^{\mathbb{C}}$ is polar, the Lie algebra of $G^{\mathbb{C}}$ is the complexification of $\mathfrak{g}$, the Lie algebra of G. As a consequence if G acts on a complex manifold one obtains a holomorphic local action of the complexification $G^{\mathbb{C}}$ by integrating the holomorphic vector fields given by the G-action. For G as above one has

Corollary 3. *Let X be a taut and Stein G-manifold. Then there exists a universal globalization X^* of the induced local $G^{\mathbb{C}}$-action.*

Proof. For $\eta \in \mathfrak{g}$, consider the $\mathbb{R}$-action on X defined by $t \cdot x := (\exp t\eta) \cdot x$ and denote by X^*_η the universal globalization of the induced local $\mathbb{C}$-action given by the above theorem. Then the corollary is a consequence of [CIT, Corollary 3.7]. □

For an action of a compact Lie group G on a Stein manifold the universal globalization X^* is automatically Stein ([H]). It would be interesting to know whether this remains true in the case where G is not compact and X^* exists. For $G = \mathbb{R}$ one has

Proposition 4. *The following statements are equivalent:*

i) *Let X be a Stein $\mathbb{R}$-manifold with universal globalization X^*. Then X^* is Stein.*
ii) *Let Y be a complex manifold and assume there exist lower semicontinuous functions $\alpha, \beta : Y \to \mathbb{R}$ such that $\Omega := \{ (\lambda, y) \in \mathbb{C} \times Y \ : \ -\beta(y) < Im\,\lambda < \alpha(y) \}$ is Stein. Then Y is Stein.*

Proof. Let Ω be as in ii) and consider the $\mathbb{R}$-action by left multiplication on the first component of $\mathbb{C} \times Y$. Then [CIT, Lemma 1.5] applies to show that $\mathbb{C} \times Y$ is the universal globalization of Ω. Thus if i) holds, then $\mathbb{C} \times Y$ is Stein and consequently so is Y, implying ii).

Conversely for X as in i) let $\mathbb{R}$ act diagonally on $\mathbb{C} \times X$ and by left multiplication on the first component of $\mathbb{C} \times X^*$. Then the map $f : \mathbb{C} \times X \to \mathbb{C} \times X^*$ given by $(\lambda, x) \to (\lambda, \lambda^{-1} \cdot x)$ is easily checked to be an $\mathbb{R}$-equivariant open embedding. In particular $f(\mathbb{C} \times X)$ is a Stein $\mathbb{R}$-invariant subdomain of $\mathbb{C} \times X^*$.

Now let $a, b : X \to \mathbb{R}^{>0}$ be as in Remark 2, fix $y \in X^*$ and choose $x \in X$ and $t \in \mathbb{R}$ such that $y = it \cdot x$. One has that

$$(\lambda, y) = (\lambda, \lambda^{-1} \cdot ((\lambda + it) \cdot x))$$

belongs to $f(\mathbb{C} \times X)$ if and only if $(\lambda + it) \cdot x \in X$, i.e., $-b(x) - t < Im\,\lambda < a(x) - t$. By defining $\alpha(y) = a(x) - t$ and $\beta(y) = b(x) + t$ (which is easily verified not to depend on the choice of x and t) for all $y \in X^*$ one has

$$f(\mathbb{C} \times X) = \{ (\lambda, y) \in \mathbb{C} \times X^* \ : \ -\beta(y) < Im\,\lambda < \alpha(y) \}$$

and statement i) follows from ii) by letting $\Omega = f(\mathbb{C} \times X)$ in $\mathbb{C} \times X^*$, which concludes the proof. $\square$

Remark 5. In the following cases it is easy to check that statement ii) holds:

1) Y is holomorphically convex.

For this, first note that for any open Stein neighborhood U in Y the restrictions of $-\alpha$ and $-\beta$ to U define the Stein domain $\Omega \cap (\mathbb{C} \times U)$ in $\mathbb{C} \times U$. It follows that $-\alpha$ and $-\beta$ are plurisubharmonic (see, e.g., [V]).

Now recall that each fiber F of the Remmert reduction of Y (cf. [GR, p. 221]) is a connected compact subspace. In particular α and β are constant on F, thus $F \cong \{z\} \times F \subset \Omega$ for any fixed z in $\mathbb{C}$ with $-\beta|_F < Im\, z < \alpha|_F$ and consequently F is holomorphically separable. By compactness and connectness it follows that F consists of a single point, hence Y is Stein.

2) Y is a domain in a Stein manifold $\hat{Y}$.

Here Ω can be regarded as an open Stein $\mathbb{R}$-invariant subdomain of $\mathbb{C} \times \hat{Y}$, where $\mathbb{R}$ acts by left multiplication on the first component. Since $\mathbb{C} \times \hat{Y}$ is Stein, then Ω is locally Stein ([DG]).

Moreover the quotient map $\mathbb{C} \times \hat{Y} \to (\mathbb{C} \times \hat{Y})/\mathbb{Z}$ is locally biholomorphic, therefore $\Omega/\mathbb{Z}$ is locally Stein in $(\mathbb{C} \times \hat{Y})/\mathbb{Z} \cong \mathbb{C}^* \times \hat{Y}$, which is Stein, and consequently so is $\Omega/\mathbb{Z}$. Finally Y is easily checked to be biholomorphic to the categorical quotient of $\Omega/\mathbb{Z}$ with respect to the natural induced S^1-action, thus it is Stein ([H, § 6.5]).

Remark 6. As already noted in the proof of Theorem 1, a complex $\mathbb{R}$-manifold admits a universal globalization X^* which is possibly non-Hausdorff. Note that the same argument used to prove Proposition 4 applies to show the analogous result in the case where X^* and Y are assumed to be in the category of possibly non-Hausdorff complex manifolds.

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