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AN APPLICATION OF BOCHNER'S TECHNIQUE TO THE DEFORMATIONS OF THE COMPLEX STRUCTURE OF $\mathbb{CP}^n$

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ABSTRACT. We consider the deformation of the complex structure of $\mathbb{CP}^n$. We show that a harmonic deformation on $\mathbb{CP}^n$ equipped with the Fubini-Study metric is trivial if its supernorm is appropriately small.

INTRODUCTION

Let $\mathbb{CP}^n$ be the n-dimensional complex projective space equipped with the Fubini-Study metric. Let T be its holomorphic tangent bundle, and let M be the underlying differentiable manifold. Suppose that φ is a smooth $(0,1)$-form on $\mathbb{CP}^n$ with value in T. If the supernorm of φ is < 1 and φ satisfies the integrability condition

$$\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi], \tag{1}$$

then such a φ determines a complex structure on M. We call φ a **deformation** and denote the deformed complex manifold by M_φ. A natural question to ask is whether M_φ is biholomorphic to $\mathbb{CP}^n$. It is well known that M_φ is biholomorphic to $\mathbb{CP}^n$ if φ and its derivatives are sufficiently small. However it seems unclear if M_φ is biholomorphic to $\mathbb{CP}^n$ for $n > 2$ if we only assume that the supernorm of φ is appropriately small. If a deformation φ also satisfies the equation

$$\bar{\partial}^*\varphi = 0, \tag{2}$$

where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$, then (1) and (2) become a strongly elliptic system if the supernorm of φ is appropriately small. Equation (2) was introduced by Kuranishi in [2] to study the versal deformation of compact complex manifolds. For convenience sake, we call a deformation φ a **harmonic deformation** if it also satisfies equation (2). One can ask if such a φ on $\mathbb{CP}^n$ is zero so that M_φ is biholomorphic to $\mathbb{CP}^n$. The arguments in [2] (see also [1]) only demonstrate that such a φ is 0 if we assume a priori that the Sobolev norm $\|\varphi\|_k$, say for $k \geq n+2$, is sufficiently small. It follows from the Sobolev embedding theorem that the $C^{1,\alpha}$ norm of such a φ must be assumed to be small. In this note we show that we can remove the assumption that the Sobolev norm of φ is sufficiently small. Using Bochner's technique, we prove that $\varphi = 0$ if the supernorm of φ is appropriately small.

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1. Preliminaries

In this section we fix some notation and list some well-known formulas needed in section 2.

Let M be an n-dimensional compact complex manifold with a Kähler metric g. Let ω be its Kähler form. Let T be the holomorphic tangent bundle of M, and let $A^{p,q}(T)$ denote the space of $C^\infty(p,q)$-forms with values in T.

In general, $e(\psi)$ denotes the exterior multiplication by ψ, i.e.,

$$e(\psi).\varphi = \psi \wedge \varphi.$$

In particular, we write $L = e(\omega)$, i.e.,

$$L\varphi = \omega \wedge \varphi,$$

so that

$$L : A^{p,q}(T) \longrightarrow A^{p+1,q+1}(T).$$

We set

$$\Lambda = *^{-1} \circ L \circ * : A^{p,q}(T) \longrightarrow A^{p-1,q-1}(T)$$

where $*$ is the Hodge star operator and Λ is the adjoint of L. We note that both L and Λ are algebraic operators.

Let ∇ be the canonical connection on T. Then

$$\nabla = \nabla' + \bar\partial$$

where

$$\nabla' : A^{p,q}(T) \longrightarrow A^{p+1,q}(T)$$

and

$$\bar\partial : A^{p,q}(T) \longrightarrow A^{p,q+1}(T).$$

We define

(a) $\quad \bar\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial,$
(b) $\quad \Box = \nabla'\nabla'^* + \nabla'^*\nabla',$

on $A^{p,q}(T)$. We have the following well-known identity:

Lemma 1. $\bar\Box = \Box + [i\Theta(T), \Lambda]$ *where* $\Theta(T)$ *is the curvature form of* T.

We also need the following result:

Lemma 2. *Let p be a point on M. Let $\{e_1, \ldots, e_n\}$ be a normal frame field of $(1,0)$ type around p. Suppose that $\varphi \in A^{0,1}(T)$. Then at p we have*

(a1) $\quad \bar\Box\varphi = -\nabla_{e_i}\nabla_{\bar e_i}\varphi + S(\varphi),$
(b1) $\quad \Box\varphi = -\nabla_{\bar e_i}\nabla_{e_i}\varphi,$

where

$$S(\varphi)(\bar X) = R(e_i, \bar X)\varphi.\bar e_i$$

for $\bar X \in T_pM^{0,1}$ *and*

$$R(e_i, \bar X) = \nabla_{e_i}\nabla_{\bar X} - \nabla_{\bar X}\nabla_{e_i} - \nabla_{[e_i, \bar X]}.$$

2. Harmonic deformations on $\mathbb{CP}^n$

In this section we prove the following result:

Theorem 1. *Suppose that φ is a harmonic deformation on $\mathbb{CP}^n$ equipped with the Fubini-Study metric. There exists a constant $\epsilon(n) > 0$ such that if $\|\varphi\|_\infty \le \epsilon(n)$, then*

$$\varphi = 0.$$

Here $\|\varphi\|_\infty$ is the supernorm of φ.

Proof. If we can show that

$$\bar{\Box}|\varphi|^2 = \Box\langle\varphi, \varphi\rangle \ge c(\|\bar{\partial}\varphi\|^2 + \|\nabla'\varphi\|^2)$$

for some constant $c > 0$ when $\|\varphi\|_\infty$ is appropriately small, then $\bar{\partial}\varphi = 0$. This together with $\bar{\partial}^*\varphi = 0$ implies $\varphi = 0$.

Suppose that p is an arbitrary point on $\mathbb{CP}^n$. Then we can find a coordinate $(z^1, \dots, z^n)$ around p such that $p = (0, \dots, 0)$ and

$$\begin{aligned}\Phi &= \log(1 + |z|^2)\\ &= |z|^2 - |z|^4/2 + |z|^6/3 - \dots\end{aligned}$$

is the Kähler potential of the Fubini-Study metric. The metric at p is given by

$$g_{i\bar{j}}(0) = \frac{\partial^2\Phi(0)}{\partial z^i \partial \bar{z}^j} = \delta_{ij}$$

and the curvature is

$$R^i_{ja\bar{b}}(0) = \frac{\partial^4(|z|^4/2)}{\partial z^a \partial \bar{z}^b \partial z^j \partial \bar{z}^i}.$$

The only nonzero $R^i_{ja\bar{b}}$ are

$$\begin{aligned}R^a_{aa\bar{a}} &= 2,\\ R^i_{ia\bar{a}} &= 1, i \ne a,\\ R^i_{ji\bar{j}} &= 1, i \ne j.\end{aligned}$$

We can find a normal frame field $\{e_1, \dots, e_n\}$ of type $(1,0)$ around p such that

$$e_i(0) = \partial/\partial z^i(0), i = 1, \dots, n.$$

Let $\{\theta^1, \dots, \theta^n\}$ be the dual frame of $\{e_1, \dots, e_n\}$. Then in a neighborhood of p, we have

$$\begin{aligned}L &= ie(\theta^k)e(\bar{\theta}^k),\\ \Lambda &= -i\tau(\bar{\theta}^k)\tau(\theta^k)\end{aligned}$$

where τ is the interior product operator. We observe

$$\begin{aligned}\Theta(T) &= \sum (R^i_{ja\bar{b}}\theta^a \wedge \bar{\theta}^b)\theta^j \otimes e_i\\ &= \Theta_1 + \Theta_2 + \Theta_3\end{aligned}$$

where

$$\Theta_1 = 2\sum(\theta^i \wedge \bar\theta^i)\theta^i \otimes e_i,$$
$$\Theta_2 = \sum_{i\neq a}(\theta^a \wedge \bar\theta^a)\theta^i \otimes e_i,$$
$$\Theta_3 = \sum_{i\neq j}(\theta^i \wedge \bar\theta^j)\theta^j \otimes e_i.$$

At p we have

$$\begin{aligned}\bar\Box\langle\varphi,\varphi\rangle &= e_i\bar e_i\langle\varphi,\varphi\rangle \\ &= e_i\langle\nabla_{\bar e_i}\varphi,\varphi\rangle + e_i\langle\varphi,\nabla_{e_i}\varphi\rangle \\ &= \langle\nabla_{e_i}\nabla_{\bar e_i}\varphi,\varphi\rangle + \langle\nabla_{\bar e_i}\varphi,\nabla_{\bar e_i}\varphi\rangle \\ &\quad + \langle\nabla_{e_i}\varphi,\nabla_{e_i}\varphi\rangle + \langle\varphi,\nabla_{\bar e_i}\nabla_{e_i}\varphi\rangle \\ &= |\nabla_{e_i}\varphi|^2 + |\nabla_{\bar e_i}\varphi|^2 + \langle\nabla_{e_i}\nabla_{\bar e_i}\varphi,\varphi\rangle \\ &\quad + \langle\varphi,\nabla_{\bar e_i}\nabla_{e_i}\varphi\rangle.\end{aligned} \tag{3}$$

It follows from the hypothesis that

$$\bar\Box\varphi = \frac{1}{2}\bar\partial^*[\varphi,\varphi]. \tag{4}$$

Lemmas 1 and 2 and (4) yield

$$\begin{aligned}&\langle\nabla_{e_i}\nabla_{\bar e_i}\varphi,\varphi\rangle + \langle\varphi,\nabla_{\bar e_i}\nabla_{e_i}\varphi\rangle \\ &= -\langle\bar\Box\varphi,\varphi\rangle + \langle\varphi,S(\varphi)\rangle - \langle\varphi,\Box\varphi\rangle \\ &= -\langle\bar\Box\varphi,\varphi\rangle - \langle\Box\varphi,\varphi\rangle + \langle\varphi,S(\varphi)\rangle \\ &= -2\langle\bar\Box\varphi,\varphi\rangle + \langle[i\Theta(T),\Lambda]\varphi,\varphi\rangle + \langle\varphi,S(\varphi)\rangle \\ &= -\langle\bar\partial^*[\varphi,\varphi],\varphi\rangle + \langle[i\Theta(T),\Lambda]\varphi,\varphi\rangle + \langle\varphi,S(\varphi)\rangle \\ &= -\langle[\varphi,\varphi],\bar\partial\varphi\rangle + \langle[i\Theta(T),\Lambda]\varphi,\varphi\rangle + \langle\varphi,S(\varphi)\rangle.\end{aligned} \tag{5}$$

$[i\Theta(T),\Lambda]\varphi$ can be written as:

$$\begin{aligned}&[i\Theta(T),\Lambda]\varphi \\ &= -i\Lambda\Theta(T)\varphi \\ &= -\sum_p \tau(\bar\theta^p)\tau(\theta^p)\Theta(T)\varphi \\ &= -\sum_{j=1}^{3}\sum_p \tau(\bar\theta^p)\tau(\theta^p)\Theta_j\varphi.\end{aligned} \tag{6}$$

Let

$$\varphi = \varphi^l_{\bar k}\bar\theta^k \otimes e_l.$$

We compute each term in (6) separately:

$$\begin{aligned}
\varphi_1 &= -\sum_p \tau(\bar\theta^p)\tau(\theta^p)\Theta_1\varphi \\
&= -\sum_p \tau(\bar\theta^p)\tau(\theta^p)\{2\sum(\theta^i\wedge\bar\theta^i)\theta^i\otimes e_i(\varphi^l_{\bar k}\bar\theta^k\otimes e_l)\} \\
&= -\sum_{i\neq k} 2\delta_{il}\varphi^l_{\bar k}\bar\theta^k\otimes e_i \\
&= -2\sum_{k\neq l}\varphi^l_{\bar k}\bar\theta^k\otimes e_l,
\end{aligned}\tag{7}$$

$$\begin{aligned}
\varphi_2 &= -\sum_p \tau(\bar\theta^p)\tau(\theta^p)\Theta_2\varphi \\
&= -\sum_p \tau(\bar\theta^p)\tau(\theta^p)\{\sum_{i\neq a}(\theta^a\wedge\bar\theta^a)\theta^i\otimes e_i(\varphi^l_{\bar k}\bar\theta^k\otimes e_l)\} \\
&= -\sum_{\substack{a\neq k\\ a\neq i}} \delta_{il}\varphi^l_{\bar k}\bar\theta^k\otimes e_i \\
&= -\sum_k (n-1)\varphi^k_{\bar k}\bar\theta^k\otimes e_k - \sum_{k\neq l}(n-2)\varphi^l_{\bar k}\bar\theta^k\otimes e_l,
\end{aligned}\tag{8}$$

$$\begin{aligned}
\varphi_3 &= -\sum_p \tau(\bar\theta^p)\tau(\theta^p)\Theta_3\varphi \\
&= -\sum_p \tau(\bar\theta^p)\tau(\theta^p)\{\sum_{i\neq j}(\theta^i\wedge\bar\theta^j)\theta^j\otimes e_i(\varphi^l_{\bar k}\bar\theta^k\otimes e_l)\} \\
&= \sum_{j\neq k} \delta_{jl}\varphi^l_{\bar k}\bar\theta^j\otimes e_k \\
&= \sum_{k\neq l}\varphi^l_{\bar k}\bar\theta^l\otimes e_k.
\end{aligned}\tag{9}$$

Summarizing (7), (8), (9), we obtain

$$[i\Theta(T),\Lambda]\varphi = -(n-1)\sum_k \bar\varphi^k_{\bar k}\bar\theta^k\otimes e_k - n\sum_{k\neq l}\varphi^l_{\bar k}\bar\theta^k\otimes e_l + \sum_{k\neq l}\varphi^l_{\bar k}\bar\theta^l\otimes e_k. \tag{10}$$

Hence

$$\begin{aligned}
&\langle[i\Theta(T),\Lambda]\varphi,\varphi\rangle \\
&\quad= -(n-1)\sum_k|\varphi^k_{\bar k}|^2 - n\sum_{k\neq l}|\varphi^l_{\bar k}|^2 + \sum_{k\neq l}\varphi^l_{\bar k}\bar\varphi^k_{\bar l}.
\end{aligned}\tag{11}$$

Now let's evaluate $S(\varphi)$:

$$\begin{aligned}
&S(\varphi)(\bar e_j) \\
&= R(e_i,\bar e_j)\varphi.\bar e_i \\
&= \varphi^l_{\bar k}\{\langle R(e_i,\bar e_j)\bar\theta^k\otimes e_l,\bar e_i\rangle + \langle\bar\theta^k\otimes R(e_i,\bar e_j)e_l,\bar e_i\rangle\} \\
&= \varphi^l_{\bar k}\{-\langle\bar\theta^k\otimes R(e_i,\bar e_j)\bar e_i\rangle e_l + \delta_{ki}R(e_i,\bar e_j)e_l\} \\
&= \varphi^l_{\bar k}\{\overline{\langle\theta^k,R(e_j,\bar e_i)e_i\rangle}e_l + \delta_{ki}R(e_i,\bar e_j)e_l\} \\
&= \varphi^l_{\bar k}\{R^k_{ij\bar i}e_l + \delta_{ki}R^p_{li\bar j}e_p\}.
\end{aligned}\tag{12}$$

Let $f(j,k,l) = R^k_{ij\bar{i}} e_l + \delta_{ki} R^p_{li\bar{j}} e_p$. When $k = l$ and $j = k$, we have

$$\begin{aligned} f(j,k,l) &= R^k_{ik\bar{i}} e_k + \delta_{ki} R^p_{ki\bar{k}} e_p \\ &= R^k_{kk\bar{k}} e_k + \sum_{i \neq k} R^k_{ik\bar{i}} e_k + R^p_{kk\bar{k}} e_p \\ &= 2e_k + (n-1)e_k + 2e_k \\ &= (n+3)e_k. \end{aligned} \tag{13}$$

When $k = l$ and $j \neq k$, we have

$$\begin{aligned} f(j,k,l) &= R^k_{ij\bar{i}} e_k + R^p_{kk\bar{j}} e_p \\ &= 0. \end{aligned} \tag{14}$$

When $k \neq l$ and $j = k$, we have

$$\begin{aligned} f(j,k,l) &= R^k_{ik\bar{i}} e_l + \delta_{ki} R^p_{li\bar{k}} e_p \\ &= R^k_{kk\bar{k}} e_l + \sum_{i \neq k} R^k_{ik\bar{i}} e_l + R^p_{lk\bar{k}} e_p \\ &= 2e_l + (n-1)e_l + e_l \\ &= (n+2)e_l. \end{aligned} \tag{15}$$

When $k \neq l$, $j \neq k$ and $j = l$, we have

$$\begin{aligned} f(j,k,l) &= R^k_{il\bar{i}} e_l + R^p_{lk\bar{l}} e_p \\ &= 0 + e_k \\ &= e_k. \end{aligned} \tag{16}$$

When $k \neq l$, $j \neq k$ and $j \neq l$, we have

$$\begin{aligned} f(j,k,l) &= 0 + R^p_{lk\bar{j}} e_p \\ &= 0. \end{aligned} \tag{17}$$

Hence we obtain

$$S(\varphi)\bar{e}_j = \sum_j (n+3)\varphi^j_{\bar{j}} e_j + \sum_{l \neq j} (n+2)\varphi^l_{\bar{j}} e_l + \sum_{k \neq j} \varphi^j_{\bar{k}} e_k. \tag{18}$$

Thus

$$\begin{aligned} S(\varphi) = &\sum_j (n+3)\varphi^j_{\bar{j}} \bar{\theta}^j \otimes e_j + \sum_{l \neq j} (n+2)\varphi^l_{\bar{j}} \bar{\theta}^j \otimes e_l \\ &+ \sum_{k \neq j} \varphi^j_{\bar{k}} \bar{\theta}^j \otimes e_k. \end{aligned} \tag{19}$$

Therefore

$$\langle \varphi, S(\varphi) \rangle = \sum_k (n+3)|\varphi^k_{\bar{k}}|^2 + \sum_{k \neq l} (n+2)|\varphi^l_{\bar{k}}|^2 + \sum_{k \neq l} \varphi^l_{\bar{k}} \bar{\varphi}^k_{\bar{l}} \tag{20}$$

and

$$\begin{aligned}
&\langle [i\Theta(T), \Lambda]\varphi, \varphi\rangle + \langle \varphi, S(\varphi)\rangle \\
&= 4\sum_k |\varphi^k_{\bar k}|^2 + 2\sum_{k\neq l} |\varphi^l_{\bar k}|^2 + 2\sum_{k\neq l} \varphi^l_{\bar k}\bar\varphi^k_{\bar l} \\
&\geq 4\sum_k |\varphi^k_{\bar k}|^2 + 2\sum_{k\neq l} |\varphi^l_{\bar k}|^2 - 2\sum_{k\neq l} |\varphi^l_{\bar k}|^2 \\
&= 4\sum_k |\varphi^k_{\bar k}|^2 \\
&\geq 0.
\end{aligned} \tag{21}$$

We observe

$$\begin{aligned}
&\|\nabla_{e_i}\varphi\|^2 + \|\nabla_{\bar e_i}\varphi\|^2 + \langle [\varphi,\varphi], \bar\partial\phi\rangle \\
&\geq \|\nabla_{e_i}\varphi\|^2 + \|\nabla_{\bar e_i}\varphi\|^2 - \|[\varphi,\varphi]\|\|\bar\partial\phi\| \\
&\geq c(\|\nabla_{e_i}\varphi\|^2 + \|\nabla_{\bar e_i}\varphi\|^2)
\end{aligned} \tag{22}$$

for some constant $c > 0$ if $\|\varphi\|_\infty$ is appropriately small. Equations (3), (5), (21) and (22) yield

$$\bar\Box\|\varphi\|^2 \geq c(\|\nabla_{e_i}\varphi\|^2 + \|\nabla_{\bar e_i}\varphi\|^2) \tag{23}$$

and this implies

$$\varphi = 0. \tag{24}$$

□

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