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ON ALTERNATING ANALOGUES OF TORNHEIM'S DOUBLE SERIES

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Abstract. In this paper, we give some evaluation formulas for alternating analogues of Tornheim's double series. These can be regarded as alternating analogues of Mordell's formulas. This gives a partial answer to the problem posed by Subbarao-Sitaramachandrarao.

1. Introduction

Tornheim considered the double series $T(p,q,r)$ defined by

$$T(p,q,r)=\sum_{m,n=1}^{\infty}\frac{1}{m^p n^q (m+n)^r}, \tag{1.1}$$

where p,q,r are nonnegative integers with $p+r>1,\ q+r>1$ and $p+q+r>2$ (see [5]). He showed that $T(p,q,N-p-q)$ is a polynomial in $\{\zeta(j)|\ 2\leq j\leq N\}$ with rational coefficients when N is odd and $N\geq 3$ (see also [2]).

In [3], Mordell gave an evaluation formula for $T(2k,2k,2k)$ for a positive integer k. Furthermore, in [4], Subbarao and Sitaramachandrarao generalized Mordell's formula, and considered alternating analogues of (1.1) defined by

$$R(p,q,r)=\sum_{m,n=1}^{\infty}\frac{(-1)^n}{m^p n^q (m+n)^r}, \tag{1.2}$$

$$S(p,q,r)=\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^p n^q (m+n)^r}. \tag{1.3}$$

They posed the problem to evaluate $R(p,p,p)$ and $S(p,p,p)$ for any positive integer p. As a partial answer to their problem, we gave an evaluation formula for $S(p,p,p)$ for any positive *odd* integer p (see [6], Corollary 3).

The purpose of this paper is to give an evaluation formula for $R(p,p,p)$ for any *odd* positive integer p with $p\geq 3$ (see Theorem 3.6). In order to prove this formula,

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we make use of the same method as we introduced in [6]. Indeed, we consider *partial* Tornheim's double series

$$\mathfrak{T}_{b_1,b_2}(p,q,r) = \sum_{m,n=0}^{\infty} \frac{1}{(2m+b_1)^p(2n+b_2)^q(2m+2n+b_1+b_2)^r}, \tag{1.4}$$

where $b_1, b_2 \in \{1,2\}$. As a result, we can write $\mathfrak{T}_{1,1}(p,p,q)$ as a rational linear combination of products of Riemann's zeta values at positive integers, when p and q are odd positive integers with $q \geq 3$ (see Proposition 3.5).

More general results on *partial* Tornheim's double series defined by (1.4) will be given in [7].

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2. Preliminaries

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, and $\mathbb{R}$ the field of real numbers. Let $i = \sqrt{-1}$. Throughout this paper we fix $\delta \in \mathbb{R}$ with $\delta > 0$. For $u \in \mathbb{R}$ with $u \in [1, 1+\delta]$ and $s \in \mathbb{R}$, we define $\rho(s;u) := \sum_{m\geq 0} (-u)^{-m}/(2m+1)^s$. If $u > 1$, then $\rho(s;u)$ is convergent for any $s \in \mathbb{Z}$. Let $\rho(s) := \rho(s;1)$. We define a set of numbers $\{\mathcal{E}_m(u)\}$ by

$$F(x;u) = \frac{2ue^x}{e^{2x}+u} = \sum_{m=0}^{\infty} \mathcal{E}_m(u)\, \frac{x^m}{m!}. \tag{2.1}$$

Note that $\{\mathcal{E}_m(1)\}$ are the Euler numbers (see, e.g., [1]). So we have $\mathcal{E}_{2j+1}(1) = 0$ $(j \in \mathbb{N}_0)$. It follows from (2.1) that if $u \in [1, 1+\delta]$, then

$$\liminf_{m\to\infty} \left(\frac{|\mathcal{E}_m(u)|}{m!} \right)^{-1/m} \geq \frac{\pi}{2}. \tag{2.2}$$

From the relation $F(x;u) = 2\sum_{n\geq 0}(-u)^{-n}e^{(2n+1)x}$, we obtain the following.

Lemma 2.1. *For $k \in \mathbb{N}_0$ and $u \in (1, 1+\delta]$,*

$$\rho(-k;u) = \frac{1}{2}\mathcal{E}_k(u). \tag{2.3}$$

For $r \in \mathbb{N}$, $p \in \mathbb{N}_0$, $u \in [1, 1+\delta]$ and $\theta \in \mathbb{R}$, we define

$$\mathcal{X}_p(\theta, r; u) := \sum_{n=0}^{\infty} \frac{(-u)^{-n} \sin^{(p)}((2n+1)\theta)}{(2n+1)^r}, \tag{2.4}$$

where we denote the l-th derivative of a function $f(\theta)$ by $f^{(l)}(\theta)$. Using the well-known relation

$$\sin^{(p)}(\theta) = \frac{i^{p-1}}{2}\left(e^{i\theta} + (-1)^{p-1}e^{-i\theta}\right) = i^{p-1}\sum_{n=0}^{\infty} \lambda_{p+1+n}\frac{(i\theta)^n}{n!}, \tag{2.5}$$

where $\lambda_j := (1+(-1)^j)/2$, we have

$$\mathcal{X}_p(\theta, r; u) = i^{p-1}\sum_{m=0}^{\infty} \rho(r-m;u)\lambda_{p+1+m}\frac{(i\theta)^m}{m!}, \tag{2.6}$$

when $u \in (1, 1+\delta]$. By (2.2) and (2.3), we see that (2.6) is uniformly convergent with respect to $u \in (1, 1+\delta]$ when $\theta \in (-\pi/2, \pi/2)$. Furthermore we define

$$\mathcal{Y}_p(\theta, r; u) := \mathcal{X}_p(\theta, r; u) - i^{p-1} \sum_{j=0}^{r} \rho(r-j; u) \lambda_{p+1+j} \frac{(i\theta)^j}{j!}, \tag{2.7}$$

for $r \in \mathbb{N}$, $p \in \mathbb{N}_0$, $u \in [1, 1+\delta]$ and $\theta \in \mathbb{R}$. When $u \in (1, 1+\delta]$,

$$\mathcal{Y}_p(\theta, r; u) = i^{p-1} \sum_{n=1}^{\infty} \rho(-n; u) \lambda_{p+1+n+r} \frac{(i\theta)^{n+r}}{(n+r)!}.$$

This is also uniformly convergent with respect to $u \in (1, 1+\delta]$ when $\theta \in (-\pi/2, \pi/2)$. So it follows from Lemma 2.1 and the fact $\mathcal{E}_{2j+1}(1) = 0$ $(j \in \mathbb{N}_0)$ that

$$\mathcal{Y}_r(\theta, r; u) \to 0 \quad (u \to 1;\ r \in \mathbb{N},\ \theta \in (-\pi/2, \pi/2)). \tag{2.8}$$

Now we define

$$\mathfrak{S}(k, s; u) := \sum_{m,n=0}^{\infty} \frac{(-u)^{-m-n}}{\{(2m+1)(2n+1)\}^{2k+1}(2m+2n+2)^s}, \tag{2.9}$$

$$\mathfrak{R}(k, s; u) := \sum_{m,n=0}^{\infty} \frac{(-u)^{-2m-n-1}}{\{(2m+1)(2m+2n+3)\}^{2k+1}(2n+2)^s}, \tag{2.10}$$

for $k \in \mathbb{N}_0$, $s \in \mathbb{Z}$, $u \in [1, 1+\delta]$. By an elementary calculation just the same as that in Lemma 3 of [6], we obtain the following.

Lemma 2.2. *For $k \in \mathbb{N}_0$, $u \in (1, 1+\delta]$ and $\theta \in \mathbb{R}$,*

$$\sum_{m=0}^{\infty} \frac{(-u)^{-m} e^{i(2m+1)\theta}}{(2m+1)^{2k+1}} \cdot \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n+1)^{2k+1}} = \sum_{m=0}^{\infty} \mathfrak{S}(k, -m; u) \frac{(i\theta)^m}{m!}, \tag{2.11}$$

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{(-u)^{-m} e^{-i(2m+1)\theta}}{(2m+1)^{2k+1}} \cdot \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n+1)^{2k+1}} \\ &\quad = \sum_{m=0}^{\infty} \mathfrak{R}(k, -m; u)\{1 + (-1)^m\} \frac{(i\theta)^m}{m!} + \sum_{m=0}^{\infty} \frac{u^{-2m}}{(2m+1)^{4k+2}}. \end{aligned} \tag{2.12}$$

For $n \in \mathbb{Z}$, $k \in \mathbb{N}_0$ and $u \in (1, 1+\delta]$, we define

$$\begin{aligned} \beta_n(k; u) :=& \frac{1}{2}\{\mathfrak{S}(k, -n; u) + (1 + (-1)^n)\mathfrak{R}(k, -n; u)\} \\ & - \sum_{\nu=0}^{k} \binom{n}{2\nu} \rho(2k+1-2\nu; u) \rho(2k+1+2\nu-n; u). \end{aligned} \tag{2.13}$$

In particular when $n \le -1$, we define $\beta_n(k; 1)$ by (2.13) with $u = 1$. By combining (2.13) and Lemma 2.2, we obtain the following.

Lemma 2.3. *For $k \in \mathbb{N}_0$, $u \in (1, 1+\delta]$ and $\theta \in \mathbb{R}$,*

$$\begin{aligned} &\mathcal{Y}_{2k+1}(\theta, 2k+1; u) \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n+1)^{2k+1}} \\ &\quad = (-1)^k \sum_{n=0}^{\infty} \beta_n(k; u) \frac{(i\theta)^n}{n!} + \frac{(-1)^k}{2} \sum_{m=0}^{\infty} \frac{u^{-2m}}{(2m+1)^{4k+2}}. \end{aligned} \tag{2.14}$$

Proof. By (2.5) and Lemma 2.2, we have

$$\begin{aligned}&\mathcal{X}_{2k+1}(\theta,2k+1;u)\sum_{n=0}^{\infty}\frac{(-u)^{-n}e^{i(2n+1)\theta}}{(2n+1)^{2k+1}}\\&=\frac{i^{2k}}{2}\sum_{n=0}^{\infty}\left\{\mathfrak{S}(k,-n;u)+(1+(-1)^n)\mathfrak{R}(k,-n;u)\right\}\frac{(i\theta)^n}{n!}\\&\quad+\frac{(-1)^k}{2}\sum_{m=0}^{\infty}\frac{u^{-2m}}{(2m+1)^{4k+2}}.\end{aligned}$$

On the other hand, by combining (2.6) and

$$\sum_{n=0}^{\infty}\frac{(-u)^{-n}e^{i(2n+1)\theta}}{(2n+1)^{2k+1}}=\sum_{n=0}^{\infty}\rho(2k+1-n;u)\frac{(i\theta)^n}{n!}, \tag{2.15}$$

we have

$$\begin{aligned}&\left\{i^{2k}\sum_{j=0}^{2k+1}\rho(2k+1-j;u)\lambda_j\frac{(i\theta)^j}{j!}\right\}\sum_{n=0}^{\infty}\frac{(-u)^{-n}e^{i(2n+1)\theta}}{(2n+1)^{2k+1}}\\&=(-1)^k\sum_{n=0}^{\infty}\sum_{\nu=0}^{k}\binom{n}{2\nu}\rho(2k+1-2\nu;u)\rho(2k+1+2\nu-n;u)\frac{(i\theta)^n}{n!}.\end{aligned}$$

By (2.13), we obtain the proof. □

Since (2.7) and (2.15) are uniformly convergent with respect to $u\in(1,1+\delta]$ when $\theta\in(-\pi/2,\pi/2)$ by (2.2), so is (2.14), and

$$\liminf_{m\to\infty}\left(\frac{|\beta_m(k;u)|}{m!}\right)^{-1/m}\geq\frac{\pi}{2}, \tag{2.16}$$

for $k\in\mathbb{N}_0$. Furthermore, by (2.8), we have

$$\lim_{u\to1}\ \beta_m(k;u)=0\qquad(m\in\mathbb{N}), \tag{2.17}$$

$$\lim_{u\to1}\ \beta_0(k;u)=-\frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{4k+2}}. \tag{2.18}$$

3. Evaluation formulas

By (2.13), we have

$$\begin{aligned}\beta_{2j+1}(k;u)=&\frac{1}{2}\mathfrak{S}(k,-2j-1;u)\\&-\sum_{\nu=0}^{k}\binom{2j+1}{2\nu}\rho(2k+1-2\nu;u)\rho(2k+2\nu-2j;u),\end{aligned} \tag{3.1}$$

for $j\in\mathbb{N}_0$.

Lemma 3.1. *For $k, l \in \mathbb{N}_0$, $p \in \{0,1\}$, $u \in (1, 1+\delta]$ and $\theta \in \mathbb{R}$,*

$$
\begin{aligned}
&\frac{1}{2}\sum_{m,n=0}^{\infty} \frac{(-u)^{-m-n}\sin^{(p)}((2m+2n+2)\theta)}{\{(2m+1)(2n+1)\}^{2k+1}(2m+2n+2)^{2l+p}} \\
&\quad - \sum_{\nu=0}^{k} \rho(2k+1-2\nu;u) \sum_{\tau=0}^{2\nu} \binom{2l+p-1+2\nu-\tau}{2l+p-1} \frac{(-\theta)^{\tau}}{\tau!} \\
&\qquad \cdot \mathcal{X}_{\tau+p}(\theta; 2k+2l+1+p+2\nu-\tau; u) \\
&\quad = i^{p-1} \sum_{j=-l-p}^{\infty} \beta_{2j+1}(k;u) \frac{(i\theta)^{2j+2l+1+p}}{(2j+2l+1+p)!}.
\end{aligned}
\tag{3.2}
$$

Proof. By (2.5) and (2.9), we have

$$
\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{(-u)^{-m-n}\sin^{(p)}((2m+2n+2)\theta)}{\{(2m+1)(2n+1)\}^{2k+1}(2m+2n+2)^{2l+p}} \\
&\qquad = i^{p-1} \sum_{m=-2l-p}^{\infty} \mathfrak{S}(k,-m;u)\lambda_{m+1} \frac{(i\theta)^{m+2l+p}}{(m+2l+p)!}.
\end{aligned}
$$

On the other hand, we use (2.5) and consider the function $f(x;d,\theta) = \sin^{(p)}(x\theta)x^{-d}$ in the argument of Lemma 6 of [6]. Then we obtain that

$$
\begin{aligned}
&\sum_{\tau=0}^{r} \binom{d-1+r-\tau}{d-1} \frac{(-\theta)^{\tau}}{\tau!} \frac{\sin^{(\tau+p)}(\theta x)}{x^{d+r+q-\tau}} \\
&\quad = i^{p-1} \sum_{m=-d}^{\infty} (-1)^r \binom{m}{r} \frac{(i\theta)^{m+d}}{(m+d)!} \lambda_{p+1+m+d}\, x^{-q-r+m},
\end{aligned}
$$

by using the well-known relation $\binom{-X}{j} = (-1)^j \binom{X+j-1}{j}$. Putting $r = 2\nu$, $q = 2k+1$ and $d = 2l+p$, we have

$$
\begin{aligned}
&\sum_{\nu=0}^{k} \rho(2k+1-2\nu;u) \sum_{\tau=0}^{2\nu} \binom{2l+p-1+2\nu-\tau}{2l+p-1} \frac{(-\theta)^{\tau}}{\tau!} \\
&\qquad \cdot \mathcal{X}_{\tau+p}(\theta; 2k+2l+1+p+2\nu-\tau; u) \\
&\quad = i^{p-1} \sum_{m=-2l-p}^{\infty} \sum_{\nu=0}^{k} \binom{m}{2\nu} \rho(2k+1-2\nu;u)\rho(2k+1+2\nu-m;u) \\
&\qquad \cdot \lambda_{m+1} \frac{(i\theta)^{m+2l+p}}{(m+2l+p)!}.
\end{aligned}
$$

Put $m = 2j+1$. Then, by (3.1), we obtain the proof. $\square$

By (2.16), we can let $u \to 1$ in both sides of (3.2) when $l \in \mathbb{N}$ and $\theta \in [-\pi/2, \pi/2]$. By (2.17), we obtain the following.

Proposition 3.2. *For $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, $p \in \{0,1\}$ and $\theta \in [-\pi/2, \pi/2]$,*

$$\begin{aligned}(3.3)\qquad &\frac{1}{2}\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\sin^{(p)}((2m+2n+2)\theta)}{\{(2m+1)(2n+1)\}^{2k+1}(2m+2n+2)^{2l+p}}\\ &-\sum_{\nu=0}^{k}\rho(2k+1-2\nu)\sum_{\tau=0}^{2\nu}\binom{2l+p-1+2\nu-\tau}{2l+p-1}\frac{(-\theta)^{\tau}}{\tau!}\\ &\qquad\cdot \mathcal{X}_{\tau+p}(\theta;2k+2l+1+p+2\nu-\tau;1)\\ &= i^{p-1}\sum_{j=-l-p}^{-1}\beta_{2j+1}(k;1)\frac{(i\theta)^{2j+2l+1+p}}{(2j+2l+1+p)!}.\end{aligned}$$

For simplicity, we let $\psi(s) := \sum_{m\geq 0} 1/(2m+1)^s = (1-2^{-s})\zeta(s)$ for $s>1$, where $\zeta(s)$ is the Riemann zeta function. It is well-known that $\sin^{(2j)}((2m+1)\pi/2) = (-1)^{j+m}$ and $\sin^{(2j+1)}((2m+1)\pi/2) = 0$ for $j,m \in \mathbb{N}_0$. Hence $\mathcal{X}_{2j}(\pi/2,s;1) = (-1)^j\psi(s)$ and $\mathcal{X}_{2j+1}(\pi/2,s;1) = 0$. Putting $p=0$, $\theta=\pi/2$ and $l=m+1$ for $m \in \mathbb{N}_0$ in (3.3), we have

$$\begin{aligned}(3.4)\qquad &\sum_{r=0}^{m}\beta_{2r-2m-1}(k;1)\frac{(i\pi/2)^{2r+1}}{(2r+1)!}\\ &= -i\sum_{\nu=0}^{k}\rho(2k+1-2\nu)\sum_{\eta=0}^{\nu}\binom{2m+1+2\nu-2\eta}{2m+1}\\ &\qquad\cdot\frac{(i\pi/2)^{2\eta}}{(2\eta)!}\psi(2k+2m+3+2\nu-2\eta),\end{aligned}$$

for $k \in \mathbb{N}_0$. We recall the following lemma which can be obtained by replacing π with $\pi/2$ in Lemma 8 of [6].

Lemma 3.3. *Suppose $\{P_m\}$ and $\{Q_m\}$ are sequences which satisfy the relation*

$$\sum_{j=0}^{m} P_{m-j}\frac{(i\pi/2)^{2j+1}}{(2j+1)!} = Q_m,$$

for any $m \in \mathbb{N}_0$. Then the relation

$$P_m = \frac{2}{i\pi}\sum_{\nu=0}^{m}\left(1-2^{2\nu+1-2m}\right)2^{2\nu+1-2m}\zeta(2m-2\nu)Q_\nu$$

holds for any $m \in \mathbb{N}_0$. Note that $\zeta(0) = -\frac{1}{2}$.

By (3.4), we can apply Lemma 3.3 with $P_m = \beta_{-2m-1}(k;1)$ and $Q_m = -i\mathcal{Z}_0(k,m)$ for $m \in \mathbb{N}_0$, where

$$\begin{aligned}(3.5)\qquad \mathcal{Z}_p(k,m) := &\sum_{\nu=p}^{k}\rho(2k+1-2\nu)\sum_{\eta=0}^{\nu-p}\binom{2m+1-2p+2\nu-2\eta}{2m+1-p}\\ &\cdot\psi(2k+2m-2p+3+2\nu-2\eta)\frac{(-1)^{\eta}(\pi/2)^{2\eta+p}}{(2\eta+p)!},\end{aligned}$$

for $p \in \{0,1\}$. Then we have

$$(3.6)\qquad \beta_{-2m-1}(k;1) = -\frac{2}{\pi}\sum_{\nu=0}^{m}\left(1-2^{2\nu+1-2m}\right)2^{2\nu+1-2m}\zeta(2m-2\nu)\mathcal{Z}_0(k,\nu),$$

for $m \in \mathbb{N}_0$.

Remark 3.4. It follows from (2.2) that both sides of

$$\sum_{m=0}^{\infty} \frac{(-u)^{-m}\cos((2m+1)\pi/2)}{(2m+1)^{2k+1}} = \sum_{n=0}^{\infty} \rho(2k+1-2n;u)\frac{(i\pi/2)^{2n}}{(2n)!}$$

are uniformly convergent with respect to $u \in (1, 1+\delta]$, when $k \in \mathbb{N}$. Letting $u \to 1$, we have

$$\sum_{j=0}^{k} \rho(2k+1-2j)\frac{(i\pi/2)^{2j}}{(2j)!} = 0,$$

because $\rho(-2m-1;u) \to 0$ $(u \to 1;\ m \in \mathbb{N}_0)$. Hence we can see that if $k \in \mathbb{N}$, then

$$\mathscr{Z}_p(k,m) = \sum_{\nu=1}^{k} \rho(2k+1-2\nu) \sum_{\eta=0}^{\nu-1} \binom{2m+1-2p+2\nu-2\eta}{2m+1-p} \cdot \psi(2k+2m-2p+3+2\nu-2\eta)\frac{(-1)^{\eta}(\pi/2)^{2\eta+p}}{(2\eta+p)!}, \tag{3.7}$$

for $p \in \{0,1\}$.

Now we can prove the following result on $\mathfrak{T}_{1,1}(2k+1,2k+1,2l+1)$ defined by (1.4).

Proposition 3.5. *For $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$,*

$$\begin{aligned}\mathfrak{T}_{1,1}&(2k+1,2k+1,2l+1)\\ &= -2\mathscr{Z}_1(k,l) + \frac{4}{\pi}\sum_{m=0}^{l}\sum_{\nu=0}^{m}\left(1-2^{2\nu+1-2m}\right)2^{2\nu+1-2m}\zeta(2m-2\nu)\\ &\qquad \cdot \mathscr{Z}_0(k,\nu)\frac{(i\pi/2)^{2l-2m}}{(2l-2m)!}.\end{aligned} \tag{3.8}$$

Proof. We put $p=1$ and $\theta=\pi/2$ in (3.3). Since $\cos((2m+2n+2)\pi/2) = (-1)^{m+n+1}$, we have

$$-\frac{1}{2}\mathfrak{T}_{1,1}(2k+1,2k+1,2l+1) = \mathscr{Z}_1(k,l) + \sum_{m=0}^{l} \beta_{-2m-1}(k;1)\frac{(i\pi/2)^{2l-2m}}{(2l-2m)!}.$$

By (3.6), we obtain the proof. □

Finally we prove an evaluation formula for $R(2k+1,2k+1,2k+1)$ for any $k \in \mathbb{N}$. By (1.1), (1.2) and (1.4), we can see that

$$\begin{aligned}R&(2k+1,2k+1,2k+1)\\ &= 2^{-6k-3}T(2k+1,2k+1,2k+1) - \mathfrak{T}_{1,1}(2k+1,2k+1,2k+1),\end{aligned} \tag{3.9}$$

for $k \in \mathbb{N}_0$. It was proved that

$$T(2k+1,2k+1,2k+1) = -4\sum_{j=0}^{k}\binom{4k-2j+1}{2k}\zeta(2j)\zeta(6k-2j+3) \tag{3.10}$$

(see [2], Eq. (1.14)). By combining (3.9), (3.10) and Proposition 3.5, we obtain the following.

Theorem 3.6. *For $k \in \mathbb{N}$,*

$$\begin{aligned} &R(2k+1, 2k+1, 2k+1) \\ &= -2^{-6k-1} \sum_{j=0}^{k} \binom{4k-2j+1}{2k} \zeta(2j)\zeta(6k-2j+3) + 2\mathcal{Z}_1(k,k) \\ &\quad - \frac{4}{\pi} \sum_{m=0}^{k} \sum_{\nu=0}^{m} \left(1 - 2^{2\nu+1-2m}\right) 2^{2\nu+1-2m} \zeta(2m-2\nu) \mathcal{Z}_0(k,\nu) \frac{(i\pi/2)^{2k-2m}}{(2k-2m)!}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{Z}_p(k,m) = \sum_{\nu=1}^{k} \rho(2k+1-2\nu) \sum_{\eta=0}^{\nu-1} \binom{2m+1-2p+2\nu-2\eta}{2m+1-p} \\ \cdot \psi(2k+2m-2p+3+2\nu-2\eta) \frac{(-1)^\eta (\pi/2)^{2\eta+p}}{(2\eta+p)!}, \end{aligned}$$

for $p \in \{0,1\}$. Note that $\rho(s) = \sum_{m\geq 0} (-1)^m/(2m+1)^s$ and $\psi(s) = (1-2^{-s})\zeta(s)$.

Example 3.7. We list several evaluation formulas for $R(2k+1, 2k+1, 2k+1)$ deduced from Theorem 3.6. Note that we use the relations

$$\rho(2j+1) = \frac{(-1)^j E_{2j}}{2(2j)!} \left(\frac{\pi}{2}\right)^{2j+1} \qquad (j \in \mathbb{N}_0),$$

where $\{E_n\}$ are the Euler numbers (see, e.g., [1]).

$$\begin{aligned} R(3,3,3) &= \frac{253}{256}\pi^2\zeta(7) - \frac{2545}{256}\zeta(9) \\ R(5,5,5) &= \frac{2039}{18432}\pi^4\zeta(11) + \frac{285565}{24576}\pi^2\zeta(13) - \frac{2056257}{16384}\zeta(15) \\ R(7,7,7) &= \frac{32639}{2211840}\pi^6\zeta(15) + \frac{913913}{491520}\pi^4\zeta(17) + \frac{40212403}{262144}\pi^2\zeta(19) \\ &\quad - \frac{896163411}{524288}\zeta(21) \\ R(9,9,9) &= \frac{522239}{275251200}\pi^8\zeta(19) + \frac{2978549}{66060288}\pi^6\zeta(21) + \frac{1194884977}{41943040}\pi^4\zeta(23) \\ &\quad + \frac{71693105055}{33554432}\pi^2\zeta(25) - \frac{1625043751045}{67108864}\zeta(27). \end{aligned}$$

Remark 3.8. More general results on *partial* Tornheim's double series $\mathfrak{T}_{b_1,b_2}(p,q,r)$ defined by (1.4) will be given in [7]. Indeed, we will be able to give more general relation formulas for $\mathfrak{T}_{b_1,b_2}(p,q,r)$.

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