

A “NONLINEAR” PROOF OF PITT’S COMPACTNESS THEOREM

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ABSTRACT. Using Stegall’s variational principle, we present a simple proof of Pitt’s theorem that bounded linear operators from ℓ_q into ℓ_p are compact for $1 \leq p < q < +\infty$.

Let φ be a bounded below lower semicontinuous function defined on a reflexive Banach space X such that $\liminf_{\|x\| \rightarrow \infty} \|x\|^{-1} \varphi(x) > 0$. Then Stegall’s variational principle yields $x_0 \in X$ and $f \in X^*$ such that $\varphi + f$ attains its minimum at x_0 (see [4], or [5], or e.g. [1, Theorem 10.20] or [3, Corollary 5.22]). Recall that a bounded linear operator T from a Banach space X into a Banach space Y is called a compact operator if $\overline{T(B_X)}$ is a compact set in Y , where B_X denotes the closed unit ball of X .

Theorem (Pitt) (see, e.g., [2, p. 54] or [1, p. 175]). *Let $1 \leq p < q < +\infty$. Then every bounded linear operator from ℓ_q into ℓ_p is compact.*

Proof. Let T be a bounded linear operator from ℓ_q into ℓ_p . Then the function

$$\varphi(x) = \|x\|_q^q - \|Tx\|_p^p, \quad x \in \ell_q$$

is bounded below and $\varphi(x) > \|x\|_q^q$ if $\|x\|_q$ is large enough. By Stegall’s variational principle there is a point $x \in \ell_q$ and there is $\xi \in \ell_q^*$ such that

$$\varphi(x+h) - \varphi(x) - \xi(h) \geq 0 \quad \text{for every } h \in \ell_q.$$

Then from the linearity of ξ ,

$$\varphi(x+h) + \varphi(x-h) - 2\varphi(x) \geq 0 \quad \text{for every } h \in \ell_q.$$

Thus

$$\|x+h\|_q^q + \|x-th\|_q^q - 2\|x\|_q^q \geq \|T(x+h)\|_p^p + \|T(x-h)\|_p^p - 2\|Tx\|_p^p \quad \text{for all } h \in \ell_q.$$

Let (x_i) be a bounded sequence in ℓ_q . By passing to a subsequence, if necessary, we may assume that (x_i) converges weakly to some $y \in \ell_q$. We will show that $\|Tx_i - Ty\| \rightarrow 0$ as $i \rightarrow \infty$. Indeed, by substituting $h = t(x_i - y)$ in the last inequality, we get

$$\begin{aligned} \|x + t(x_i - y)\|_q^q + \|x - t(x_i - y)\|_q^q - 2\|x\|_q^q \\ \geq \|Tx + tT(x_i - y)\|_p^p + \|Tx - tT(x_i - y)\|_p^p - 2\|Tx\|_p^p \end{aligned}$$

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for all $i = 1, 2, \dots$ and all $t > 0$.

Thus we get that for all $t > 0$,

$$\limsup_{i \rightarrow \infty} \|x \pm t(x_i - y)\|_q^q = \|x\|_q^q + t^q \limsup_{i \rightarrow \infty} \|x_i - y\|_q^q.$$

In order to see this, we use the fact that if $z \in \ell_q$ and $w_i \rightarrow 0$ weakly, then

$$\limsup \|z + w_i\|^q = \|z\|^q + \limsup \|w_i\|^q.$$

Indeed, first assume that $z_i = 0$ for all $i \geq i_0$ for some i_0 . Choose $\|\tilde{w}_i - w_i\|_q \rightarrow 0$ where $\tilde{w}_i = 0$ for all $i \leq i_0$. Since the desired equality trivially holds for \tilde{w}_i and the q -th power of the norm function is Lipschitz on bounded sets, we get that the equality holds for all finitely supported z . The conclusion then follows from the Lipschitz property of the function in question on bounded sets and from the density of finitely supported elements in ℓ_q . For the same reason, since $Tx_i \rightarrow Ty$ weakly, we get for all $t > 0$,

$$\limsup_{i \rightarrow \infty} \|Tx \pm tT(x_i - y)\|_p^p = \|Tx\|_p^p + t^p \limsup_{i \rightarrow \infty} \|T(x_i - y)\|_p^p.$$

Thus

$$2t^q \limsup_{i \rightarrow \infty} \|x_i - y\|_q^q \geq 2t^p \limsup_{i \rightarrow \infty} \|T(x_i - y)\|_p^p$$

for all $t > 0$, and therefore, $\|T(x_i - y)\|_p \rightarrow 0$ as $i \rightarrow \infty$. \square

The usual proofs of Pitt's theorem involve the theory of Schauder bases in ℓ_p spaces (cf., e.g., [1, p. 175] or [2, p. 54]).

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