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A q-ANALOGUE OF THE WHITTAKER-SHANNON-KOTEL'NIKOV SAMPLING THEOREM

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Abstract. The Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem plays an important role not only in harmonic analysis and approximation theory, but also in communication engineering since it enables engineers to reconstruct analog signals from their samples at a discrete set of data points. The main aim of this paper is to derive a q-analogue of the Whittaker-Shannon-Kotel'nikov sampling theorem. The proof uses recent results in the theory of q-orthogonal polynomials and basic hypergeometric functions, in particular, new results on the addition theorems for q-exponential functions.

1. Introduction

Let $\sigma > 0$ and $1 \leq p \leq \infty$, and denote by B_σ^p the set of all entire functions f of exponential type with type at most σ that belong to $L^p(\mathcal{R})$ when restricted to the real line. That is, $f \in B_\sigma^p$ if and only if f is an entire function satisfying

$$|f(z)| \leq \sup_{x\in\mathcal{R}} |f(x)| \exp(\sigma|y|), \quad z = x + iy,$$

and

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx < \infty, \quad \text{if } 1 \leq p < \infty, \quad \sup_{x\in\mathcal{R}} |f(x)| < \infty, \quad \text{if } p = \infty.$$

A function f is said to be *band-limited* to $[-\sigma, \sigma]$ if and only if $f \in B_\sigma^2$. A nice characterization of the space B_σ^2 is given by the Paley-Wiener theorem [11]. It may be stated as follows: A function f belongs to B_σ^2 if and only if it is representable in the form

$$f(t) = \int_{-\sigma}^{\sigma} e^{ixt} g(x) \, dx \quad (t \in \mathcal{R}), \quad \text{for some function } g \in L^2(-\sigma, \sigma). \tag{1.1}$$

The Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem states that if a function f is band-limited to $[-\sigma, \sigma]$, then f can be reconstructed from its samples,

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$f(k\pi/\sigma)$, that are taken at the equally spaced nodes $k\pi/\sigma$ on the time axis $\mathcal{R}$. The construction formula is

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin(\sigma t - k\pi)}{(\sigma t - k\pi)} \qquad (t \in \mathcal{R}), \tag{1.2}$$

the series being absolutely and uniformly convergent on $\mathcal{R}$. See, e.g., [13, p. 16].

The series in (1.2) can be put in the form

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{G(t)}{G'(t_k)(t - t_k)}, \tag{1.3}$$

where $G(t) = \sin\sigma t$, and $t_k = k\pi/\sigma$. We shall call any series of the form (1.3) a *Lagrange-type interpolation series* whether $G(t) = \sin\sigma t$ or not. This series is reminiscent of the *Lagrange interpolation formula*

$$\sum_{k=0}^{n} f(t_k) \frac{G_n(t)}{G_n'(t_k)(t - t_k)}, \quad \text{where} \quad G_n(t) = \prod_{k=0}^{n} (t - t_k),$$

which gives a polynomial of degree n that coincides with f at the points t_k, $k = 0, 1, 2, \ldots, n$.

One of the important generalizations of the WSK sampling theorem is the Paley-Wiener-Levinson Sampling Theorem, which can be stated as follows: Let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that

$$D = \sup_{k\in\mathbb{Z}} |t_k - k| < \frac{1}{4},$$

and let

$$G(t) = (t - t_0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right).$$

Then for any $f \in B_\pi^2$, we have

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{G(t)}{G'(t_k)(t - t_k)} \qquad (t \in \mathcal{R}), \tag{1.4}$$

the series being uniformly convergent on compact sets ([10], [13, p. 24]). When $t_k = k$, $G(t)$ reduces to $\sin\pi t/\pi$ and we obtain the WSK theorem.

The purpose of this paper is to derive a q-analogue of the WSK sampling theorem. Although the proof of the WSK is simple, finding its q-analogue was at first far from obvious, because there are several q-generalizations of the exponential function that can play the role of the exponential function in (1.1). Even after we found the right kind of exponential function, we needed a multiplication formula for it, as well as an integration formula for the product. Fortunately, these formulae have recently been discovered by one of the authors and Stanton [8].

The paper is organized as follows. In the next section we introduce some of the preliminary material that will be used in the sequel. The main result will then be presented in Section 3.

2. Preliminaries

We use the notation

$$(2.1)\qquad (a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}),$$

$$(2.2)\qquad (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha,q)_\infty}, \quad (a_1,...,a_m;q)_n = \prod_{l=1}^{m}(a_l;q)_n, \quad |q|<1,$$

where $n = 1,2,\dots$ or ∞, as in [4], [1]. There are a number of q-analogues of the cosine and sine functions (see [3], [7], [4]); however, the ones we shall adopt in this paper are the ones introduced by Ismail and Zhang in [7] and use the latter notation as in [8]. The q-cosine and sine functions are defined by

$$C_q(\cos\theta;\omega) = \frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\ {}_2\phi_1(-qe^{2i\theta},-qe^{-2i\theta};q;q^2,-\omega^2)$$

and

$$S_q(\cos\theta;\omega) = \frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\left(\frac{2q^{1/4}\omega}{1-q}\right)\cos\theta$$
$$\times {}_2\phi_1(-q^2e^{2i\theta},-q^2e^{-2i\theta};q^3;q^2,-\omega^2).$$

The symbol ${}_{r+1}\phi_r$ stands for the function

$${}_{r+1}\phi_r(a_1,\dots,a_{r+1};b_1,\dots,b_r;q,z) = \sum_{n=0}^{\infty}\frac{(a_1,\dots,a_{r+1};q)_n}{(q,b_1,\dots,b_r;q)_n}z^n.$$

Set

$$(2.3)\ w(\cos\theta) = \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{\sin\theta\,(q^{1/2}e^{2i\theta},q^{1/2}e^{-2i\theta};q)_\infty}, \quad W(x) = \sqrt{1-x^2}\ w(x).$$

It has been shown that, [3], [6],

$$\int_{-1}^{1} C_q(x;\omega)C_q(x;\omega')w(x)\ dx = \int_{-1}^{1} S_q(x;\omega)S_q(x;\omega')w(x)\ dx = 0\,,$$
$$\int_{-1}^{1} C_q(x;\omega)S_q(x;\omega')w(x)\,dx = 0,$$
$$\int_{-1}^{1} C_q^2(x;\omega)w(x)\,dx = \int_{-1}^{1} S_q^2(x;\omega)w(x)\,dx$$
$$= \pi\frac{(q^{1/2},-q^{1/2}\omega^2;q)_\infty}{(q,-\omega^2;q)_\infty}\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\ {}_2\phi_1(q^{1/2},-\omega^2;-q^{1/2}\omega^2;q,q)$$

where ω and ω' are different solutions of the equation

$$(2.4)\qquad S_q\left(\frac{1}{2}(q^{1/4}+q^{1/4});\omega\right) = \frac{\left(-i\omega;\sqrt{q}\right)_\infty - \left(i\omega;\sqrt{q}\right)_\infty}{2i(-q\omega^2;q^2)_\infty} = 0.$$

We have denoted the nonnegative zeros of the above equation by ω_n where $\omega_0 = 0$, $\omega_1 < \omega_2 < \dots$.

Ismail and Zhang [7] defined the q-sine and q-cosine functions through their q-exponential function in the standard way, i.e.,

$$(2.5)\qquad \mathcal{E}_q(x;i\omega) = C_q(x;\omega) + iS_q(x;\omega).$$

Bustoz and Suslov [3] proved that $\{\mathcal{E}_q(x; i\omega_n)\}_{n=-\infty}^{\infty}$ is a complete orthogonal system in $L^2(-1,1)$ with respect to the weight function $w(x)$ and a simple proof and generalizations thereof were given by Ismail in [6]. More precisely, the system $\{\mathcal{E}_q(x; i\omega_n)\}_{n=-\infty}^{\infty}$ is a complete orthogonal system whose orthogonality relation is

$$\int_{-1}^{1} \mathcal{E}_q(x; i\omega_m)\ \mathcal{E}_q(x; -i\omega_n) w(x) dx = 2k(\omega_n)\delta_{m,n} \tag{2.6}$$

and

$$k(\omega) = \pi \frac{(q^{1/2}, -q^{1/2}\omega^2; q)_\infty (-\omega^2; q^2)_\infty}{(q, -\omega^2; q)_\infty (-q\omega^2; q^2)_\infty} {}_2\phi_1(q^{1/2}, -\omega^2; -q^{1/2}\omega^2; q, q). \tag{2.7}$$

To derive the classical Shannon sampling theorem, recall that if $f(x)$ is band-limited to $[-\pi, \pi]$, then f is an entire function of order one by the Paley-Wiener theorem and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\omega x} d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} \hat{f}_n e^{-in\omega} \right) e^{i\omega x} d\omega$$

where

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{in\omega} d\omega = \frac{1}{\sqrt{2\pi}} f(n).$$

Thus,

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f(n) \int_{-\pi}^{\pi} e^{i\omega(x-n)} d\omega = \sum_{n=-\infty}^{\infty} f(n) \mathrm{Sinc}\,(x-n)\,, \tag{2.8}$$

where

$$\mathrm{Sinc}\, x = \frac{\sin \pi x}{\pi x}. \tag{2.9}$$

If f is bandlimited to $[-1, 1]$, then

$$f(x) = \sum_{n=-\infty}^{\infty} f(n\pi) \frac{\sin(x - n\pi)}{(x - n\pi)}.$$

In the special case where $\hat{f}$ is even or odd, then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \hat{f}(\omega) \cos \omega x d\omega, \text{or} \qquad f(x) = i\sqrt{\frac{2}{\pi}} \int_0^{\pi} \hat{f}(\omega) \sin \omega x d\omega,$$

respectively. In the former case the sampling expansion takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)} = f(0)\frac{\sin \pi x}{\pi x} + \sum_{n=1}^{\infty} f(n) \frac{2x \sin \pi(x-n)}{\pi(x^2 - n^2)},$$

because $f(n) = f(-n)$. If f is band-limited to $[-1, 1]$, then

$$f(x) = f(0)\frac{\sin x}{x} + \sum_{n=1}^{\infty} f(n\pi) \frac{2x \sin(x - n\pi)}{x^2 - (n\pi)^2}.$$

Similar results hold if $\hat{f}$ is odd, and we have

$$f(x) = \sum_{n=1}^{\infty} f(n\pi) \frac{(2n\pi)\ \sin(x - n\pi)}{x^2 - (n\pi)^2}.$$

For the q-cosine and sine functions, we have similar results. It is clear that $\theta \to \pi - \theta$ maps x to $-x$. Hence $C_q(x;\omega)$ is even in x and $S_q(x;\omega)$ is odd in x. Furthermore, the weight function $w(x)$ is also even in x. Therefore, it is more general to work with $\mathcal{E}_q(x;i\omega)$ than with $C_q(x;i\omega)$ or $S_q(x;i\omega)$.

3. A q-sampling theorem

We start by defining a q-analogue of band-limitedness.

Definition 1. We say that a function $f(x)$ is q-band-limited to $[-1,1]$ if it can be written in the form

$$f(t) = \int_{-1}^{1} g(x)\mathcal{E}_q(x;it)w(x)dx = \int_0^{\pi} g(\cos\theta)\mathcal{E}_q(\cos\theta;it)W(\cos\theta)d\theta, \tag{3.1}$$

for some $g \in L^2(-1,1)$ or $g \in L^1(-1,1)$, where $w(x)$ and $W(x)$ are as in (2.3).

The function $g(x)$ plays the role of the Fourier transform of $f(t)$. We may call g the q-Fourier transform of f. This should not be confusing despite the fact that there are other definitions of the q-Fourier transform of f; see [4], [9].

Since $\{\mathcal{E}_q(x;i\omega_n)\}_{n=-\infty}^{\infty}$ is a complete orthogonal set in a weighted $L^2(-1,1)$, we have

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{g}_n \mathcal{E}_q(x;-i\omega_n), \tag{3.2}$$

where

$$\hat{g}_n = \frac{1}{2k(\omega_n)} \int_{-1}^{1} g(x)\mathcal{E}_q(x,i\omega_n)\, w(x)\, dx. \tag{3.3}$$

Substituting (3.2) into (3.1) and using Parseval's relation, we have

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{g}_n \int_0^{\pi} \mathcal{E}_q(\cos\theta;-i\omega_n)\mathcal{E}_q(\cos\theta;it)W(\cos\theta)\, d\theta. \tag{3.4}$$

But in view of (3.1) and (3.3), Equation (3.4) becomes

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2k(\omega_n)} f(\omega_n)\widetilde{\mathrm{Sinc}}_q(t,n), \tag{3.5}$$

where

$$\widetilde{\mathrm{Sinc}}_q(t,n) = \int_0^{\pi} \mathcal{E}_q(\cos\theta;-i\omega_n)\mathcal{E}_q(\cos\theta;it)\, W(\cos\theta)\, d\theta. \tag{3.6}$$

The function, $\widetilde{\mathrm{Sinc}}_q(t,n)$, is an analogue of the standard sinc function defined by (2.9). To evaluate the last integral, we use formula (5.5) in [8], which states that

$$\begin{aligned} &\int_0^{\pi} \mathcal{E}_q(\cos\theta;\alpha)\mathcal{E}_q(\cos\theta;\beta)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}\, d\theta \\ &= \frac{2\pi(\gamma,q\gamma,-\alpha\beta q^{1/2};q)_\infty}{(q,\gamma^2;q)_\infty(q\alpha^2,q\beta^2;q^2)_\infty}\, {}_2\phi_2\left(\begin{matrix} -q^{1/2}\alpha/\beta, -q^{1/2}\beta/\alpha \\ q\gamma,, -\alpha\beta\gamma q^{1/2} \end{matrix} \,\middle|\, q, -\alpha\beta\gamma q^{1/2}\right). \end{aligned} \tag{3.7}$$

With $\gamma = q^{1/2}, \alpha = it, \beta = -i\omega_n$, we have

$$\begin{aligned}&\int_0^\pi \mathcal{E}_q(\cos\theta; it)\mathcal{E}_q(\cos\theta; -i\omega_n)W(\cos\theta)d\theta\\ &= \frac{2\pi(q^{1/2}, q^{3/2}, -t\omega_n q^{1/2}; q)_\infty}{(q,q;q)_\infty(-qt^2, -q\omega_n^2; q^2)_\infty}\, {}_2\phi_2\left(\begin{matrix} q^{1/2}t/\omega_n, q^{1/2}\omega_n/t \\ q^{3/2}, -t\omega_n q^{1/2}\end{matrix}\,\middle|\, q, -t\omega_n q\right).\end{aligned} \tag{3.8}$$

To simplify (3.8), we use Formula (III.4) in Appendix III [4]:

$${}_2\phi_2(A, C/B; C, AZ; q, BZ) = \frac{(Z;q)_\infty}{(AZ;q)_\infty}\, {}_2\phi_1(A, B; C; q, Z)$$

where $A = q^{1/2}t/\omega_n, B = qt/\omega_n, C = q^{3/2}, Z = -\omega_n^2$ to get

(3.9)

$$\begin{aligned}&\int_0^\pi \mathcal{E}_q(\cos\theta; it)\mathcal{E}_q(\cos\theta; -i\omega_n)W(\cos\theta)\,d\theta\\ &= \frac{2\pi(q^{1/2}, q^{3/2}, -t\omega_n q^{1/2}; q)_\infty}{(q,q;q)_\infty(-qt^2, -q\omega_n^2; q^2)_\infty}\frac{(-\omega_n^2;q)_\infty}{(-t\omega_n q^{1/2};q)_\infty}\, {}_2\phi_1\left(\begin{matrix} q^{1/2}t/\omega_n, qt/\omega_n \\ q^{3/2}\end{matrix}\,\middle|\, q, -\omega_n^2\right)\\ &= \frac{2\pi(q^{1/2};q)_\infty(q^{3/2};q)_\infty(-\omega_n^2;q)_\infty}{(q;q)_\infty(q;q)_\infty(-qt^2;q^2)_\infty(-q\omega_n^2;q^2)_\infty}\, {}_2\phi_1\left(\begin{matrix} q^{1/2}t/\omega_n, qt/\omega_n \\ q^{3/2}\end{matrix}\,\middle|\, q, -\omega_n^2\right).\end{aligned}$$

This formula is initially valid for $|\omega_n| < 1$, but it can be analytically continued for all ω_n. We begin by ${}_2\phi_1$ on the right-hand side of (3.9):

$$\begin{aligned}&{}_2\phi_1\left(\begin{matrix} q^{1/2}t/w_n, qt/\omega_n \\ q^{3/2}\end{matrix}\,\middle|\, q, -\omega_n^2\right)\\ &= \sum_{k=0}^\infty \frac{\left(\sqrt{q}t/\omega_n;q\right)_k (qt/\omega_n;q)_k}{(q, q^{3/2};q)_k}(-\omega_n^2)^k\\ &= \frac{\left(1-\sqrt{q}\right)}{i\omega_n}\sum_{k=0}^\infty \frac{\left(\sqrt{q}t/\omega_n;\sqrt{q}\right)_{2k}}{\left(\sqrt{q};\sqrt{q}\right)_{2k+1}}(i\omega_n)^{2k+1},\end{aligned}$$

or

$$\begin{aligned}&{}_2\phi_1\left(\begin{matrix} q^{1/2}t/w_n, qt/\omega_n \\ q^{3/2}\end{matrix}\,\middle|\, q, -\omega_n^2\right)\\ &= A_q(t,\omega_n)\left[\sum_{k=0}^\infty \frac{\left(t/\omega_n;\sqrt{q}\right)_k (i\omega_n)^k}{\left(\sqrt{q};\sqrt{q}\right)_k} - \sum_{k=0}^\infty \frac{\left(t/\omega_n;\sqrt{q}\right)_k (-i\omega_n)^k}{\left(\sqrt{q};\sqrt{q}\right)_k}\right],\end{aligned}$$

because $\left(h;\sqrt{q}\right)_{2k+1} = (1-h)\left(\sqrt{q}h;\sqrt{q}\right)_{2k}$, where

$$A_q(t,\omega_n) = \frac{\left(1-\sqrt{q}\right)}{2i\omega_n\left(1-t/\omega_n\right)}.$$

But in view of the q-binomial theorem,

$${}_1\phi_0(a, -; q, z) = \sum_{k=0}^\infty \frac{(a;q)_k}{(q,q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1,$$

we conclude that

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} q^{1/2}t/\omega_n, qt/\omega_n \\ q^{3/2} \end{matrix} \,\middle|\, q, -\omega_n^2\right) \\
&\quad = \frac{\left(1-\sqrt{q}\right)}{2i(\omega_n - t)}\left[\frac{\left(it;\sqrt{q}\right)_\infty}{\left(i\omega_n;\sqrt{q}\right)_\infty} - \frac{\left(-it;\sqrt{q}\right)_\infty}{\left(-i\omega_n;\sqrt{q}\right)_\infty}\right] \\
&\quad = \frac{\left(1-\sqrt{q}\right)}{2i(\omega_n - t)} \frac{\left(it, -i\omega_n;\sqrt{q}\right)_\infty - \left(-it, i\omega_n;\sqrt{q}\right)_\infty}{\left(i\omega_n, -i\omega_n;\sqrt{q}\right)_\infty} \\
&\quad = \frac{\left(1-\sqrt{q}\right)}{2i(\omega_n - t)} \frac{\left(it, -i\omega_n;\sqrt{q}\right)_\infty - \left(-it, i\omega_n;\sqrt{q}\right)_\infty}{(-\omega_n^2;q)_\infty}.
\end{aligned} \tag{3.10}
$$

Thus, by substituting (3.10) into (3.9), we obtain

$$
\begin{aligned}
&\int_0^\pi \mathcal{E}_q(\cos\theta; it)\mathcal{E}_q(\cos\theta; -i\omega_n)\, W(\cos\theta)\, d\theta \\
&\quad = \frac{\pi(q^{1/2};q)_\infty^2 \left[\left(it, -i\omega_n;\sqrt{q}\right)_\infty - \left(-it, i\omega_n;\sqrt{q}\right)_\infty\right]}{i(\omega_n - t)(q;q)_\infty^2(-qt^2, -q\omega_n^2;q^2)_\infty}.
\end{aligned} \tag{3.11}
$$

Notice that $t = \omega_n$ is not a pole since the quantity in the square brackets, [..], also vanishes when $t = \omega_n$. Other possible poles of the right-hand side are at $\omega_n^2 = -q^{-(2k+1)}$ and $t^2 = -q^{(2k+1)}$, for $k = 0, 1, \ldots,$ which are ruled out since ω_n and t are real. Thus, the left-hand side is defined for all ω_n and it is analytic for t real.

Since

$$\frac{(q^{1/2};q)_\infty}{(q;q)_\infty} = (q^{1/2}, q)_{1/2},$$

we have

$$
\begin{aligned}
&\int_0^\pi \mathcal{E}_q(\cos\theta; it)\mathcal{E}_q(\cos\theta; -i\omega_n)\, W(\cos\theta)\, d\theta \\
&\quad = \frac{2\pi(q^{1/2};q)_\infty^2 \,\mathrm{Im}\,\mathrm{h}(\mathrm{t}, \omega_\mathrm{n}, \mathrm{q})}{(q,q;q)_\infty(\omega_n - t)(-qt^2, -q\omega_n^2;q^2)_\infty},
\end{aligned} \tag{3.12}
$$

where $h(t, \omega_n, q) = (it, -i\omega_n; \sqrt{q})_\infty$.

Thus,

$$
\begin{aligned}
\widetilde{\mathrm{Sinc}}_q(t,n) &= \int_0^\pi \mathcal{E}_q(\cos\theta; it)\mathcal{E}_q(\cos\theta; -i\omega_n)\, W(\cos\theta)\, d\theta \\
&= \frac{2\pi(q^{1/2};q)_{1/2}^2 \,\mathrm{Im}\,\mathrm{h}(\mathrm{t}, \omega_\mathrm{n}, \mathrm{q})}{(\omega_n - t)(-qt^2, -q\omega_n^2;q^2)_\infty}.
\end{aligned} \tag{3.13}
$$

Substituting (3.13) into (3.5) and noting that

$$k(\omega_n) = \frac{\pi(q^{1/2}, -q^{1/2}\omega_n^2;q)_\infty(-\omega_n^2;q^2)_\infty}{(q, -\omega_n^2;q)_\infty(-q\omega_n^2, q^2)_\infty}\, {}_2\phi_1\left(\begin{matrix} q^{1/2}, -\omega_n^2 \\ -q^{1/2}\omega_n^2 \end{matrix} \,\middle|\, q, q\right),$$

we obtain

$$f(t) = \sum_{n=-\infty}^{\infty} f(\omega_n) \frac{(q^{1/2};q)_{1/2}(-\omega_n^2;q)_\infty \,\mathrm{Im}\, h(t,\omega_n,q)}{(-q^{1/2}\omega_n^2;q)_\infty(-\omega_n^2, -qt^2;q^2)_\infty(\omega_n - t)\Phi(\omega_n,q)}, \tag{3.14}$$

where

$$\Phi(\omega_n, q) = {}_2\phi_1\left(\begin{matrix} q^{1/2}, -\omega_n^2 \\ -q^{1/2}\omega_n^2 \end{matrix} \,\middle|\, q, q\right).$$

To write (3.14) in a more symmetric form, we observe that

$$\begin{aligned} 2i\mathrm{Im}\, h(t,\omega_n,q) &= (it, -i\omega_n; \sqrt{q})_\infty - (-it, i\omega_n; \sqrt{q})_\infty \\ &= \left[(it;\sqrt{q})_\infty - (-it;\sqrt{q})_\infty\right](-i\omega_n;\sqrt{q})_\infty \\ &\quad + \left[(-i\omega_n;\sqrt{q})_\infty - (i\omega_n;\sqrt{q})_\infty\right](-it;\sqrt{q})_\infty, \end{aligned}$$

which in view of (2.4) is zero if $t = \omega_m$ for any m. Therefore, from (2.6) and (3.13), the constant $k(\omega_n)$ can be written in the form

$$k(\omega_n) = \frac{\pi(q^{1/2};q)_{1/2}^2}{(-q\omega_n^2, -q\omega_n^2; q^2)_\infty} \frac{\partial}{\partial t}\mathrm{Im}\, h(t,\omega_n,q)\Big|_{t=\omega_n}, \tag{3.15}$$

which is obtained by taking the limit in (3.13) as $t \to \omega_n$.

Thus, formula (3.14), with the aid of (3.5) and (3.15), can be written in the form

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f(\omega_n) \frac{(-q\omega_n^2;q^2)_\infty \,\mathrm{Im}\, h(t,\omega_n,q)}{(-qt^2;q^2)_\infty(\omega_n - t)\frac{\partial}{\partial t}\,\mathrm{Im}\, h(t,\omega_n q)\Big|_{t=\omega_n}} \\ &= \sum_{n=-\infty}^{\infty} f(\omega_n)\mathrm{Sinc}_q(t,n), \end{aligned} \tag{3.16}$$

where

$$\mathrm{Sinc}_q(t,n) = \frac{(-q\omega_n^2;q^2)_\infty \,\mathrm{Im}\, h(t,\omega_n,q)}{(-qt^2;q^2)_\infty(\omega_n - t)\frac{\partial}{\partial t}\,\mathrm{Im}\, h(t,\omega_n,q)\Big|_{t=\omega_n}}.$$

Notice that $\mathrm{Sinc}_q(t,n)$ is an analogue of the *Sinc* function defined in (2.9) and satisfies the relation

$$\mathrm{Sinc}_q(\omega_m, n) = \delta_{m,n}.$$

Now we state the q-analogue of the Whittaker-Shannon-Kotel'nikov sampling theorem.

Theorem 3.1. *Let $f(t)$ be a q-band-limited function according to Definition* 1. *Then the function*

$$F(t) = (-qt^2;q^2)_\infty f(t)$$

is an entire function of order zero that can be reconstructed by the formula

$$\begin{aligned} F(t) &= \sum_{n=-\infty}^{\infty} F(\omega_n) \frac{\mathrm{Im}\, h(t,\omega_n,q)}{(\omega_n - t)\frac{\partial}{\partial t}\mathrm{Im}\, h(t,\omega_n,q)\Big|_{t=\omega_n}} \\ &= \sum_{n=-\infty}^{\infty} F(\omega_n)\frac{G_n(t)}{(\omega_n - t)G_n'(\omega_n)}, \end{aligned} \tag{3.17}$$

where $G_n(t) = \mathrm{Im}\, h(t,\omega_n,q)$.

Proof. To show that $F(t)$ is an entire function of order zero, we first observe that the function $e_q(\cos\theta;t) = (-qt^2;q^2)_\infty \mathcal{E}_q(\cos\theta;t)$ is an entire function of order zero (see [3]). Thus,

$$F(t) = (-qt^2;q^2)_\infty \int_0^\pi g(\cos\theta)\mathcal{E}_q(\cos\theta;it)(e^{2i\theta},e^{-2i\theta};q)_{1/2}d\theta$$
$$= \int_0^\pi g(\cos\theta)e_q(\cos\theta;t)(e^{2i\theta},e^{-2i\theta};q)_{1/2}d\theta.$$

The integral converges uniformly in t over compact sets whenever $g(\cos\theta) \in L^1([0,\pi],\tilde{W})$, where $\tilde{W}(\theta) = (e^{2i\theta},e^{-2i\theta};q)_{1/2}$. Hence F is analytic on any compact set, i.e., F is entire. The order of F is the same as that of $e_q(\cos\theta,t)$.

By setting $f(t) = F(t)/(-qt^2;q^2)_\infty$ in (3.5) we obtain (3.17). □

Because formula (3.17) resembles (1.4) and the sampling points are not equally spaced, Theorem 1 may be considered as a q-analogue of the Paley-Wiener-Levinson (PWL) Sampling Theorem. As is known, the PWL Sampling Theorem reduces to the WSK Sampling Theorem when $t_n = n$. It remains an open question as to whether Theorem 1 reduces to the WSK Sampling Theorem when $q \to 1^-$.

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