PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 2, Pages 523–534
S 0002-9939(03)07051-5
Article electronically published on June 5, 2003

STRONG MIXING COEFFICIENTS FOR NON-COMMUTATIVE GAUSSIAN PROCESSES

WŁODZIMIERZ BRYC AND VICTOR KAFTAL

(Communicated by David R. Larson)

ABSTRACT. Bounds for non-commutative versions of two classical strong mixing coefficients for q-Gaussian processes are found in terms of the angle between the underlying Hilbert spaces. As a consequence, we construct a ψ-mixing q-Gaussian stationary sequence with growth conditions on variances of partial sums. If classical processes with analogous properties were to exist, they would provide a counter-example to the Ibragimov conjecture.

1. INTRODUCTION

The long-standing Ibragimov conjecture in (classical) probability ([13], [12], and [9, Section 13.1]) involves the validity of the Central Limit Theorem for a stationary sequence of random variables X_k which are ϕ-mixing, i.e., such that there is a sequence $\phi_N \to 0$ such that for every $N, m, n \in \mathbb{N}$,

$$|\mathrm{cov}(V_1, V_2)| \leq \phi_N \|V_1\|_1 \|V_2\|_\infty$$

for all bounded random variables V_1, V_2 such that V_1 is $\sigma(X_1, \ldots, X_n)$-measurable and V_2 is $\sigma(X_{n+N}, \ldots, X_{m+n+N})$-measurable. Related to the Ibragimov conjecture are Bradley's conjecture [8, page 226], Iosifescu's conjecture [14], and works by M. Peligrad [18], and Berkes and Philipp [1].

Here we investigate the same notions in the non-commutative setting introduced by Voiculescu [19] for the free probability case ($q = 0$), and by Bozejko and Speicher [6] in the $-1 < q < 1$ case. Many classical (i.e., commutative) probability results have already been extended to these settings. In this paper we obtain a result, Theorem 4, which does not yet have a classical precursor. If a classical version of this theorem were to hold, it would settle in the negative Ibragimov's conjecture and all the other mentioned conjectures ([7]).

Non-commutative q-Gaussian random variables

$$\mathbf{X}_h := \mathbf{a}_h + \mathbf{a}_h^* \tag{1}$$

are defined in terms of a bounded real-linear mapping $\mathbf{a} : \mathbb{H} \mapsto \mathcal{B}(\mathcal{H}_q)$ from a real Hilbert space $\mathbb{H}$ into the algebra of all bounded operators on a complex separable

Received by the editors September 12, 2002.
2000 *Mathematics Subject Classification.* Primary 81S05; Secondary 60E99.
Key words and phrases. Non-commutative uniform strong mixing, Ibragimov's conjecture, covariance estimates.

Hilbert space $\mathcal{H}_q$ that satisfies the q-commutation relations

$$\mathbf{a}_g\mathbf{a}_h^* - q\mathbf{a}_h^*\mathbf{a}_g = \langle h|g\rangle\mathbf{I} \tag{2}$$

which were introduced in [11].

The von Neumann algebra $\mathcal{A}$ generated by these variables $\mathbf{X}_h$ (i.e., the weak-operator limits of non-commutative polynomials in the variables $\mathbf{X}_h$) has a tracial state $\mathbb{E}$. For $1 \le p < \infty$, this trace permits us to define the L_p-norms

$$\|\mathbf{X}\|_p := \left(\mathbb{E}\left((\mathbf{X}^*\mathbf{X})^{p/2}\right)\right)^{1/p}, \tag{3}$$

and the non-commutative L_p space $L_p(\mathcal{A},\mathbb{E})$ is the closure of the von Neumann algebra $\mathcal{A}$ in this norm; see [17, Section 3]. We also use the standard conventions: $L_\infty(\mathcal{A},\mathbb{E})$ is $\mathcal{A}$ with the operator norm and $L_2(\mathcal{A},\mathbb{E})$ is a Hilbert space with the scalar product $(\mathbf{Y}|\mathbf{X}) := \mathbb{E}(\mathbf{X}^*\mathbf{Y})$.

The main results we obtain are as follows. We first extend to the non-commutative setting a theorem of Kolmogorov and Rozanov [16] stating that for classical Gaussian sequences the "linear dependence coefficients" coincide with the "maximal correlation coefficients". In our setting, the linear dependence coefficient r of two subspaces $\mathbb{H}_1, \mathbb{H}_2 \subset \mathbb{H}$ is defined as

$$r = r(\mathbb{H}_1,\mathbb{H}_2) := \sup\left\{\frac{|\mathrm{cov}(\mathbf{X}_f,\mathbf{X}_g)|}{\|\mathbf{X}_f\|_2\|\mathbf{X}_g\|_2} : \mathbf{X}_f \neq 0, \mathbf{X}_g \neq 0, f \in \mathbb{H}_1, g \in \mathbb{H}_2\right\}; \tag{4}$$

compare [9, Section 8.7]. Here,

$$\mathrm{cov}(\mathbf{X},\mathbf{Y}) := \mathbb{E}(\mathbf{X}^*\mathbf{Y}) - \mathbb{E}(\mathbf{X}^*)\,\mathbb{E}(\mathbf{Y}).$$

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are the von Neumann algebras generated by $\{\mathbf{X}_f : f \in \mathbb{H}_1\}$ and $\{\mathbf{X}_g : g \in \mathbb{H}_2\}$ respectively, then the maximal correlation coefficient is

$$\rho(\mathbb{H}_1,\mathbb{H}_2) := \sup\left\{\frac{|\mathrm{cov}(\mathbf{X},\mathbf{Y})|}{\|\mathbf{X}\|_2\|\mathbf{Y}\|_2} : \mathbf{X} \neq 0, \mathbf{Y} \neq 0, \mathbf{X} \in L_2(\mathcal{A}_1,\mathbb{E}), \mathbf{Y} \in L_2(\mathcal{A}_2,\mathbb{E})\right\}.$$

Theorem 1.

$$\rho(\mathbb{H}_1,\mathbb{H}_2) = r(\mathbb{H}_1,\mathbb{H}_2). \tag{5}$$

We then obtain an upper bound for the non-commutative analog of the ψ-mixing coefficient

$$\psi(\mathbb{H}_1,\mathbb{H}_2) := \sup\left\{\frac{|\mathrm{cov}(\mathbf{X},\mathbf{Y})|}{\|\mathbf{X}\|_1\|\mathbf{Y}\|_1} : \mathbf{X} \neq 0, \mathbf{Y} \neq 0, \mathbf{X} \in L_2(\mathcal{A}_1,\mathbb{E}), \mathbf{Y} \in L_2(\mathcal{A}_2,\mathbb{E})\right\}$$

(cf. [9, Theorem 3.10]). This result is somewhat unexpected since for the classical Gaussian random variables the ψ-mixing coefficient can only be zero (independent case) or infinity.

Theorem 2. *If $r = r(\mathbb{H}_1,\mathbb{H}_2) < 1$, then*

$$\psi(\mathbb{H}_1,\mathbb{H}_2) \le C_q^2\, r\frac{r^2-3r+4}{(1-r)^3}, \tag{6}$$

where $C_q = \prod_{m=1}^{\infty}(1-|q|^m)^{-3/2}$.

This upper bound is sharp in the free probability case, i.e., if $q = 0$; for a related result, see also [2, Corollary 3].

Theorem 3. *If $q = 0$ and $r = r(\mathbb{H}_1,\mathbb{H}_2) < 1$, then $\psi(\mathbb{H}_1,\mathbb{H}_2) = r\frac{r^2-3r+4}{(1-r)^3}$.*

As a consequence of Theorem 2 we can adapt a classical probability construction of Bradley [7] to obtain the following non-commutative result.

Theorem 4. *For every $\epsilon > 0$ and $-1 < q < 1$ there exists a q-Gaussian sequence $\{\mathbf{X}_k\}$ such that the following statements hold true:*

(i) $\mathbb{E}(\mathbf{X}_j) = 0$, $\|\mathbf{X}_1 + \cdots + \mathbf{X}_n\|_2 \to \infty$ *as* $n \to \infty$, *and* $\frac{1}{n}\|\mathbf{X}_1 + \cdots + \mathbf{X}_n\|_2^2 \to 0$ *as* $n \to \infty$.

(ii) $\{\mathbf{X}_k\}$ *is strictly stationary, i.e.,*

$$\mathbb{E}\left(\mathbf{X}_{i(1)} \dots \mathbf{X}_{i(m)}\right) = \mathbb{E}\left(\mathbf{X}_{i(1)+t} \dots \mathbf{X}_{i(m)+t}\right) \tag{7}$$

for all $t, m \in \mathbb{N}$, *and all sequences of integers* $i(1), i(2), \dots, i(m) \in \mathbb{N}$.

(iii) $\{\mathbf{X}_k\}$ *is ψ-mixing, i.e., there is a monotone sequence of numbers* $\psi_N \to 0$ *such that* $0 < \psi_1 < \epsilon$, *and for all* $m, n, N \in \mathbb{N}$,

$$|cov(\mathbf{V}_1, \mathbf{V}_2)| \leq \psi_N \|\mathbf{V}_1\|_1 \|\mathbf{V}_2\|_1$$

for all random variables $\mathbf{V}_1$ *in the von Neumann algebra generated by* $\mathbf{X}_1, \dots, \mathbf{X}_n$, *and* $\mathbf{V}_2$ *in the von Neumann algebra generated by* $\mathbf{X}_{n+N}, \dots, \mathbf{X}_{m+n+N}$.

Our proof of Theorem 2 is based on the proof of Theorem 1 and, via a duality argument, on the main theorem in Bozejko [4]. In the free case which corresponds to $q = 0$, a more self-contained proof along the lines of [3] is given in Section 3 where we also present the proof of Theorem 3.

2. Proofs

We will be working with the q-Fock space representation of q-Gaussian processes, adapted from [5]; see also [19, Section 1.5] for the $q = 0$ (free) case. For a real Hilbert space $\mathbb{H}$ with complexification $\mathbb{H}_c := \mathbb{H} \oplus i\mathbb{H}$, the associated q-Fock space $\mathcal{H}_q$ is the closure of $\bigoplus_{n=0}^{\infty} \mathbb{H}_c^{\otimes n}$ with respect to the scalar product obtained as the sesquilinear extension of

$$\langle g_1 \otimes \cdots \otimes g_n | h_1 \otimes \cdots \otimes h_m \rangle_q = \begin{cases} \sum_{\sigma \in S_n} q^{|\sigma|} \prod_{j=1}^n \langle g_j | h_{\sigma(j)} \rangle & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{8}$$

Here, $\mathbb{H}_c^{\otimes 0} := \mathbb{C}\mathbf{1}$, where $\mathbf{1}$ is called the vacuum vector, S_n is the set of all the permutations of $\{1, \dots, n\}$ and $|\sigma| := \mathrm{card}\{(i,j) : i < j, \sigma(i) > \sigma(j)\}$ is the number of inversions of $\sigma \in S_n$.

We denote by $\|\cdot\|_{\mathcal{H}_q}$ the corresponding norm. We denote by $\overline{\mathbb{H}^{\otimes n}}$ the $\|\cdot\|_{\mathcal{H}_q}$-closure of the algebraic tensor product $\mathbb{H}_c^{\otimes n}$ so that $\mathcal{H}_q = \bigoplus_{n=0}^{\infty} \overline{\mathbb{H}^{\otimes n}}$. In this setting, for $h \in \mathbb{H}$, the annihilation operator $\mathbf{a}_h : \mathcal{H}_q \to \mathcal{H}_q$ and its adjoint, the creation operator $\mathbf{a}_h^* : \mathcal{H}_q \to \mathcal{H}_q$, are the bounded linear extensions of

$$\mathbf{a}_h \mathbf{1} := 0,$$

$$\mathbf{a}_h g_1 \otimes \cdots \otimes g_n := \sum_{j=1}^n q^{j-1} \langle h | g_j \rangle g_1 \otimes \cdots \otimes g_{j-1} \otimes g_{j+1} \otimes \cdots \otimes g_n \tag{9}$$

and

$$\mathbf{a}_h^* \mathbf{1} = h,$$

$$\mathbf{a}_h^* g_1 \otimes \cdots \otimes g_n := h \otimes g_1 \otimes \cdots \otimes g_n \tag{10}$$

for $g_1, g_2, \ldots, g_n \in \mathbb{H}_c$, and satisfy relations (2) (see [6], [5]; cf. also [19, Example 1.5.8] for $q = 0$).

Let $\mathcal{A}$ be the von Neumann algebra generated by the variables $\{\mathbf{X}_h : h \in \mathbb{H}\}$ given by (1). It is known that the vacuum expectation state $\mathbb{E} : \mathcal{A} \to \mathbb{C}$ defined by

$$\mathbb{E}(\mathbf{X}) := \langle \mathbf{X}\mathbf{1}|\mathbf{1}\rangle_{\mathcal{H}_q}$$

is a faithful normal finite trace on $\mathcal{A}$; see [5, Proposition 2.3], or [19, Theorem 2.6.2 (ii)] when $q = 0$.

For $g_1, g_2, \ldots, g_n \in \mathbb{H}$, the Wick product $\Psi(g_1 \otimes \cdots \otimes g_n) \in \mathcal{A}$ is defined recursively by $\Psi(\mathbf{1}) := \mathbf{I}$, $\Psi(h) := \mathbf{X}_h$, and

$$\Psi(h \otimes g_1 \otimes \cdots \otimes g_n) := \mathbf{X}_h \Psi(g_1 \otimes \cdots \otimes g_n)$$

$$-\sum_{j=1}^{n} q^{j-1} \langle h|g_j\rangle \Psi(g_1 \otimes \cdots \otimes g_{j-1} \otimes g_{j+1} \otimes \cdots \otimes g_n). \tag{11}$$

By definition, $\mathbf{X}_h \mathbf{1} = h$, so

$$\mathbb{E}(\mathbf{X}_h) = 0 \tag{12}$$

and

$$\|\mathbf{X}_h\|_2 = \|h\| \tag{13}$$

for all $h \in \mathbb{H}$.

By (11), $\Psi(h_1 \otimes \cdots \otimes h_n) = \mathbf{X}_{h_1}\mathbf{X}_{h_2} \ldots \mathbf{X}_{h_n} + \cdots$, where the dots represent a polynomial in $\mathbf{X}_{h_1}, \ldots, \mathbf{X}_{h_n}$ of degree lower than n. Thus it is clear that every non-commutative polynomial in the variables $\mathbf{X}_{h_1}, \ldots, \mathbf{X}_{h_n}$ can be expressed as a linear combination of Wick products. We will need to make this relation more precise in Lemma 1.

Denote by $\mathbf{i}$ the multi-index $\mathbf{i} := (i(1), \ldots, i(N)) \in \mathbb{N}^N$ and denote by $|\mathbf{i}|$ the length N of the multi-index $\mathbf{i}$. Let $(\mathbf{i}, \mathbf{j})$ denote the concatenation of the multi-indices $\mathbf{i}, \mathbf{j}$:

$$(\mathbf{i}, \mathbf{j}) = (i(1), i(2), \ldots, i(L), j(1), (j(2), \ldots, j(M))).$$

Thus $|(\mathbf{i}, \mathbf{j})| = |\mathbf{i}| + |\mathbf{j}|$. Denote by $\mathbf{i}[a \ldots b]$ the subindex $(i(a), i(a+1), \ldots, i(b))$.

For a sequence of vectors $g_1, g_2, \cdots \in \mathbb{H}$ write

$$g^{\otimes \mathbf{i}} = g_{i(1)} \otimes g_{i(2)} \otimes \cdots \otimes g_{i(m)}$$

so that $g^{\otimes(\mathbf{i},\mathbf{j})} = g^{\otimes \mathbf{i}} \otimes g^{\otimes \mathbf{j}}$.

Lemma 1. *For every $m \in \mathbb{N}$ and all multi-indices $\mathbf{i}$ of length $0 \le |\mathbf{i}| \le m$ there are polynomials $P^m_{\mathbf{i}}$ in m^2 variables $\{x_{i,j} : i, j \le m\}$ such that for any $g_1, g_2, \ldots, g_m \in \mathbb{H}$,*

$$\mathbf{X}_{g_m}\mathbf{X}_{g_{m-1}} \ldots \mathbf{X}_{g_1} = \sum_{|\mathbf{i}| \le m} P^m_{\mathbf{i}}(x_{s,t} : s, t \le m)\Psi(g^{\otimes \mathbf{i}}), \tag{14}$$

where $x_{s,t} = \langle g_s|g_t\rangle$, and if $|\mathbf{i}| = 0$, then $g^{\otimes \emptyset} = \mathbf{1}$.

Proof. We proceed by induction with respect to $m \ge 1$. If $m = 1$, then $\mathbf{X}_{g_1} = \Psi(g_1)$ proving (14) with $P^1_\emptyset = 0, P^1_1 = 1, P^1_i = 0$ for $i > 1$.

Suppose that formula (14) holds true for some $m \in \mathbb{N}$. Then from (11) we get

$$\begin{aligned}\mathbf{X}_{g_{m+1}}\mathbf{X}_{g_m}\dots\mathbf{X}_{g_1} &= \sum_{|\mathbf{i}|\le m} P^m_{\mathbf{i}}(x_{s,t}: s,t\le m)\mathbf{X}_{g_{m+1}}\Psi(g^{\otimes \mathbf{i}})\\ &= \sum_{|\mathbf{i}|\le m} P^m_{\mathbf{i}}(x_{s,t}: s,t\le m)\Psi(g^{\otimes(m+1,\mathbf{i})})\\ &+ \sum_{|\mathbf{i}|\le m} P^m_{\mathbf{i}}(x_{s,t}: s,t\le m)\sum_{k=1}^{|\mathbf{i}|} q^{k-1}\langle g_{m+1}|g_{i(k)}\rangle \Psi(g^{\otimes(\mathbf{i}[0\dots k-1],\mathbf{i}[k+1\dots|\mathbf{i}|])}).\end{aligned}$$

Notice that in the last sum the same multi-index can be obtained from more than one concatenation $(\mathbf{i}[0\dots k-1], \mathbf{i}[k+1\dots|\mathbf{i}|])$. Grouping all of them together and noticing that $\langle g_{m+1}|g_s\rangle = x_{m+1,s}$, we get the polynomials in the right-hand side of (14). □

From (11) and (2),

$$\Psi(h_1\otimes\cdots\otimes h_n)\mathbf{1} = h_1\otimes\cdots\otimes h_n,$$

and thus $\|\Psi(h_1\otimes\cdots\otimes h_n)\|_2 = \|h_1\otimes\cdots\otimes h_n\|_{\mathcal{H}_q}$, which extends (13). Therefore, the mapping

$$\sum \alpha_{i_1,\dots,i_k} h_{i_1}\otimes\cdots\otimes h_{i_k} \mapsto \sum \alpha_{i_1,\dots,i_k}\Psi(h_{i_1}\otimes\cdots\otimes h_{i_k})$$

is an isometry in the L_2-norm (3) from a dense subset of $\mathcal{H}_q$ onto all the polynomials in $\{\mathbf{X}_h : h\in\mathbb{H}\}$ and hence it extends to a unitary mapping $\widetilde{\Psi}$ of $\mathcal{H}_q$ onto the Hilbert space $L_2(\mathcal{A},\mathbb{E})$. Thus $\widetilde{\Psi}$ induces the orthogonal decomposition

$$L_2(\mathcal{A},\mathbb{E}) = \bigoplus_{n=0}^{\infty}\widetilde{\Psi}\left(\overline{\mathbb{H}^{\otimes n}}\right). \tag{15}$$

Furthermore,

$$\widetilde{\Psi}(\xi)\mathbf{1} = \xi \tag{16}$$

for all $\xi\in\mathcal{H}_q$.

Proof of Theorem 1. First, we give a Hilbert space theoretic characterization of the linear dependence coefficient $r = r(\mathbb{H}_1,\mathbb{H}_2)$ refined by (4). By (12)

$$\operatorname{cov}(\mathbf{X}_f,\mathbf{X}_g) = \mathbb{E}(\mathbf{X}_f^*\mathbf{X}_g) = (\mathbf{X}_g|\mathbf{X}_f) = (\widetilde{\Psi}(g)|\widetilde{\Psi}(f)) = \langle f|g\rangle_{\mathcal{H}_q} = \langle f|g\rangle.$$

Hence taking into account (13) we obtain

$$r = \sup\{\langle f|g\rangle : f\in\mathbb{H}_1, g\in\mathbb{H}_2, \|f\|_{\mathbb{H}} = \|g\|_{\mathbb{H}} = 1\}. \tag{17}$$

Now let $P_j : \mathbb{H}_c \to \mathbb{H}_c$ denote the orthogonal projection onto $\mathbb{H}_j \subset \mathbb{H}_c$, j=1,2. It is easy to verify that $\|P_1P_2\| = r$.

The n-fold tensor product $P_j^{\otimes n}$ of the projection P_j with itself is clearly a linear idempotent operator on $\mathbb{H}_c^{\otimes n}$. It is also selfadjoint with respect to the scalar product (8). Indeed, if $g_1 \otimes g_2 \otimes \cdots \otimes g_n$ and $h_1 \otimes h_2 \otimes \cdots \otimes h_n$ are in $\mathbb{H}_c^{\otimes n}$, then

$$\begin{aligned}\left\langle P_j^{\otimes n} g_1 \otimes \cdots \otimes g_n | h_1 \otimes \cdots \otimes h_m \right\rangle_q &= \langle P_j g_1 \otimes \cdots \otimes P_j g_n | h_1 \otimes \cdots \otimes h_m \rangle_q \\ &= \sum_{\sigma \in S_n} q^{|\sigma|} \prod_{k=1}^n \langle P_j g_k | h_{\sigma(k)} \rangle = \sum_{\sigma \in S_n} q^{|\sigma|} \prod_{k=1}^n \langle g_k | P_j h_{\sigma(k)} \rangle \\ &= \left\langle g_1 \otimes \cdots \otimes g_n | P_j^{\otimes n} h_1 \otimes \cdots \otimes h_m \right\rangle_q .\end{aligned}$$

Moreover, it is easy to see that $P_j^{\otimes n}$, and hence $(P_2P_1)^{\otimes n} = P_2^{\otimes n} P_1^{\otimes n}$, commute with the unitary operations of permuting the components of $\mathbb{H}_c^{\otimes n}$. Therefore, by [5, Lemma 1.4], the norm $\|(P_2P_1)^{\otimes n}\|$ of $(P_2P_1)^{\otimes n}$ with respect to the norm $\|\cdot\|_{\mathcal{H}_q}$ coincides with the norm with respect to the Hilbert space tensor norm. Therefore, by [15, Section 2.6.12 Eqn. (16)] $\|(P_2P_1)^{\otimes n}\| = \|P_1P_2\|^n$, where $\|P_1P_2\|$ is the usual operator norm in $B(\mathbb{H}_c)$ which, as we observed above, coincides with r.

Thus for $n \geq 1$, $\xi \in \overline{\mathbb{H}_1^{\otimes n}}$, $\eta \in \overline{\mathbb{H}_2^{\otimes n}}$, we have

$$|\mathbb{E}(\widetilde{\Psi}(\eta)^* \widetilde{\Psi}(\xi))| \leq r^n \|\widetilde{\Psi}(\xi)\|_2 \|\widetilde{\Psi}(\eta)\|_2. \tag{18}$$

Indeed,

$$\begin{aligned}\left|\mathbb{E}(\widetilde{\Psi}(\eta)^* \widetilde{\Psi}(\xi))\right| &= \left|\langle \widetilde{\Psi}(\xi)\mathbf{1} | \widetilde{\Psi}(\eta)\mathbf{1}\rangle_{\mathcal{H}_q}\right| = \left|\langle \xi | \eta \rangle_{\mathcal{H}_q}\right| \\ &= \left|\langle P_1^{\otimes n}\xi | P_2^{\otimes n}\eta\rangle_{\mathcal{H}_q}\right| = \left|\langle P_2^{\otimes n} P_1^{\otimes n}\xi | \eta\rangle_{\mathcal{H}_q}\right| = \left|\langle (P_2P_1)^{\otimes n}\xi | \eta\rangle_{\mathcal{H}_q}\right| \\ &\leq \|(P_2P_1)^{\otimes n}\| \, \|\xi\|_{\mathcal{H}_q} \|\eta\|_{\mathcal{H}_q} = r^n \, \|\xi\|_{\mathcal{H}_q} \|\eta\|_{\mathcal{H}_q} = r^n \|\widetilde{\Psi}(\xi)\|_2 \|\widetilde{\Psi}(\eta)\|_2,\end{aligned}$$

where the last equality follows because $\widetilde{\Psi}$ is an isometry.

Now denote by $\mathbf{X}^{(n)}, \mathbf{Y}^{(n)}$ the components of $\mathbf{X}, \mathbf{Y}$ in the direct sum decomposition (15). Since $\mathbf{X} \in L_2(\mathcal{A}_1, \mathbb{E})$, then $\mathbf{X}^{(n)}$ is in the closed subspace $\widetilde{\Psi}\left(\overline{\mathbb{H}_1^{\otimes n}}\right)$ of $\widetilde{\Psi}\left(\overline{\mathbb{H}^{\otimes n}}\right)$, and similarly $\mathbf{Y}^{(n)} \in \widetilde{\Psi}\left(\overline{\mathbb{H}_2^{\otimes n}}\right)$ for all n. So from (18) we get for $n \geq 1$ that

$$|\mathbb{E}(\mathbf{X}^{(n)*}\mathbf{Y}^{(n)})| \leq r^n \|\mathbf{X}^{(n)}\|_2 \|\mathbf{Y}^{(n)}\|_2. \tag{19}$$

From (16) we see that $\mathbf{X}^{(n)}\mathbf{1} \in \overline{\mathbb{H}_1^{\otimes n}}$ and hence $\mathbb{E}(\mathbf{X}^{(n)}) = 0$ for $n \geq 1$. It is easy to verify that $\mathbb{E}(\mathbf{X}) = \mathbb{E}(\mathbf{X}^{(0)}), \mathbb{E}(\mathbf{Y}) = \mathbb{E}(\mathbf{Y}^{(0)})$, and $\mathbb{E}(\mathbf{X}^{(0)^*}\mathbf{Y}^{(0)}) = \mathbb{E}(\mathbf{X}^{(0)^*})\mathbb{E}(\mathbf{Y}^{(0)}) = \mathbb{E}(\mathbf{X})\mathbb{E}(\mathbf{Y})$. Keeping in mind that $\mathbb{E}(\mathbf{X}^*\mathbf{Y})$ is the scalar product of $\mathbf{Y}$ and $\mathbf{X}$ in $L_2(\mathcal{A}, \mathbb{E})$ we have

$$\mathbb{E}(\mathbf{X}^*\mathbf{Y}) = \sum_{n=0}^{\infty} \mathbb{E}(\mathbf{X}^{(n)*}\mathbf{Y}^{(n)}). \tag{20}$$

Therefore

$$|\mathrm{cov}(\mathbf{X}, \mathbf{Y})| = |\mathbb{E}(\mathbf{X}^*\mathbf{Y}) - \mathbb{E}(\mathbf{X}^*)\mathbb{E}(\mathbf{Y})| \leq \sum_{n=1}^{\infty} |\mathbb{E}(\mathbf{X}^{(n)*}\mathbf{Y}^{(n)})|,$$

and inequality (19) gives

$$|\mathrm{cov}(\mathbf{X}, \mathbf{Y})| \leq \sum_{n=1}^{\infty} r^n \|\mathbf{X}^{(n)}\|_2 \|\mathbf{Y}^{(n)}\|_2. \tag{21}$$

As $r^n \le r$, by the Cauchy-Schwarz inequality we have

$$|\mathrm{cov}(\mathbf{X},\mathbf{Y})| \le r\left(\sum_{n=1}^{\infty}\|\mathbf{X}^{(n)}\|_2^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|\mathbf{Y}^{(n)}\|_2^2\right)^{1/2} \le r\|\mathbf{X}\|_2\|\mathbf{Y}\|_2,$$

which proves the theorem. □

Proof of Theorem 2. Let $\mathbf{X} \in L_2(\mathcal{A},\mathbb{E})$. As in the proof of Theorem 1, denote by $\mathbf{X}^{(n)}$ the n-th term in the expansion (15) of $\mathbf{X}$. Since $L_2(\mathcal{A},\mathbb{E})$ is a Hilbert space,

$$\|\mathbf{X}^{(n)}\|_2 = \sup\{|\mathbb{E}(\mathbf{Z}^*\mathbf{X}^{(n)})| : \mathbf{Z} \in L_2(\mathcal{A},\mathbb{E}), \|\mathbf{Z}\|_2 \le 1\}.$$

By (20), $\mathbb{E}(\mathbf{Z}^*\mathbf{X}^{(n)}) = \mathbb{E}(\mathbf{Z}^{(n)*}\mathbf{X}^{(n)}) = \mathbb{E}(\mathbf{Z}^{(n)*}\mathbf{X})$, where $\mathbf{Z}^{(n)}$ is the component of $\mathbf{Z}$ in $\widetilde{\Psi}\left(\overline{\mathbb{H}^{\otimes n}}\right)$. As $\mathcal{A}$ is dense in $L_2(\mathcal{A},\mathbb{E})$, we get

$$\|\mathbf{X}^{(n)}\|_2 = \sup\{|\mathbb{E}(\mathbf{Z}^*\mathbf{X})| : \mathbf{Z} \in \mathcal{A}\cap\widetilde{\Psi}\left(\overline{\mathbb{H}^{\otimes n}}\right), \|\mathbf{Z}\|_2 \le 1\}.$$

For $\mathbf{Z} \in \mathcal{A}\cap\widetilde{\Psi}\left(\overline{\mathbb{H}^{\otimes n}}\right)$ and $\|\mathbf{Z}\|_2 \le 1$, by Hölder's inequality ([17, (23)]) we get

$$|\mathbb{E}(\mathbf{Z}^*\mathbf{X})| \le \|\mathbf{Z}^*\|_\infty\|\mathbf{X}\|_1 = \|\mathbf{Z}\|_\infty\|\mathbf{X}\|_1.$$

By [4, Proposition 2.1(b)],

$$\|\mathbf{Z}\|_\infty \le C_q(n+1)\|\mathbf{Z}\|_2 \le C_q(n+1). \tag{22}$$

Hence

$$\|\mathbf{X}^{(n)}\|_2 \le C_q(n+1)\|\mathbf{X}\|_1.$$

The same inequality holds for any $\mathbf{Y} \in L_2(\mathcal{A},\mathbb{E})$.

Applying these inequalities to each term on the right-hand side of (21) we get

$$|\mathrm{cov}(\mathbf{X},\mathbf{Y})| \le C_q^2\sum_{n=1}^{\infty}(n+1)^2 r^n\|\mathbf{X}\|_1\|\mathbf{Y}\|_1 = C_q^2 r\frac{r^2-3r+4}{(1-r)^3}\|\mathbf{X}\|_1\|\mathbf{Y}\|_1,$$

which completes the proof. □

Proof of Theorem 4. To prove this theorem, we need to construct an appropriate sequence of vectors h_k in a real Hilbert space $\mathbb{H}$. The construction relies on [7] (and hence, indirectly, on results of Helson and Sarason on Toeplitz forms); according to [7, Lemma 3], for every $\epsilon > 0$ there is a sequence h_k of (real) classical Gaussian random variables on a probability space $(\Omega,\mathcal{F},P)$ with the following properties:

(i′) $\|h_1+\cdots+h_n\|_2 \to \infty$ and $\frac{1}{n}\|h_1+\cdots+h_n\|_2^2 \to 0$.
(ii′) $\langle h_t|h_{t+m}\rangle = \langle h_0|h_m\rangle$ for all $m,t \in \mathbb{N}$.
(iii′) There exists a monotone sequence $\epsilon_N \to 0$ such that $\epsilon_1 < \min(1,\epsilon)$ and for every (real) linear combination $v_1 = \sum_{j=1}^n a_j h_j$, $v_2 = \sum_{j=n+N}^{n+N+m} b_j h_j$ we have

$$|\langle v_1|v_2\rangle| \le \epsilon_N\|v_1\|_2\|v_2\|_2,$$

where $\langle g|h\rangle$ is the scalar product in $L_2(\Omega,\mathcal{F},P)$.

We define $\mathbb{H}$ as the closure of the real span of h_k in $L_2(\Omega,\mathcal{F},P)$. For any $-1 < q < 1$, let $\mathcal{H}_q$ be the q-Fock space based on $\mathbb{H}$, with the creation and annihilation operators $\mathbf{a}_h, \mathbf{a}_h^*$ defined by (9), (10) and the q-Gaussian random variables $\mathbf{X}_h$ defined in (1). We now verify that the q-Gaussian sequence $\mathbf{X}_k := \mathbf{X}_{h_k}$ has the properties (i)-(iii).

Statement (i) follows from (i′) by (12), and

$$\|\mathbf{X}_1+\cdots+\mathbf{X}_n\|_2^2=\mathbb{E}(|\mathbf{X}_1+\cdots+\mathbf{X}_n|^2)=\mathbb{E}(|\mathbf{X}_{h_1+\cdots+h_n}|^2)=\|h_1+\cdots+h_n\|_{\mathbb{H}}^2,$$

where the second equality follows from the linearity of $\mathbf{a}:\mathbb{H}\mapsto\mathcal{B}(\mathcal{H}_q))$ and the third one holds true by (13).

Statement (ii) follows from (ii′) as follows. Since $\mathbb{E}(\Psi(h^{\otimes\mathbf{i}}))=0$ for $|\mathbf{i}|>0$, by (14)

$$\mathbb{E}\left(\mathbf{X}_{i(1)+t}\dots\mathbf{X}_{i(m)+t}\right)=P_\emptyset^m(x_{r,s}:r,s\le m)$$

is a polynomial in the m^2 variables $x_{r,s}=\langle h_{i(r)+t}|h_{i(s)+t}\rangle$. Since (ii′) implies that $\langle h_{i(r)+t}|h_{i(s)+t}\rangle=\langle h_{i(r)}|h_{i(s)}\rangle$, $r,s\in\mathbb{N}$, therefore (7) follows.

Statement (iii) is a consequence of Theorem 2 and (iii′). In this context, fix $n,m,N\in\mathbb{N}$ and let $\mathbb{H}_1$ be spanned by vectors $\{h_1,\dots,h_n\}$ and $\mathbb{H}_2$ be spanned by vectors $\{h_{n+N},\dots,h_{m+n+N}\}$. Thus by (17), we have $r(\mathbb{H}_1,\mathbb{H}_2)\le\epsilon_N$. By (6) and the monotonicity in r of the right-hand side of (6) we get (iii) with $\psi_N=C_q^2\frac{4\epsilon_N}{(1-\epsilon_N)^3}$. □

3. Free processes

Proof of Theorem 3. By Theorem 2, $\psi(\mathbb{H}_1,\mathbb{H}_2)\le r\frac{r^2-3r+4}{(1-r)^3}$. Since $\psi(\mathbb{H}_1,\mathbb{H}_2)\ge 0$, we can assume without loss of generality that $0<r<1$. Fix $\epsilon\in(0,r)$. Then there are unit vectors $f\in\mathbb{H}_1,g\in\mathbb{H}_2$ such that $r_0:=\langle f|g\rangle>r-\epsilon>0$. Then

$$\psi(\mathbb{H}_1,\mathbb{H}_2)\ge\sup\frac{\operatorname{cov}(v(\mathbf{X}_f),w(\mathbf{X}_g))}{\|v(\mathbf{X}_f)\|_1\|w(\mathbf{X}_g)\|_1},\tag{23}$$

where the supremum is taken over all real continuous functions v,w. The joint distribution of $\mathbf{X}_f,\mathbf{X}_g$ is known, and has the density

$$p(x,y)=\frac{1-r_0^2}{4\pi^2}\frac{\sqrt{4-x^2}\sqrt{4-y^2}}{(1-r_0^2)^2-r_0(1+r_0^2)xy+r_0^2(x^2+y^2)},$$

i.e., $\mathbb{E}(v(\mathbf{X}_f)w(\mathbf{X}_g))=\int_{-2}^2\int_{-2}^2 v(x)w(y)p(x,y)\,dxdy$; see [5, Theorem 1.10]. The one-dimensional distributions of $\mathbf{X}_f,\mathbf{X}_g$ have the same density $p(x)=\frac{1}{2\pi}\sqrt{4-x^2}$. Thus the right-hand side of (23) becomes

$$\sup\frac{\left|\int v(x)w(y)(p(x,y)-p(x)p(y))dxdy\right|}{\int|v(x)|p(x)dx\int|w(y)|p(y)dy},$$

which is equal to

$$\begin{aligned}&\sup_{|x|,|y|\le 2}\left|1-\frac{p(x,y)}{p(x)p(y)}\right|\\&=\sup_{|x|,|y|\le 2}\left|1-\frac{1-r_0^2}{(1-r_0^2)^2-r_0(1+r_0^2)xy+r_0^2(x^2+y^2)}\right|=r_0\frac{r_0^2-3r_0+4}{(1-r_0)^3}.\end{aligned}$$

Since $r-\epsilon<r_0\le r$ and $\epsilon>0$ is arbitrary, this concludes the proof. □

In the remaining part of this section we present the simplifications in the proofs of Theorem 1 and Theorem 2 which occur in the free case $q=0$. Here (9) simplifies to

$$\mathbf{a}_h g_1\otimes\cdots\otimes g_n:=\langle h|g_1\rangle g_2\otimes\cdots\otimes g_n\tag{24}$$

and the commutation relation (2) reduces to

$$\mathbf{a}_g \mathbf{a}_h^* = \langle h|g\rangle \mathbf{I}. \tag{25}$$

The scalar product in formula (8) becomes the regular symmetric scalar product in the tensor product of the Hilbert spaces

$$\langle g_1 \otimes \cdots \otimes g_n | h_1 \otimes \cdots \otimes h_m \rangle = \begin{cases} \prod_{j=1}^n \langle g_j | h_j \rangle & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Definition (11) of the Wick product simplifies to

$$\Psi(h \otimes g_1 \otimes \cdots \otimes g_n) := \mathbf{X}_h \Psi(g_1 \otimes \cdots \otimes g_n) - \langle h|g_1\rangle \Psi(g_2 \otimes \cdots \otimes g_n). \tag{26}$$

From (25) follows the so-called normal ordered representation of Wick products

$$\Psi(g_1 \otimes \cdots \otimes g_n) = \sum_{m=0}^{n} \mathbf{a}^*(g_1) \dots \mathbf{a}^*(g_{n-m}) \mathbf{a}(g_{n-m+1}) \dots \mathbf{a}(g_n); \tag{27}$$

compare [4, Proposition 1.1]. For example $\Psi(g) = \mathbf{a}_g + \mathbf{a}_g^*, \Psi(f \otimes g) = \mathbf{a}_f \mathbf{a}_g + \mathbf{a}_f^* \mathbf{a}_g + \mathbf{a}_f^* \mathbf{a}_g^*$.

In the proof of Theorem 1 we no longer need to invoke [5, Lemma 1.4] to obtain a bound for the norm of $P^{\otimes n}$, as that is a standard tensor product result [15, Section 2.6.12 Eqn. (16)]. With these simplifications, the proof of Theorem 1 is now self-contained and more transparent.

A key step in the proof of Theorem 2, i.e., (22), can be obtained more directly in the case of free processes. This result can also be derived from Bozejko [3]. We add for completeness the proof in our notation and setting.

Direct proof of (22). Let $\{e_j : j = 1, 2, \dots\}$ be an orthonormal basis of $\mathbb{H}$. Then $\{e^{\otimes \mathbf{j}} : |\mathbf{j}| = 0, 1, \dots\}$ forms an orthonormal basis of $\mathcal{H}_q$. Since $\mathbf{Z} \in \mathcal{A} \cap \widetilde{\Psi}\left(\overline{\mathbb{H}^{\otimes n}}\right)$, we have the expansion $\mathbf{Z} = \sum_{|\mathbf{i}|=n} \alpha_{\mathbf{i}} \Psi(e^{\otimes \mathbf{i}})$. Then

$$\|\mathbf{Z}\|_2 = \|\mathbf{Z}\mathbf{1}\|_{\mathcal{H}_q} = \left\| \sum_{|\mathbf{i}|=n} \alpha_{\mathbf{i}} e^{\otimes \mathbf{i}} \right\|_{\mathcal{H}_q} = \left(\sum_{|\mathbf{i}|=n} |\alpha_{\mathbf{i}}|^2 \right)^{1/2}.$$

Take $\xi \in \mathcal{H}_q$ of norm 1 and expand it into the orthonormal basis

$$\xi = \sum_{\mathbf{j}} \beta_{\mathbf{j}} e^{\otimes \mathbf{j}}.$$

Using the normal ordered expansion (27) we have

$$\mathbf{Z}\xi = \sum_{|\mathbf{i}|=n} \sum_{\mathbf{j}} \sum_{m=0}^{n} \alpha_{\mathbf{i}} \beta_{\mathbf{j}} \mathbf{a}^*_{e_{i(1)}} \mathbf{a}^*_{e_{i(n-m)}} \mathbf{a}_{e_{i(n-m+1)}} \dots \mathbf{a}_{e_{i(n)}} e^{\otimes \mathbf{j}}.$$

The expression

$$\mathbf{a}^*_{e_{i(1)}} \mathbf{a}^*_{e_{i(n-m)}} \mathbf{a}_{e_{i(n-m+1)}} \dots \mathbf{a}_{e_{i(n)}} e^{\otimes \mathbf{j}}$$

is zero, except when the first m components of $\mathbf{j}$ coincide with the last m components of $\mathbf{i}$ in reverse order. Therefore, we keep only the multi-indices in the sum that have the form $\mathbf{i} = (\mathbf{i}', \mathbf{k})$, $\mathbf{j} = (\overline{\mathbf{k}}, \mathbf{j}')$, where $\mathbf{j}'$ is arbitrary, $\mathbf{i}'$ is an arbitrary multi-index of length $|\mathbf{i}'| = n - m$, $\mathbf{k}$ is arbitrary multi-index of length $|\mathbf{k}| = m$, and $\overline{\mathbf{k}}$ is the reverse of $\mathbf{k}$, i.e., $\overline{k}(s) = k(m - s + 1)$. Dropping the primes, we get

$$\mathbf{Z}\xi = \sum_{m=0}^{n} \sum_{|\mathbf{i}|=n-m} \sum_{\mathbf{j}} \sum_{|\mathbf{k}|=m} \alpha_{(\mathbf{i},\mathbf{k})} \beta_{(\overline{\mathbf{k}},\mathbf{j})} e^{\otimes \mathbf{i}} \otimes e^{\otimes \mathbf{j}}.$$

By the Cauchy-Schwarz inequality for $a_m \in \mathbb{C}, m = 0, 1, \dots, n$, we have

$$\left(\sum_{m=0}^{n} |a_m| \right)^2 \leq (n+1) \sum_{m=0}^{n} |a_m|^2,$$

which together with the triangle inequality gives

$$\|\mathbf{Z}\xi\|_{\mathcal{H}_q}^2 \leq (n+1) \sum_{m=0}^{n} \left\| \sum_{|\mathbf{i}|=n-m} \sum_{\mathbf{j}} \sum_{|\mathbf{k}|=m} \alpha_{(\mathbf{i},\mathbf{k})} \beta_{(\overline{\mathbf{k}},\mathbf{j})} e^{\otimes \mathbf{i}} \otimes e^{\otimes \mathbf{j}} \right\|_{\mathcal{H}_q}^2 .$$

Notice that for a fixed $m \in \mathbb{N}$, different pairs of multi-indices $\mathbf{i}, \mathbf{j}$ of lengths $|\mathbf{i}| = n - m, |\mathbf{j}| \geq 0$ generate different concatenations $(\mathbf{i}, \mathbf{j})$. Thus the corresponding vectors $e^{\otimes \mathbf{i}} \otimes e^{\otimes \mathbf{j}}$ are orthogonal, and we get

$$\|\mathbf{Z}\xi\|_{\mathcal{H}_q}^2 \leq (n+1) \sum_{m=0}^{n} \sum_{|\mathbf{i}|=n-m} \sum_{\mathbf{j}} \left| \sum_{|\mathbf{k}|=m} \alpha_{(\mathbf{i},\mathbf{k})} \beta_{(\overline{\mathbf{k}},\mathbf{j})} \right|^2 .$$

By the Cauchy-Schwarz inequality, this gives

$$\|\mathbf{Z}\xi\|_{\mathcal{H}_q}^2 \leq (n+1) \sum_{m=0}^{n} \sum_{|\mathbf{i}|=n-m, |\mathbf{k}|=m} |\alpha_{(\mathbf{i},\mathbf{k})}|^2 \sum_{\mathbf{j}, |\mathbf{k}|=m} |\beta_{(\mathbf{k},\mathbf{j})}|^2 \leq (n+1)^2 \|\mathbf{Z}\|_2^2 \, \|\xi\|_{\mathcal{H}_q}^2 .$$

Therefore (22) follows with constant $C_q = 1$. The rest of the proof of Theorem 2 then follows unchanged. □

4. Open questions

(1) A classical version of a non-commutative process is defined as a classical process that has the same sequence of mixed moments of all orders as the non-commutative process. It would be interesting to clarify if this concept could link Theorem 4 with the Ibragimov conjecture.

(1) Does the q-Gaussian sequence in Theorem 4 have a classical version?
(2) If a q-Gaussian process is ψ-mixing, and has a classical version, does the classical version satisfy the classical ψ-mixing condition?

A sufficient condition for the existence of a classical version is given in [5, Section 4]; for a necessary condition, see [10, Theorem 3]. Definitions and properties of the classical (commutative) mixing conditions can be found in [9].

(2) Bradley [8] shows that commutative (not necessarily stationary) Markov chains X_k with small values of the ψ-mixing coefficient ψ_1 satisfy a mixing condition which implies that there are positive constants c, C which depend only on ψ_1 and such that

$$c \sum E(|X_k|^2) \leq E(|\sum X_k|^2) \leq C \sum E(|X_k|^2). \tag{28}$$

Since the Markov property is well-defined in the non-commutative context, it would be interesting to know if Bradley's result, or its implication (28), has a non-commutative version. Theorem 4 shows that without the Markov property the non-commutative version of the left-hand side of (28) fails.

Acknowledgements

The first author thanks M. Bozejko for several discussions, an invitation to a workshop, and references. He also thanks R. Bradley for an early copy of [9], and for encouragement. Both authors benefitted from discussions with M. Peligrad.

References

[1] István Berkes and Walter Philipp. Limit theorems for mixing sequences without rate assumptions. *Ann. Probab.*, 26(2):805–831, 1998. MR **99d:**60022

[2] Philippe Biane. Free hypercontractivity. *Comm. Math. Phys.*, 184(2):457–474, 1997. MR **98g:**46097

[3] Marek Bożejko. A q-deformed probability, Nelson's inequality and central limit theorems. In Garbaczewski and Popowicz, editors, *Nonlinear fields: classical, random, semiclassical (Karpacz, 1991)*, pages 312–335. World Sci. Publishing Co., River Edge, NJ, 1991. MR **93a:**81098

[4] Marek Bożejko. Ultracontractivity and strong Sobolev inequality for q-Ornstein-Uhlenbeck semigroup $(-1 < q < 1)$. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 2(2):203–220, 1999. MR **2002d:**47058

[5] Marek Bożejko, Burkhard Kümmerer, and Roland Speicher. q-Gaussian processes: Non-commutative and classical aspects. *Comm. Math. Physics*, 185:129–154, 1997. MR **98h:**81053

[6] Marek Bożejko and Roland Speicher. An example of a generalized Brownian motion. *Comm. Math. Phys.*, 137(3):519–531, 1991. MR **92m:**46096

[7] Richard C. Bradley. A remark on the central limit question for dependent random variables. *J. Appl. Probab.*, 17(1):94–101, 1980. MR **82j:**60031

[8] Richard C. Bradley. Every "lower psi-mixing" Markov chain is "interlaced rho-mixing". *Stochastic Process. Appl.*, 72(2):221–239, 1997. MR **98m:**60053

[9] Richard C. Bradley. Introduction to strong mixing conditions. Technical report, Indiana University, Bloomington, 2002. ISBN 1-58902-566-0.

[10] Włodzimierz Bryc. Classical versions of q-Gaussian processes: conditional moments and Bell's inequality. *Comm. Math. Physics*, 219:259–270, 2001. MR **2002h:**81129

[11] U. Frisch and R. Bourret. Parastochastics. *J. Mathematical Physics*, 11(2):364–390, 1970. MR **41:**4979

[12] I. A. Ibragimov and Ju. V. Linnik. *Nezavisimye stalionarno svyazannye velichiny*. Izdat. "Nauka", Moscow, 1965. MR **34:**2049

[13] I. A. Ibragimov and Yu. V. Linnik. *Independent and stationary sequences of random variables*. Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian edited by J. F. C. Kingman. MR **48:**1287

[14] Marius Iosifescu. Limit theorems for ϕ-mixing sequences. A survey. In *Proceedings of the Fifth Conference on Probability Theory (Braşov, 1974)*, pages 51–57. Editura Acad. R.S.R., Bucharest, 1977. MR **57:**1609

[15] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras*. Interscience Publishers, Inc., Academic Press, 1983. MR **85j:**46099

[16] A. N. Kolmogorov and Ju. A. Rozanov. On a strong mixing condition for stationary Gaussian processes. *Teor. Verojatnost. i Primenen.*, 5:222–227, 1960. MR **24:**A3009

[17] Edward Nelson. Notes on non-commutative integration. *J. Functional Analysis*, 15:103–116, 1974. MR **50:**8102

[18] Magda Peligrad. An invariance principle for ϕ-mixing sequences. *Ann. Probab.*, 13(4):1304–1313, 1985. MR **87b:**60056

[19] D. V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables.* American Mathematical Society, Providence, RI, 1992. MR **94c:**46133

Department of Mathematics, University of Cincinnati, P.O. Box 210025, Cincinnati, Ohio 45221–0025

E-mail address: `Wlodzimierz.Bryc@UC.edu`

Department of Mathematics, University of Cincinnati, P.O. Box 210025, Cincinnati, Ohio 45221–0025

E-mail address: `Victor.Kaftal@UC.edu`