PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 2, Pages 381–384
S 0002-9939(03)07052-7
Article electronically published on June 11, 2003

# WEAKLY SEQUENTIAL COMPLETENESS OF THE PROJECTIVE TENSOR PRODUCT $L^p[0,1]\hat{\otimes}X$, $1 < p < \infty$

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(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. D. R. Lewis (1977) proved that for a Banach space $X$ and $1 < p < \infty$, $L^p[0,1]\hat{\otimes}X$, the projective tensor product of $L^p[0,1]$ and $X$, is weakly sequentially complete whenever $X$ is weakly sequentially complete. In this note, we give a short proof of Lewis's result, based on our sequential representation (2001) of $L^p[0,1]\hat{\otimes}X$.

For any Banach space $X$, we will denote its topological dual by $X^*$ and its closed unit ball by $B_X$. From [4, page 3] and [5, page 155], we know that the Haar system $\{\chi_i\}_{i=1}^\infty$ is an unconditional basis of $L^p[0,1]$, $1 < p < \infty$. Let us use $K_p$ to denote the unconditional basis constant of the basis $\{\chi_i\}_{i=1}^\infty$. Now renorm $L^p[0,1]$ by

$$\|f\|_p^{new} = \sup\left\{ \left\| \sum_{i=1}^\infty \theta_i a_i \chi_i \right\|_p : \theta_i = \pm 1, i = 1, 2, \cdots \right\}, \qquad f = \sum_{i=1}^\infty a_i \chi_i \in L^p[0,1].$$

Then

$$\|\cdot\|_p \le \|\cdot\|_p^{new} \le K_p \cdot \|\cdot\|_p.$$

With this new norm, $L^p[0,1]$ is also a Banach space. Furthermore, $\{\chi_i\}_{i=1}^\infty$ is a monotone, unconditional basis with respect to this new norm. Now let

$$e_i = \frac{\chi_i}{\|\chi_i\|_p^{new}}, \qquad i = 1, 2, \cdots.$$

Then $\{e_i\}_{i=1}^\infty$ is a normalized, unconditional basis of $(L^p[0,1], \|\cdot\|_p^{new})$ whose unconditional basis constant is 1. From [4, pp. 18–19] we have the following

**Proposition 1.** *Let $u = \sum_{i=1}^\infty \alpha_i e_i \in L^p[0,1]$, $1 < p < \infty$. Then*

(i) *For each subset $\sigma$ of $\mathbb{N}$, $\|\sum_{i\in\sigma} \alpha_i e_i\|_p^{new} \le \|u\|_p^{new}$.*
(ii) *For each choice of signs $\theta = \{\theta_i\}_1^\infty$, $\|\sum_{i=1}^\infty \theta_i \alpha_i e_i\|_p^{new} \le \|u\|_p^{new}$.*
(iii) *For each $\lambda = (\lambda_i)_i \in \ell_\infty$, $\|\sum_{i=1}^\infty \lambda_i \alpha_i e_i\|_p^{new} \le 2 \cdot \|\lambda\|_{\ell_\infty} \cdot \|u\|_p^{new}$.*

Received by the editors May 7, 2002 and, in revised form, September 12, 2002.
2000 *Mathematics Subject Classification.* Primary 46M05, 46B28, 46E40.
*Key words and phrases.* Projective tensor product, function space, weakly sequential completeness.

For any Banach space $X$ and $1 < p < \infty$ with $1/p + 1/p\prime = 1$, define

$$L^p_{weak}(X) = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_i x^*(x_i)e_i \text{ converges in } L^p[0,1] \text{ for } \forall x^* \in X^* \right\},$$

$$L^p\langle X\rangle = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i^*(x_i)| < \infty \ \forall \ (x_i^*)_i \in L^{p\prime}_{weak}(X^*) \right\},$$

and define norms on $L^p_{weak}(X)$ and $L^p\langle X\rangle$, respectively, to be

$$\|\bar{x}\|_{weak} = \sup\left\{ \left\| \sum_{i=1}^{\infty} x^*(x_i)e_i \right\|_p^{new} : x^* \in B_{X^*} \right\}, \quad \bar{x} \in L^p_{weak}(X),$$

$$\|\bar{x}\|_{L^p\langle X\rangle} = \sup\left\{ \sum_{i=1}^{\infty} |x_i^*(x_i)| : (x_i^*)_i \in B_{L^{p\prime}_{weak}(X^*)} \right\}, \quad \bar{x} \in L^p\langle X\rangle.$$

With their own norm respectively, $L^p_{weak}(X)$ and $L^p\langle X\rangle$ are Banach spaces [1], [2]. Moreover, from [1] we have the following

**Proposition 2.** (i) *For each* $\bar{x} = (x_i)_i \in L^p\langle X\rangle$,

$$\lim_n \|(0, \cdots, 0, x_n, x_{n+1}, \cdots)\|_{L^p\langle X\rangle} = 0.$$

(ii) $L^p[0,1]\hat{\otimes}X$ *is isomorphic to* $(L^p[0,1], \|\cdot\|_p^{new})\hat{\otimes}X$ *which is isometrically isomorphic to* $L^p\langle X\rangle$.

**Proposition 3.** $(L^p\langle X\rangle)^* = L^{p\prime}_{weak}(X^*)$, *where the dual operation is defined by*

$$\langle \bar{x}, \bar{x}^* \rangle = \sum_{i=1}^{\infty} x_i^*(x_i)\,,$$

*for each* $\bar{x} = (x_i)_i \in L^p\langle X\rangle$ *and each* $\bar{x}^* = (x_i^*)_i \in L^{p\prime}_{weak}(X^*)$.

*Proof.* Let $F \in (L^p\langle X\rangle)^*$. For each $i \in \mathbb{N}$, define an $x_i^* \in X^*$ by

$$x_i^*(x) = \langle (0, \cdots, 0, \overset{(i)}{x}, 0, 0, \cdots), F\rangle.$$

Then one can associate a linear operator $\tilde{F} : (L^p[0,1], \|\cdot\|_p^{new}) \longrightarrow X^*$ by

$$\tilde{F}(f)(x) = \langle (e_i^*(f)x)_i, F\rangle,$$

for each $f \in (L^p[0,1], \|\cdot\|_p^{new})$ and $x \in X$, i.e.,

$$\tilde{F}(f)(x) = \sum_{i=1}^{\infty} e_i^*(f)x_i^*(x).$$

The operator $\tilde{F}$ is bounded, in fact

$$\|\tilde{F}\| \le \|F\|,$$

and it is easy to check that

$$\|(x_i^*)_i\|_{L^{p\prime}_{weak}(X^*)} = \|\tilde{F}\|.$$

This shows that $F = (x_i^*)_i \in L^{p\prime}_{weak}(X^*)$. The other inclusion is obvious. $\square$

**Lemma 4.** *For each $\bar{x}^* \in (L^p\langle X\rangle)^* = L^{p'}_{weak}(X^*)$, define*

$$\begin{array}{rccl} I_{\bar{x}^*}: & L^p\langle X\rangle & \longrightarrow & \ell_1 \\ & \bar{x} & \longmapsto & (x_i^* x_i)_i \,. \end{array}$$

*Then $I_{\bar{x}^*}$ is a continuous linear map.*

*Proof.* For each $(s_i)_i \in \ell_\infty$, by Proposition 1,

$$\begin{aligned} |\sum_{i=1}^{\infty} s_i x_i^* x_i| &= |\langle (s_i x_i)_i, \bar{x}^*\rangle \le \|(s_i x_i)_i\|_{L^p\langle X\rangle} \cdot \|\bar{x}^*\|_{(L^p\langle X\rangle)^*} \\ &\le 2 \cdot \|(s_i)_i\|_{\ell_\infty} \cdot \|\bar{x}\|_{L^p\langle X\rangle} \cdot \|\bar{x}^*\|_{(L^p\langle X\rangle)^*}. \end{aligned}$$

So

$$\|(x_i^* x_i)_i\|_{\ell_1} \le 2 \cdot \|\bar{x}\|_{L^p\langle X\rangle} \cdot \|\bar{x}^*\|_{(L^p\langle X\rangle)^*}.$$

Therefore, $I_{\bar{x}^*}$ is continuous. The proof is complete. □

**Theorem 5.** *$L^p[0,1]\hat{\otimes}X$ is weakly sequentially complete if $X$ is weakly sequentially complete.*

*Proof.* By Proposition 2, it is enough to show that $L^p\langle X\rangle$ is weakly sequentially complete if $X$ is weakly sequentially complete. Let $\{\bar{x}^{(n)}\}_1^\infty$ be a weakly Cauchy sequence in $L^p\langle X\rangle$. It is easy to see that $\{x_i^{(n)}\}_{n=1}^\infty$ are weakly Cauchy sequences in $X$ for each $i \in \mathbb{N}$. Therefore there are $x_i \in X$ such that

$$\text{weak-}\lim_n x_i^{(n)} = x_i\,, \qquad i = 1, 2, \cdots . \tag{1}$$

Denote $M = \sup_n \|\bar{x}^{(n)}\|_{L^p\langle X\rangle} < \infty$. For each $\bar{x}^* = (x_i^*)_i \in L^{p'}_{weak}(X^*) = (L^p\langle X\rangle)^*$ and each fixed $m \in \mathbb{N}$, from (1) there exists an $n_0 \in \mathbb{N}$ such that

$$|x_i^*(x_i^{(n_0)} - x_i)| \le 1/m\,, \qquad i = 1, 2, \cdots, m.$$

Thus,

$$\begin{aligned} \sum_{i=1}^{m} |x_i^*(x_i)| &\le \sum_{i=1}^{m} |x_i^*(x_i^{(n_0)} - x_i)| + \sum_{i=1}^{m} |x_i^*(x_i^{(n_0)})| \\ &\le 1 + \sum_{i=1}^{\infty} |x_i^*(x_i^{(n_0)})| \\ &\le 1 + \|\bar{x}^*\|_{(L^p\langle X\rangle)^*} \cdot \|\bar{x}^{(n_0)}\|_{L^p\langle X\rangle} \\ &\le 1 + M\|\bar{x}^*\|_{(L^p\langle X\rangle)^*}. \end{aligned}$$

Letting $m \longrightarrow \infty$,

$$\sum_{i=1}^{\infty} |x_i^*(x_i)| \le 1 + M\|\bar{x}^*\|_{(L^p\langle X\rangle)^*} < \infty.$$

Therefore $\bar{x} = (x_i)_i \in L^p\langle X\rangle$. Next, we want to show that $\bar{x}^{(n)}$ converges to $\bar{x}$ weakly in $L^p\langle X\rangle$.

Fix $\bar{x}^* \in (L^p\langle X\rangle)^*$. By Lemma 4, $I_{\bar{x}^*}$ is continuous and hence, weakly – weakly continuous. Since $\{\bar{x}^{(n)}\}_1^\infty$ is a weakly Cauchy sequence in $L^p\langle X\rangle$, $\{I_{\bar{x}^*}(\bar{x}^{(n)}\}_1^\infty$ is a weakly Cauchy sequence in $\ell_1$, and hence, relatively weakly sequentially compact.

By the Schur property, $\{I_{\bar{x}^*}(\bar{x}^{(n)}\}_1^\infty$ is a relatively sequentially compact subset of $\ell_1$. Thus, for each $\varepsilon > 0$, there exists an $m_1 \in \mathbb{N}$ such that

$$\sum_{i=m_1+1}^{\infty} |x_i^*(x_i^{(n)})| \leq \varepsilon/3\,, \qquad n = 1, 2, \cdots\,. \tag{2}$$

By Proposition 3, there exists an $m_2 > m_1$ such that

$$\sum_{i=m_2+1}^{\infty} |x_i^*(x_i)| \leq \varepsilon/3. \tag{3}$$

Now, from (1), there exists an $n_0 \in \mathbb{N}$ such that for each $n > n_0$,

$$|x_i^*(x_i^{(n)} - x_i)| < \varepsilon/3, \qquad i = 1, 2, \cdots, m_2\,. \tag{4}$$

Thus, from (2), (3), and (4), for each $n > n_0$,

$$\begin{aligned}
|\langle \bar{x}^{(n)} - \bar{x}, \bar{x}^* \rangle| &= |\sum_{i=1}^{\infty} x_i^*(x_i^{(n)} - x_i)| \\
&\leq \sum_{i=1}^{m_2} |x_i^*(x_i^{(n)} - x_i)| \\
&+ \sum_{i=m_2+1}^{\infty} |x_i^*(x_i^{(n)})| + \sum_{i=m_2+1}^{\infty} |x_i^*(x_i)| \\
&\leq \varepsilon.
\end{aligned}$$

Hence, $\bar{x}^{(n)}$ converges to $\bar{x}$ weakly in $L^p\langle X\rangle$. Therefore, $L^p\langle X\rangle$ is weakly sequentially complete. The proof is complete. □

## Acknowledgement

The author thanks Professor Joe Diestel for his many useful suggestions concerning this paper. The author also thanks the referee of this paper for a good suggestion for Proposition 3 and its proof.

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