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AN EXISTENCE THEOREM OF HARMONIC FUNCTIONS WITH POLYNOMIAL GROWTH

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ABSTRACT. We prove the existence of nonconstant harmonic functions with polynomial growth on manifolds with nonnegative Ricci curvature, Euclidean volume growth and unique tangent cone at infinity.

INTRODUCTION

For a noncompact, complete Riemannian manifold (M^n, p) with nonnegative Ricci curvature,

$$\mathrm{Ric}_{M^n} \geq 0, \tag{0.1}$$

we have the notion of tangent cone at infinity, which is *any* pointed Gromov-Hausdorff limit of *some* sequence $M_i = (M^n, R_i^{-2} dx^2)$ with $R_i \to \infty$.

The almost rigidity theorem of Cheeger and Colding [4] implies that if M^n has Euclidean volume growth, i.e., there is some $V_\infty > 0$ such that for all $R > 0$,

$$\mathrm{Vol}(B_R(p)) \geq V_\infty R^n, \tag{0.2}$$

then every tangent cone at infinity is a metric cone, i.e., $\mathbf{R}_+ \times X$ with the metric $dr^2 + r^2 dx^2$; here (X, dx^2) is a metric space with diameter no more than π.

In this paper we will prove

Theorem 0.1. *Assume that M^n is a complete Riemannian manifold satisfying* (0.1) *and* (0.2). *Assume that M has a unique tangent cone $C(X)$ at infinity. Then the dimension of the space of harmonic functions on M^n with*

$$|u(y)| \leq C(1 + d(p, y)^N) \tag{0.3}$$

is at least $C(V_\infty)N^{n-1}$; here $C(V_\infty) > 0$.

For each $N > 0$, the space of harmonic functions u with (0.3) on manifolds with (0.1) is finite dimensional; this was conjectured by Yau and proved by Colding and Minicozzi in [11]. See, for example, [12], [16] for further developments.

The tangent cone at infinity may not be unique; see [19], [5]. However, it is unique if we assume that the sectional curvature is nonnegative. Moreover, the example of Menguy [18] shows that even if M^n has unique tangent cone, M^n can have infinite topological type.

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By inspecting the proof of Theorem 0.1, we have, when the tangent cones are not unique,

Theorem 0.2. *Assume that M^n is a complete Riemannian manifold satisfying* (0.1) *and* (0.2). *Assume there exists $\lambda > 0$, such that for all tangent cones $C(X)$, λ is greater than $\lambda_1(X)$, the first eigenvalue of the Laplacian on X, and λ is not an eigenvalue of any X. Then there exists a nonconstant polynomial growth harmonic function on M^n.*

It seems the example in [5] satisfies the assumption above and so admits a nonconstant polynomial growth harmonic function.

There are manifolds that do not admit nonconstant harmonic functions with polynomial growth. For example, the manifold obtained by rounding off the end of $\mathbf{R}_+ \times S^{n-1}$; one can check this directly or by [20]. Note this example satisfies (0.1) but not (0.2).

In [13], the author showed that there is a separation of variables formula for the Laplacian on $C(X)$. In particular, there exist many harmonic functions on $C(X)$. We will transplant these harmonic functions back to balls on M^n; we then construct the desired harmonic functions by the Arzela-Ascoli theorem. In order to control the growth of these functions, we use a monotonicity Lemma 1.2, which is a generalization of the monotonicity of frequency for harmonic functions on $\mathbf{R}^n$ (see [1], [10], [9]).

Suppose that $(M_i^n, \mathrm{Vol}_i) \xrightarrow{d_{GH}} (M_\infty, \mu_\infty)$ in the measured Gromov-Hausdorff sense, i.e., the sequence $\{M_i^n\}$ converges in the Gromov-Hausdorff sense to M_∞, and for any $x_i \to x_\infty$ ($x_i \in M_i^n$) and $R > 0$, we have $\mathrm{Vol}_i(B_R(x_i)) \to \mu_\infty(B_R(x_\infty))$. In fact, for any sequences of manifolds with Ricci curvature bounded from below, after possible renormalization of the measures when $\{M_i^n\}$ is collapsing, there is a subsequence that converges in the measured Gromov-Hausdorff sense; moreover, under assumption (0.2), μ_∞ is just the n-Hausdorff measure on M_∞. See [5].

Definition 0.3. Suppose $K_i \subset M_i^n \xrightarrow{d_{GH}} K_\infty \subset M_\infty$ in the measured Gromov-Hausdorff sense. f_i is a function on M_i^n, $i = 1, 2, ...$; f_∞ is a continuous function on M_∞. Assume that $\Phi_i : K_\infty \to K_i$ are ϵ_i-Gromov-Hausdorff approximations, $\epsilon_i \to 0$. If $f_i \circ \Phi_i$ converge to f_∞ uniformly, we say that $f_i \to f_\infty$ *uniformly* over $K_i \xrightarrow{d_{GH}} K_\infty$.

For a Lipschitz function f on M_∞, one can define a norm

$$\|f\|_{H_{1,2}}^2 = \|f\|_{L^2}^2 + \int_{M_\infty} |\operatorname{Lip} f|^2, \tag{0.4}$$

where $\operatorname{Lip} f$ is the pointwise Lipschitz constant

$$\operatorname{Lip} f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y,x)}. \tag{0.5}$$

In [3], a Sobolev space $H_{1,2}$ is constructed by taking the closure of the norm (0.4). Moreover, one can define the *differential* df for $H_{1,2}$ functions f. In [6] it is proved that M_∞ is μ_∞-rectifiable, and, as a corollary, (0.4) comes from an inner product $\langle \cdot, \cdot \rangle$. Thus $H_{1,2}$ transforms to a Hilbert space. Now by the standard theory of Dirichlet forms, one gets a *positive* self-adjoint Laplacian Δ on M_∞,

$$\int_{M_\infty} \langle df, dg \rangle = \int_{M_\infty} (\Delta f) g; \tag{0.6}$$

see Theorem 6.25 of [6].

The general philosophy is that the Laplacian Δ_i over M_i "converge" to the operator Δ on M_∞. We have the *persistence of Poisson's equation* [3], [6], [14]:

Lemma 0.4. *Assume that* $\Delta u_i = f_i$ *on (a subset of)* M_i, $\mathbf{Lip} u_i, \mathbf{Lip} f_i \leq L$ *for some* $L > 0$. *Assume that* $u_i \to u_\infty$, $f_i \to f_\infty$ *uniformly. Then on* M_∞ *we have* $\Delta u_\infty = f_\infty$.

We use some standard notation. Write

$$\fint_W f = \frac{1}{\mathrm{Vol}(W)} \int_W f. \tag{0.7}$$

Denote by $A(p, R_1, R_2)$ the metric annulus $\{x | R_1 \leq d(p,x) \leq R_2\}$. For any function u_i we denote by $u_{i,p,R}$ the average of u_i over $A(p, R/2, R)$:

$$u_{i,p,R} = \fint_{A(p,\, R/2,\, R)} u_i. \tag{0.8}$$

The Laplacian operators are assumed to be *positive*.

After finishing this manuscript, Professor Colding pointed out to the author a paper of Zhang [22], in which nonconstant harmonic functions of polynomial growth can be constructed in the case when $C(X)$ is a smooth cone. Our construction turns out to be a generalization of [22] and applies to the case when $C(X)$ is not a smooth cone (so there are no coordinate systems available).

1. Analysis on metric cones

It is easy to see ([13]) that the $(n-1)$-Hausdorff measure on the cross section X satisfies a doubling condition and the Poincare inequality. Moreover, the rectifiability as in [6] holds on X as well; so one can define a Laplacian Δ_X on X. We have an eigenfunction expansion $\{\phi_i\}$ with $\Delta_X \phi_i = \lambda_i \phi_i$ on X. By the standard Moser iteration, the ϕ_i are Hölder continuous; later we will see that they are Lipschitz.

On a metric cone $C(X)$, there is a separation of variable formula [13]:

$$\Delta u = -\frac{\partial^2 u}{\partial r^2} - \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_X u. \tag{1.1}$$

Therefore, if ϕ_i is the i-th eigenfunction of Δ_X on X with eigenvalue λ_i, then $r^{\alpha_i}\phi_i(x)$ is harmonic; here α_i is the unique positive number with

$$\lambda_i = \alpha_i(n + \alpha_i - 2). \tag{1.2}$$

We normalize so that $\|\phi_i\|_{L^2(X)} = 1$. Assume u is harmonic on $B_2(p) \subset C(X)$. Then we can write (see [2], [8])

$$u = \sum_{i=0}^{\infty} c_i r^{\alpha_i} \phi_i. \tag{1.3}$$

Define

$$I(r) = \frac{1}{\mathrm{Vol}(\partial B_r(p_\infty))} \int_{\partial B_r(p_\infty)} u^2; \tag{1.4}$$

here Vol is the $(n-1)$-Hausdorff measure; see [5]. p_∞ is the pole of $C(X)$. Then

$$I(r) = \sum_{i=0}^{\infty} c_i^2 r^{2\alpha_i}. \tag{1.5}$$

Similarly to the Euclidean case ([14]), we have

Lemma 1.1. *There is a $k > 1$ that depends only on X such that for $\epsilon > 0$ sufficiently small, if u is harmonic, then*

$$I(r) \le (2^{\alpha_1+\epsilon})^2 I(r/2) \tag{1.6}$$

implies

$$I(r/2) < (2^{\alpha_1+\frac{\epsilon}{k}})^2 I(r/4). \tag{1.7}$$

Proof. By (1.5), (1.6) is equivalent to

$$\sum_{\alpha_i \ne \alpha_1} c_i^2 r^{2\alpha_i}\Big(1 - \frac{2^{2\alpha_1+2\epsilon}}{2^{2\alpha_i}}\Big) \le \sum_{\alpha_i=\alpha_1} c_i^2 r^{2\alpha_1}(2^{2\epsilon}-1). \tag{1.8}$$

On the other hand, (1.7) is equivalent to

$$\sum_{\alpha_i \ne \alpha_1} \frac{1}{2^{2\alpha_i}} c_i^2 r^{2\alpha_i}\Big(1 - \frac{2^{2\alpha_1+\frac{2}{k}\epsilon}}{2^{2\alpha_i}}\Big) \le \sum_{\alpha_i=\alpha_1} \frac{1}{2^{2\alpha_1}} c_i^2 r^{2\alpha_1}(2^{\frac{2}{k}\epsilon}-1). \tag{1.9}$$

Thus, it suffices to show for $\alpha_i \ne \alpha_1$,

$$\frac{1}{2^{2\alpha_i}}(2^{2\alpha_i} - 2^{2\alpha_1+\frac{2}{k}\epsilon})/(2^{2\alpha_i} - 2^{2\alpha_1+2\epsilon}) < \frac{1}{2^{2\alpha_1}}(2^{\frac{2}{k}\epsilon}-1)/(2^{2\epsilon}-1). \tag{1.10}$$

Since there is a definite gap (that depends on X) between α_1 and those $\alpha_i \ne \alpha_1$, the above holds when $k > 1$ is sufficiently close to 1 and ϵ sufficiently small. □

Corollary 1.11. *Assume u is harmonic. If*

$$\fint_{A(p_\infty, r/2, r)} u^2 \le (2^{\alpha_1+\epsilon})^2 \fint_{A(p_\infty, r/4, r/2)} u^2, \tag{1.12}$$

then

$$\fint_{A(p_\infty, r/4, r/2)} u^2 < (2^{\alpha_1+\frac{\epsilon}{k}})^2 \fint_{A(p_\infty, r/8, r/4)} u^2. \tag{1.13}$$

Lemma 1.2. *For ϵ small enough (as in Corollary 1.11), there exist $\delta, H > 0, k > 1$ depending only on ϵ such that if a manifold (M,p) satisfies* (0.1),

$$d_{GH}(B_4(p), B_4(p_\infty)) < \delta \tag{1.14}$$

($B_2(p_\infty) \subset C(X)$)*, then for any harmonic function u over $B_2(p)$, the inequality*

$$\fint_{A(p,1/2,1)} |u - u_{p,1}|^2 \le (2^{\alpha_1+\epsilon})^2 \fint_{A(p,1/4,1/2)} |u - u_{p,1/2}|^2 \tag{1.15}$$

implies

$$\fint_{A(p,1/4,1/2)} |u - u_{p,1/2}|^2 < (2^{\alpha_1+\frac{\epsilon}{k}})^2 \fint_{A(p,1/8,1/4)} |u - u_{p,1/4}|^2. \tag{1.16}$$

Proof. The proof is similar to the arguments in [14]. Assume the lemma is not true; then for $\delta_j \to 0$, we can find a sequence of harmonic functions u_i that satisfies (1.15) but not (1.16). After suitable renormalization, by the Cheng-Yau gradient estimate, a subsequence of u_i will converge to a function u_∞ on $C(X)$ satisfying (1.15) but not (1.16). Now by Lemma 0.4, u_∞ is harmonic, so we get a contradiction to Corollary 1.11. □

Lemma 1.3. *For all ϵ small enough, there exists δ such that if a manifold (M, p) satisfies* (0.1) *and* (0.2)*, and*

$$d_{GH}(B_2(p), B_2(p_\infty)) < \delta \tag{1.17}$$

$(B_2(p_\infty) \subset C(X))$, then for any nonconstant harmonic function u over $B_2(p)$,

$$⨍_{A(p,\,1/2,\,1)} |u - u_{p,1}|^2 \geq (2^{\alpha_1 - \epsilon})^2 ⨍_{A(p,\,1/4,\,1/2)} |u - u_{p,\,1/2}|^2. \tag{1.18}$$

Proof. This is clearly true for harmonic functions on the metric cone $C(X)$. The proof follows from a compactness argument like the previous lemma. □

Similarly, we have

Lemma 1.4. *For $\epsilon < 1$, there exist $\delta > 0$, $k > 1$ such that if a manifold (M, p) satisfies* (0.1) *and* (0.2)*, and*

$$d_{GH}(B_4(p), B_4(p_\infty)) < \delta \tag{1.19}$$

$(B_2(p_\infty) \subset C(X))$, then for any harmonic function u over $B_2(p)$, the inequality

$$\Big| ⨍_{A(p,\,1,\,2)} u \Big| \leq \epsilon \Big(⨍_{A(p,\,1,\,2)} |u|^2 \Big)^{\frac{1}{2}} \tag{1.20}$$

implies

$$\Big| ⨍_{A(p,\,2,\,4)} u \Big| \leq \frac{\epsilon}{k} \Big(⨍_{A(p,\,2,\,4)} |u|^2 \Big)^{\frac{1}{2}}. \tag{1.21}$$

2. The Barrier and applications

Theorem 2.1. *Assume u_∞ is harmonic on the closed ball $B_R(p) \subset C(X)$. Then u_∞ is the uniform limit of a sequence of harmonic functions u_i on $B_R(p_i) \subset M_i$.*

Proof. We approximate $u_\infty|_{\partial B_R(p_\infty)}$ by Lipschitz functions, then by the transplantation theorem of Cheeger (Lemma 10.7 of [3]) we transplant it back to M_i to a Lipschitz function β_i on $\partial B_R(p_i) \subset M_i$,

$$\beta_i \to u_\infty|_{\partial B_R(p_\infty)}. \tag{2.1}$$

Solve the Dirichlet problem

$$\begin{cases} \Delta u_i &= 0, \\ u_i &= \beta_i \quad \text{on} \quad \partial B_R(p_i). \end{cases} \tag{2.2}$$

Since $M_i \xrightarrow{d_{GH}} C(X)$, when i is getting bigger we see the ball $B_R(p_i)$ almost satisfies an *exterior sphere condition*; see [15].

Fix $X_\infty \in \partial B_R(p_\infty)$. Pick $x_i \in \partial B_R(p_i)$ with $x_i \to x_\infty$. On the cone $C(X)$ there is a unique ray starting from the pole p_∞, passing through x_∞. Pick a point q_∞ on this ray with $d(p_\infty, q_\infty) > d(p_\infty, x_\infty)$. Pick $q_i \in M_i$ with $q_i \to q_\infty$.

Consider $b_i(x) = d(q_i, x_i)^{2-n} - d(q_i, x)^{2-n}$. By the Laplacian comparison theorem,

$$\Delta b_i \leq 0. \tag{2.3}$$

Thus exactly as in Chapter 2 of [15] we get two side bounds of u_i near the boundary. Precisely, for all $\epsilon > 0$ there exists δ such that for $x_i \in \partial B_R(p_i)$, $d(x, x_i) \le \delta$ implies $|u_i(x) - u_i(x_i)| \le \epsilon$, when i is sufficiently large.

Now by the Arzela-Ascoli theorem, (a subsequence of) u_i converges to some limit function v_∞ on $C(X)$. By our estimate near the boundary and the maximum principle on $C(X)$, [3], $v_\infty = u_\infty$. □

Note our argument does not imply that u_i is continuous at the boundary.

By the Cheng-Yau gradient estimate we have

Corollary 2.4. *Harmonic functions on $C(X)$ are Lipschitz. The eigenfunctions ϕ_i on X are Lipschitz.*

Corollary 2.5. *The first eigenvalue λ_1 of Δ_X on X satisfies $\lambda_1 \ge n-1$.*

Proof. The first eigenvalue λ gives a harmonic function $r^{\alpha_1}\phi_i(x)$ on $C(X)$. Since it is Lipschitz, $\alpha_1 \ge 1$. By (1.2) we have $\lambda_1 \ge n-1$. □

This is a generalization of the Lichnerowicz theorem. However, the Obata theorem does not hold: any X such that $C(X)$ splits off some $\mathbf{R}$ satisfies $\lambda_1 = n-1$.

3. Proof of Theorem 0.1

We now prove Theorem 0.1. Pick any sequence $R_i \to \infty$.

By the almost rigidity theorem of Cheeger-Colding [4], there exists a *critical radius* R_c for α_1 such that for *all* $r > R_c$, the assumptions of Lemma 1.2, Lemma 1.3 and Lemma 1.4, i.e., (0.1), (0.2), (1.14), hold on the rescaled manifold $(M^n, r^{-2}dx^2)$.

As in the previous section we transplant $u_\infty = r_1^\alpha \phi_1(x)$ back to harmonic functions u_i on $B_2(p_i) \subset M_i = (M^n, R_i^{-2}dx^2)$ so that $u_i \to u_\infty$ uniformly.

We scale back and view u_i as functions on M^n. By Theorem 2.1, for R_i sufficiently large, at scale R_i the harmonic function u_i is close to some function $u_\infty = cr_1^\alpha \phi_1(x)$. Here and below, *close* means L^∞-close, after an obvious rescale.

So, in particular, we can apply the monotonicity Lemma 1.2; in fact, we iterate it until the scale of critical radius R_c when (the rescaled version of) (1.14) fails. So for all R with $R_c \le R \le R_i$,

$$\tag{3.1} \fint_{A(p,\,R/2,\,R)} |u_i - u_{i,p,R}|^2 \le (2^{\alpha_1+\epsilon})^2 \fint_{A(p,\,R/4,\,R/2)} |u_i - u_{i,p,\,R/2}|^2;$$

here recall $u_{i,p,R}$ is the average of u_i on $A(p,\,R/2,\,R)$.

Clearly u_i is not a constant. We first subtract a constant and then multiply by a constant so that we can assume

$$\tag{3.2} \fint_{A(p,R_c/2,R_c)} u_i = 0, \quad \fint_{A(p,R_c/2,R_c)} u_i^2 = 1.$$

So by iterating Lemma 1.4, for all R with $R_c \le R \le R_i$,

$$\tag{3.3} |u_{i,p,R}| = \Big|\fint_{A(p,R,R)} u_i\Big| \le \epsilon\Big(\fint_{A(p,R/2,R)} u_i^2\Big)^{1/2}.$$

We have

$$\left(\fint_{A(p,R_c,2R_c)} u_i^2\right)^{1/2} \le \left(\fint_{A(p,R_c,2R_c)} |u_i - u_{i,2R_c}|^2\right)^{1/2} + |u_{i,2R_c}| \tag{3.4}$$
$$\le 2^{\alpha_1+\epsilon}\left(\fint_{A(p,R_c/2,R_c)} u_i^2\right)^{1/2} + \epsilon 2^{\alpha_1+\epsilon}\left(\fint_{A(p,R_c/2,R_c)} u_i^2\right)^{1/2}$$
$$\le 2^{\alpha_1+2\epsilon}\left(\fint_{A(p,R_c/2,R_c)} u_i^2\right)^{1/2}.$$

Iterating this, we have

$$\left(\fint_{A(p,2^{j-1}R_c,2^jR_c)} u_i^2\right)^{1/2} \le 2^{(\alpha_1+2\epsilon)j}\left(\fint_{A(p,R_c/2,R_c)} u_i^2\right)^{1/2}. \tag{3.5}$$

So u_i (defined on $B_{R_i}(p)$, with $R_i \gg R_c$) is of polynomial growth,

$$|u_i| \le Cr^{\alpha_1+2\epsilon}. \tag{3.6}$$

Combining with the Cheng-Yau gradient and the Arzela-Ascoli theorem, u_i converges to a nonconstant polynomial growth harmonic function $u^{(1)}$ on M.

Next, we indicate how to construct a second harmonic function when there is *another* eigenfunction for λ_1. By construction, $u^{(1)}$ satisfies (3.1) and (3.3) at every scale $R > R_c$. So by Lemma 0.4 on any sufficiently large scale, $u^{(1)}$ is close to a function of the form

$$\sum_{\alpha_i=\alpha_1} c_i r^{\alpha_1}\phi_i(x) \tag{3.7}$$

on $C(X)$. Note that we have no control over the constants c_i. By assumption, λ_1 has more than one multiple; so there is a function of the form

$$\sum_{\alpha_i=\alpha_1} b_i r^{\alpha_1}\phi_i(x) \tag{3.8}$$

that is perpendicular to (3.7) on $C(X)$. Like the construction of $u^{(1)}$, we transplant (3.8) back to M_i, solve the Dirichlet problem as in (2.2), and get a sequence of harmonic functions $w_i^{(2)}$. Now adjust $w_i^{(2)}$ by a tiny constant, then subtract $cu^{(1)}$, a multiple of our first harmonic function $u^{(1)}$, so that

$$u_i^{(2)} := (w_i^{(2)} - cu^{(1)}) \perp u^{(1)} \quad \text{on} \quad A(p, R_c, 2R_c). \tag{3.9}$$

Note that we have no control over the constant c, but this is not important since all we need is that on scale R_i we have the inequality (3.1), and $u_i^{(2)}$ is not a constant. Then as before we construct our second function $u^{(2)}$. It is independent of $u^{(1)}$ since it is perpendicular to $u^{(1)}$ on $u^{(1)}$ $A(p, R_c, 2R_c)$.

The constructions of all the other harmonic functions follow the same pattern. Note then we need a revised version of Lemma 1.2 in which α_1 is substituted by α_i. The generalization is straightforward.

Clearly, if we have N eigenvalues of X with $\lambda \le \Lambda = N(N + n - 2)$, then we have at least N independent nonconstant harmonic functions $u^{(j)}$ with

$$|u^{(j)}(y)| \le C(j, \epsilon)(1 + d(p, y)^{N+\epsilon}). \tag{3.10}$$

Now we can count them. By a well-known argument in estimating upper bounds of eigenvalues (similar to p. 105 of [21]), we have

$$\lambda_j \leq C(n)\Big(\frac{j}{H^{n-1}(X)}\Big)^{\frac{2}{n-1}}; \tag{3.11}$$

here $H^{n-1}(X)$ is the $(n-1)$-Hausdorff measure of X. Actually, we can take V_∞ in (0.2) for it; see [5]. So there are at least $C(V_\infty)\Lambda^{\frac{n-1}{2}}$ many eigenvalues less than Λ, and the dimension of harmonic functions with

$$|u(y)| \leq C(1 + d(p, y)^N) \tag{3.12}$$

is at least $C(V_\infty)N^{n-1}$. □

Finally, we remark that the technical assumption in Theorem 0.2 is needed to guarantee that Lemma 1.2 works when $C(X)$ is not unique.

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