

TWO NONTRIVIAL SOLUTIONS FOR QUASILINEAR PERIODIC EQUATIONS

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ABSTRACT. In this paper we study a nonlinear periodic problem driven by the ordinary scalar p -Laplacian and with a Carathéodory nonlinearity. We establish the existence of at least two nontrivial solutions. Our approach is variational based on the smooth critical point theory and using the “Second Deformation Theorem”.

1. INTRODUCTION

In the last decade there has been increasing interest for periodic problems driven by the ordinary p -Laplacian differential operator. We refer to the works of Del Pino-Manasevich-Murua [5], Fabry-Fayyad [6], Guo [7], Dang-Oppenheimer [4] (scalar problems), and Manasevich-Mawhin [10], Mawhin [11], Kyritsi-Matzakos-Papageorgiou [9] (vector problems). In all these works the method of analysis is based on degree theoretic arguments or on the theory of nonlinear operators of monotone type and on fixed point results (see Kyritsi-Matzakos-Papageorgiou [9]). The problem of existence of multiple periodic solutions was addressed only by Del Pino-Manasevich-Murua [5], where the forcing term $f(t, x)$ is continuous, the map $x \rightarrow \hat{f}(t, x) = f(t, \|x\|^{p-2}x)$ is locally Lipschitz, and if $p > 2$, $f(t, x) \neq 0$ for all $x \neq 0$. Their approach is based on conditions on the interaction between the Fučík spectrum of the ordinary p -Laplacian and the nonlinearity f .

In this paper we prove a multiplicity result for problems with a Carathéodory nonlinearity. We assume that the equation is strongly resonant at the first (zero) eigenvalue of the negative ordinary scalar p -Laplacian with periodic boundary conditions (i.e., $f(t, x) \rightarrow 0$ as $|x| \rightarrow \infty$ and the potential $F(t, x) = \int_0^x f(t, r)dr$ has finite limits as $|x| \rightarrow \infty$, i.e., the potential has a small rate of increase as $|x| \rightarrow \infty$). The term “strong resonance” (describing the situation just mentioned) was coined by Bartolo-Benci-Fortunato [2]. Our approach is variational and uses smooth critical point theory (see Chang [3] and Mawhin-Willem [12]) and the so-called “second deformation theorem” (see Chang [3], p.23).

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2. MULTIPLICITY RESULT

The problem under consideration is the following:

$$(1) \quad \left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = f(t, x(t)) \text{ a.e on } T = [0, b] \\ x(0) = x(b), x'(0) = x'(b), 1 < p < \infty. \end{array} \right\}$$

Our hypotheses on the nonlinearity f are the following:

$H(f)$. $f : T \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that:

- (i) for all $x \in \mathbb{R}$, $t \mapsto f(t, x)$ is measurable;
- (ii) for almost all $t \in T$, the function $x \mapsto f(t, x)$ is continuous;
- (iii) for almost all $t \in T$ and all $x \in \mathbb{R}$, we have

$$|f(t, x)| \leq a_1(t) + c_1(t)|x|^{r-1},$$

$$1 \leq r < +\infty \text{ with } a_1, c_1 \in L^{r'}(T), \frac{1}{r} + \frac{1}{r'} = 1;$$

- (iv) if $F(t, x) = \int_0^x f(t, r)dr$, there exist $F_{\pm} \in L^1(T)$ such that $F(t, x) \rightarrow F_{\pm}(t)$ as $x \rightarrow \pm\infty$ uniformly for almost all $t \in T$, $\int_0^b F_{\pm}(t)dt \leq 0$;

- (v) for almost all $t \in T$ and all $x \in \mathbb{R}$, $F(t, x) \leq \frac{1}{b^p} |x|^p$;

- (vi) $\liminf_{x \rightarrow 0} \frac{pF(t, x)}{|x|^p} \geq \theta(t)$ uniformly for almost all $t \in T$ with $\theta \in L^1(T)$ and $\int_0^b \theta(t)dt > 0$.

We introduce the space $W_{per}^{1,p}(T) = \{x \in W^{1,p}(T) : x(0) = x(b)\}$ and the functional $\varphi : W_{per}^{1,p}(T) \rightarrow \mathbb{R}$ defined by $\varphi(x) = \frac{1}{p} \|x'\|_p^p - \int_0^b F(t, x(t))dt$. It is well known that $\varphi \in C^1(W_{per}^{1,p}(T))$. Recall that if X is a Banach space and $\varphi \in C^1(X)$, we say that φ satisfies the Palais-Smale condition at level c ((PS) $_c$ -condition for short) if every sequence $\{x_n\}_{n \geq 1}$ such that $\varphi(x_n) \rightarrow c$ and $\varphi'(x_n) \rightarrow 0$ has a strongly convergent subsequence. In what follows for $\varphi \in C^1(X)$, we set $K = \{x \in X : \varphi'(x) = 0\}$ (the set of critical points of φ) and for $c \in \mathbb{R}$, $K_c = \{x \in X : \varphi'(x) = 0 \text{ and } \varphi(x) = c\}$ and $\varphi^c = \{x \in X : \varphi(x) \leq c\}$. The so-called “second deformation theorem” (see Chang [3], p. 23) that we shall use says the following:

Theorem 1. *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the (PS) $_c$ -condition for every $c \in [a, d]$, a is the only critical value of φ on $[a, d]$ and $\varphi^{-1}(\{a\}) \cap K$ consists of isolated critical points, then there exists $h \in C([0, 1] \times \varphi^d \setminus K_d, X)$ such that $h(t, \cdot)|_{\varphi^a} = \text{identity}$ for all $t \in T$, $h(0, \cdot) = \text{identity}$ and $h(1, \varphi^d \setminus K_d) \subseteq \varphi^a$. In addition $\varphi(h(s, x)) \leq \varphi(h(t, x))$ for all $t < s$ and all $x \in \varphi^d \setminus K_d$.*

Remark. In the terminology of Chang [3] and Mawhin-Willem [12] (p. 171), the first part of the conclusion of Theorem 1 says that φ^a is a strong deformation retract of $\varphi^d \setminus K_d$.

Proposition 2. *If hypotheses $H(f)$ hold, then for every $c < -\int_0^b F_{\pm}(t)dt$, φ satisfies the (PS) $_c$ -condition.*

Proof. Let $\{x_n\}_{n \geq 1} \subseteq W_{per}^{1,p}(T)$ be a sequence such that $\varphi(x_n) \rightarrow c$ and $\varphi'(x_n) \rightarrow 0$. We have $\varphi'(x_n) = A(x_n) - N(x_n)$ where $A : W_{per}^{1,p}(T) \rightarrow W_{per}^{1,p}(T)^*$ is the nonlinear operator defined by $\langle A(x), y \rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt$ for all $x, y \in W_{per}^{1,p}(T)$ (by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W_{per}^{1,p}(T), W_{per}^{1,p}(T)^*)$) and $N : L^r(T) \rightarrow L^{r'}(T)$ is the Nemitsky operator corresponding to f , i.e., $N(x)(\cdot) = f(\cdot, x(\cdot))$. It is easy to check that A is monotone, demicontinuous, hence maximal

monotone (see Hu-Papageorgiou [8], p. 309) and N is continuous (Krasnoselkii's theorem).

We claim that the sequence $\{x_n\}_{n \geq 1} \subseteq W_{per}^{1,p}(T)$ is bounded. Suppose that this is not the case. By passing to a subsequence if necessary, we may assume that $\|x_n\| \rightarrow \infty$. Set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that $y_n \xrightarrow{w} y$ in $W_{per}^{1,p}(T)$ and $y_n \rightarrow y$ in $C(T)$ (recall that by the Sobolev embedding theorem $W_{per}^{1,p}(T)$ is embedded compactly in $C(T)$). From the choice of the sequence $\{x_n\}_{n \geq 1} \subseteq W_{per}^{1,p}(T)$, we have

$$(2) \quad \frac{|\varphi(x_n)|}{\|x_n\|^p} = \frac{1}{p} \|y'_n\|_p^p - \int_0^b \frac{F(t, x_n(t))}{\|x_n\|^p} dt \leq \frac{M_1}{\|x_n\|^p}$$

for some $M_1 > 0$ and all $n \geq 1$.

Because of hypothesis $H(f)(iv)$, we can find $M_2 > 0$ such that for almost all $t \in T$, we have that

$$|F(t, x)| \leq |F_+(t)| + 1 \text{ if } x \geq M_2 \text{ and } |F(t, x)| \leq |F_-(t)| + 1 \text{ if } x \leq -M_2.$$

From the mean value theorem we know that for almost all $t \in T$ and all $x \in \mathbb{R}$ we can find $\lambda = \lambda(t, x) \in (0, 1)$ such that $F(t, x) = f(t, \lambda x)x$. Then using hypothesis $H(f)(iii)$, we see that for almost all $t \in T$ and all $x \in (-M_2, M_2)$ we have $|F(t, x)| \leq a_2(t)$ with $a_2 \in L^{r'}(T)$. So finally for almost all $t \in T$ and all $x \in \mathbb{R}$, we have $|F(t, x)| \leq a_3(t)$ with $a_3 \in L^1(T)$. Therefore $|\int_0^b \frac{F(t, x_n(t))}{\|x_n\|^p} dt| \leq \int_0^b \frac{a_3(t)}{\|x_n\|^p} dt \rightarrow 0$ as $n \rightarrow \infty$. Hence if we pass to the limit in (2), we obtain $\frac{1}{p} \|y'\|_p^p = 0$ and so $y = \xi \in \mathbb{R}$. If $\xi = 0$, then $y'_n \rightarrow 0$ in $L^p(T)$ and so $y_n \rightarrow 0$ in $W_{per}^{1,p}(T)$, a contradiction since $\|y_n\| = 1$ for all $n \geq 1$. So $\xi \neq 0$ and without any loss of generality we may assume that $\xi > 0$ (the analysis is similar if instead we assume that $\xi < 0$). From the choice of the sequence $\{x_n\}_{n \geq 1} \subseteq W_{per}^{1,p}(T)$, we know that there exists $n_0 \geq 1$ such that for all $n \geq n_0$, we have $|\varphi(x_n) - c| \leq \varepsilon$, hence $\varphi(x_n) \leq c + \varepsilon$ and so $-\int_0^b F(t, x_n(t)) dt \leq c + \varepsilon$. Note that $x_n(t) \rightarrow +\infty$ uniformly in $t \in T$ and so $F(t, x_n(t)) \rightarrow F_+(t)$ uniformly for almost all $t \in T$ (hypothesis $H(f)(iv)$). Therefore $-\int_0^b F_+(t) dt \leq c + \varepsilon$. Let $\varepsilon \downarrow 0$ to obtain $-\int_0^b F_+(t) dt \leq c$, a contradiction to the choice of c . Similarly we reach a contradiction if we assume that $\xi < 0$.

Therefore we have proved that $\{x_n\}_{n \geq 1} \subseteq W_{per}^{1,p}(T)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W_{per}^{1,p}(T)$ and $x_n \rightarrow x$ in $C(T)$. From the choice of the sequence $\{x_n\}_{n \geq 1} \subseteq W_{per}^{1,p}(T)$ we have

$$(3) \quad \langle A(x_n), x_n - x \rangle - \int_0^b f(t, x_n(t))(x_n(t) - x(t)) dt \leq \varepsilon_n \|x_n - x\| \text{ with } \varepsilon_n \downarrow 0.$$

Note that by virtue of hypothesis $H(f)(iii)$ and the dominated convergence theorem, we have that $\int_0^b f(t, x_n(t))(x_n(t) - x(t)) dt \rightarrow 0$. So from (3) we obtain that $\limsup \langle A(x_n), x_n - x \rangle \leq 0$. But A being maximal monotone, it is generalized pseudomonotone (see Hu-Papageorgiou [8], p. 365) and so $\langle A(x_n), x_n \rangle \rightarrow \langle A(x), x \rangle$, hence $\|x'_n\|_p \rightarrow \|x'\|_p$. Also $x'_n \xrightarrow{w} x'$ in $L^p(T)$ and since the latter is uniformly convex, by the Kadec-Klee property we have $x'_n \rightarrow x'$ in $L^p(T)$. Therefore $x_n \rightarrow x$ in $W_{per}^{1,p}(T)$. \square

Now we are ready for the multiplicity result.

Theorem 3. *If hypotheses $H(f)$ hold, then there exist two nontrivial solutions $x_1, x_2 \in C^1(T)$ of (1) such that $|x'_i(\cdot)|^{p-2}x'_i(\cdot) \in W_{per}^{1,\tau}(T)$ with $\tau = \min\{q, r'\}$ and $i = 1, 2$.*

Proof. Note that because of hypotheses $H(f)$ (iii) and (iv), φ is bounded below. Also hypothesis $H(f)$ (vi) implies that given $\varepsilon > 0$, we can find $\delta(\varepsilon) > 0$ such that for almost all $t \in T$ and all $|x| \leq \delta$, we have

$$(4) \quad (\theta(t) - \varepsilon)|x|^p \leq pF(t, x).$$

For every $\xi \in \mathbb{R}$, we have

$$\varphi(\xi) = - \int_0^b F(t, \xi) dt \leq \frac{|\xi|^p}{p} \int_0^b (\varepsilon - \theta(t)) dt = \frac{|\xi|^p \varepsilon b}{p} - \frac{|\xi|^p}{p} \int_0^b \theta(t) dt.$$

From the properties of θ (see hypothesis $H(f)$ (vi)), we see that if $\varepsilon > 0$ is small, $\varphi(\xi) < 0$. So we infer that $\inf[\varphi(x) : x \in W_{per}^{1,p}(T)] = m_0 < 0 \leq - \int_0^b F_{\pm}(t) dt$ (see hypothesis $H(f)$ (iv)). According to Proposition 2, φ satisfies the $(PS)_{m_0}$ -condition. Thus we can find $x_1 \in W_{per}^{1,p}(T)$ such that

$$\varphi(x_1) = m_0 < 0 = \varphi(0), \quad \text{hence } \varphi'(x_1) = 0 \quad \text{and } x_1 \neq 0.$$

Suppose that x_1 and 0 are the only critical points of φ . From the above argument we know that given $\varepsilon > 0$, we can find $r_1 > 0$ such that if $|\xi| \leq r_1$, we have

$$\varphi(\xi) \leq \frac{|\xi|^p}{p} (\varepsilon b - \int_0^b \theta(t) dt).$$

If $\varepsilon > 0$ is small, we have that $\eta = \varepsilon b - \int_0^b \theta(t) dt < 0$ and so

$$(5) \quad \varphi(\xi) \leq \frac{|\xi|^p}{p} \eta < 0.$$

Consider the direct sum decomposition $W_{per}^{1,p}(T) = \mathbb{R} \oplus V$ with $V = \{v \in W_{per}^{1,p}(T) : \int_0^b v(t) dt = 0\}$. By virtue of hypothesis $H(f)$ (v), for every $v \in V$, we have

$$\varphi(v) \geq \frac{1}{p} \|v'\|_p^p - \frac{1}{pb^p} \|v\|_p^p.$$

From the Wirtinger inequality (see Mawhin-Willem [12], p. 8), we have $\|v\|_p^p \leq b\|v\|_\infty^p \leq b^p\|v'\|_p^p$ and so

$$\varphi(v) \geq \frac{1}{p} \|v'\|_p^p - \frac{1}{p} \|v'\|_p^p = 0, \quad \text{i.e., } \inf_V \varphi = 0.$$

From the previous considerations (see (5)), we know that

$$(6) \quad \mu = \sup_{\bar{B}_r \cap \mathbb{R}} \varphi < 0.$$

Here $B_r = \{x \in W_{per}^{1,p}(T) : \|x\| < r\}$. Let $\Gamma = \{\gamma \in C(\bar{B}_r \cap \mathbb{R}, W_{per}^{1,p}(T)) : \gamma|_{\partial \bar{B}_r \cap \mathbb{R}} = \text{identity}\}$. If h is the homotopy postulated by Theorem 1, we define the map $\gamma_0 : \bar{B}_r \cap \mathbb{R} \rightarrow W_{per}^{1,p}(T)$ by

$$\gamma_0(x) = \begin{cases} x_1 & \text{if } \|x\| < \frac{r}{2}, \\ h(\frac{2(r-\|x\|)}{r}, \frac{rx}{\|x\|}) & \text{if } \|x\| \geq \frac{r}{2}. \end{cases}$$

Recall that we have assumed that $\{x_1, 0\}$ are the only critical points of φ . Then x_1 is the only minimizer of φ and so from Theorem 1, it follows that $h(1, y) = x_1$

for all $y \in \varphi^0 \setminus \{0\}$. Hence we infer that for $|x| = \frac{r}{2}$, we have $h(\frac{2(r-|x|)}{r}, \frac{rx}{\|x\|}) = h(1, 2x) = x_1$, which proves the continuity of γ_0 . In addition, from Theorem 1, $h(0, \cdot) = \text{identity}$ and so $\gamma_0|_{\partial \bar{B}_r \cap \mathbb{R}} = \text{identity}$. Thus it follows that $\gamma_0 \in \Gamma$. Moreover, since h is φ -decreasing (see Theorem 1), for all $x \in \varphi^0 \setminus \{0\}$ and all $t, s \in [0, 1]$ with $t < s$, we have $\varphi(h(s, x)) \leq \varphi(h(t, x))$. From this and (6), we see that

$$(7) \quad \varphi(\gamma_0(x)) < 0 \quad \text{for all } x \in \bar{B}_r \cap \mathbb{R}.$$

From Struwe [13] (p. 116), we know that the sets $\partial \bar{B}_r \cap \mathbb{R}$ and V link. So we have that $\gamma(\bar{B}_r \cap \mathbb{R}) \cap V \neq \emptyset$ for all $\gamma \in \Gamma$, hence $\sup[\varphi(\gamma(x)) : x \in \bar{B}_r \cap \mathbb{R}] \geq 0$ for all $\gamma \in \Gamma$ (recall that $\inf_V \varphi = 0$) and so

$$(8) \quad \sup[\varphi(\gamma_0(x)) : x \in \bar{B}_r \cap \mathbb{R}] = \varphi(\gamma_0(x^*)) \geq 0 \quad \text{for some } x^* \in \bar{B}_r \cap \mathbb{R}.$$

From (7) and (8), we have a contradiction. Therefore φ has another critical point $x_2 \neq x_1$, $x_2 \neq 0$. Now let $y = x_k$, $k = 1, 2$. We have $\varphi'(y) = 0$ and so

$$(9) \quad A(y) = N(y);$$

hence $\langle A(y), \psi \rangle = \int_0^b f(t, y(t))\psi(t)dt$ for all $\psi \in C_0^\infty(0, b)$.

Note that $|y'(\cdot)|^{p-2}y'(\cdot) \in W^{-1,q}(T) = W_0^{1,p}(T)^*$ (see Adams [1], p. 50). So from the definition of distributional derivative we have $\langle -(|y'|^{p-2}y')', \psi \rangle_0 = \langle N(y), \psi \rangle_0$, with $\langle \cdot, \cdot \rangle_0$ denoting the duality brackets for the pair $(W_0^{1,p}(T), W^{-1,q}(T))$. Since $C_0^\infty(0, b)$ is dense in $W_0^{1,p}(T)$, we have

$$(10) \quad -(|y'(t)|^{p-2}y'(t))' = f(t, y(t)) \quad \text{a.e. on } T, \quad y(0) = y(b).$$

So $|y'(\cdot)|^{p-2}y'(\cdot) \in W^{1,r}(T)$ with $r = \min\{q, r'\}$, hence $|y'(\cdot)|^{p-2}y'(\cdot) \in C(T)$, from which we have $y' \in C(T)$. Therefore $y \in C^1(T)$. Also from (9), for every $u \in W_{per}^{1,p}(T)$ we have

$$\begin{aligned} \int_0^b |y'(t)|^{p-2}y'(t)u'(t)dt &= \int_0^b f(t, y(t))u(t)dt \\ &\Rightarrow |y'(b)|^{p-2}y'(b)u(b) - |y'(0)|^{p-2}y'(0)u(0) \\ &\quad - \int_0^b (|y'(t)|^{p-2}y'(t))'u(t)dt = \int_0^b f(t, y(t))u(t)dt \quad \text{integration by parts} \\ &\Rightarrow |y'(0)|^{p-2}y'(0)u(0) = |y'(b)|^{p-2}y'(b)u(b) \quad \text{for all } u \in W_{per}^{1,p}(T) \quad (\text{see (10)}) \\ &\Rightarrow |y'(0)|^{p-2}y'(0) = |y'(b)|^{p-2}y'(b) \Rightarrow y'(0) = y'(b). \end{aligned}$$

Therefore $x_1, x_2 \in C^1(T)$ are the two distinct nontrivial solutions of (1) with $|x'_i(\cdot)|^{p-2}x'_i(\cdot) \in W_{per}^{1,r}(T)$. \square

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