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EIGENVALUE ESTIMATES FOR OPERATORS ON SYMMETRIC BANACH SEQUENCE SPACES

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ABSTRACT. Using abstract interpolation theory, we study eigenvalue distribution problems for operators on complex symmetric Banach sequence spaces. More precisely, extending two well-known results due to König on the asymptotic eigenvalue distribution of operators on ℓ_p-spaces, we prove an eigenvalue estimate for Riesz operators on ℓ_p-spaces with $1 < p < 2$, which take values in a p-concave symmetric Banach sequence space $E \hookrightarrow \ell_p$, as well as a dual version, and show that each operator T on a 2-convex symmetric Banach sequence space F, which takes values in a 2-concave symmetric Banach sequence space E, is a Riesz operator with a sequence of eigenvalues that forms a multiplier from F into E. Examples are presented which among others show that the concavity and convexity assumptions are essential.

1. Introduction

As usual, we assign to every Riesz operator T on a complex Banach space X (in particular, to every compact operator) its sequence of eigenvalues $(\lambda_n(T))$, arranged in order of non-decreasing magnitude, $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \dots \geq 0$, and each eigenvalue is counted according to its algebraic multiplicity (if T possesses less than n eigenvalues, then $\lambda_n(T) = 0$). Based on now classical inequalities due to Grothendieck as well as Bennett and Carl, König proved the following two remarkable eigenvalue estimates for operators in complex ℓ_p-spaces (see [10, 2.b.11] or [11, Proposition 20]): *Let $1 \leq q < p \leq \infty$ and $1/r = 1/q - 1/p$. Then every operator T on ℓ_p with values in ℓ_q is a Riesz operator, and*

(i) $(\lambda_n(T)) \in \ell_{r,\infty}$ *, provided* $1 \leq q < p < 2$ *or* $2 < q < p \leq \infty$*; in other terms,* $|\lambda_n(T)| \leq c\, n^{1/p-1/q}$*, where the constant* $c > 0$ *is independent of* n*.*

(ii) $(\lambda_n(T)) \in \ell_r$ *, provided* $1 \leq q \leq 2 \leq p \leq \infty$*,* $q \neq p$*.*

Recall that each such operator T is even compact whenever $1 \leq q < p < \infty$ (by Pitt's theorem) or $1 \leq q < 2$ and $p = \infty$. The aim of this note is to prove two extensions of König's results within the setting of complex symmetric Banach sequence spaces. More precisely, extending (i), we estimate single eigenvalues of Riesz operators on ℓ_p with values in E (resp., on E with values in ℓ_p), E a p-concave (resp., p-convex) symmetric Banach sequence space, $1 < p < 2$ (resp., $2 < p < \infty$), and, extending (ii), we show that the sequence of eigenvalues of an operator on

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F with values in E, F a 2-convex symmetric Banach sequence space and E a 2-concave symmetric Banach sequence space different from the first one, belongs to the space of multipliers between these spaces. Examples show that the convexity and concavity assumptions in both results are not superfluous.

2. Preliminaries

If (a_n) and (b_n) are real-valued sequences, we write $a_n \prec b_n$ whenever there is some $c > 0$ such that $a_n \leq c\, b_n$ for all $n \in \mathbb{N}$, and $a_n \asymp b_n$ whenever $a_n \prec b_n$ and $b_n \prec a_n$.

We use standard notation and notions from Banach space theory (see e.g. [14]). If X is a Banach space, then B_X is its (closed) unit ball and E' its dual space. As usual, $\mathcal{L}(X, Y)$ denotes the Banach space of all (bounded and linear) operators from X into Y endowed with the operator norm.

Throughout the paper by a Banach sequence space we mean a real(!) Banach lattice E modelled on the set of positive integers $\mathbb{N}$ which contains an element x with $\operatorname{supp} x = \mathbb{N}$. If E is a Banach sequence space, then we denote by $E(\mathbb{C})$ the vector space of all complex sequences x such that $|x| \in E$ which together with the norm $\|x\|_{E(\mathbb{C})} = \||x|\|_E$ forms a Banach space. If we consider problems related to eigenvalues, then we will carefully distinguish between real sequence spaces E and their complexifications $E(\mathbb{C})$, in particular, between ℓ_p and $\ell_p(\mathbb{C})$.

Any time the term complex Banach sequence space is used, it refers to the complexification of a real Banach sequence space in the above sense. We write E_n for the linear span of the first n standard unit vectors e_k of a Banach sequence space E. As usual, we denote by (x_n^*) the decreasing rearrangement of a bounded scalar sequence (x_n), and define (x_n^{**}) by $x_n^{**} := \frac{1}{n}\sum_{k=1}^n x_k^*$. A Banach sequence space E is said to be symmetric provided that $\|(x_n)\|_E = \|(x_n^*)\|_E$ for all $x \in E$. It is maximal if the unit ball B_E is closed in the pointwise convergence topology of the space $\omega := \mathbb{R}^{\mathbb{N}}$ of all real sequences. Recall that if E is separable, then E' can be identified in a natural way with the Köthe dual $E^\times$ of E defined by

$$E^\times := \big\{x = (x_n) \in \omega;\ \textstyle\sum_{n=1}^\infty |x_n y_n| < \infty \text{ for all } y \in E\big\}.$$

Note also that $E^\times$ is always a maximal Banach sequence space under the norm $\|x\| := \sup\{\Sigma_{n=1}^\infty |x_n y_n|;\ \|y\|_E \leq 1\}$ (symmetric, provided that E is) and $(E^\times)^\times = E$ with equality of norms if and only if E is maximal.

The fundamental function of a symmetric Banach sequence space E is defined by

$$\lambda_E(n) := \|\textstyle\sum_{i=1}^n e_i\|_E, \quad n \in \mathbb{N}.$$

Recall that a Banach sequence space E is p-convex, $1 \leq p \leq \infty$, and q-concave, $1 \leq q \leq \infty$, if there is a constant $C > 0$ such that for each choice of finitely many $x_1, ..., x_n \in E$,

$$\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p}\Big\|_E \leq C\,\Big(\sum_{k=1}^n \|x_k\|_E^p\Big)^{1/p}$$

and

$$\Big(\sum_{k=1}^n \|x_k\|_E^q\Big)^{1/q} \leq C\,\Big\|\Big(\sum_{k=1}^n |x_k|^q\Big)^{1/q}\Big\|_E\,,$$

respectively, and the best constant $C > 0$ is denoted by $\mathbf{M}^{(\mathbf{p})}(E)$ and $\mathbf{M}_{(\mathbf{q})}(E)$, respectively; see [14] for all information needed on these notions. Notice that a

p-concave Banach sequence space, $1 < p < \infty$, is automatically both maximal and separable since it cannot contain a copy of c_0.

As usual, we denote by $\mathcal{S}_E$ the unitary ideal of all compact operators $T : \ell_2(\mathbb{C}) \to \ell_2(\mathbb{C})$ for which the sequence $(s_n(T))$ of singular numbers belongs to E. Together with the norm $\|(s_n(T))\|_E$ this space forms a Banach space.

For two Banach sequence spaces F and E the space of multipliers $M(F,E)$ from F into E consists of all real sequences x such that the associated multiplication operator $(y_n) \mapsto (x_n\, y_n)$ is defined and bounded from F into E. Note that $M(F,E)$ equipped with the norm

$$\|x\|_{M(F,E)} := \sup\{\|xy\|_E\, ;\, y \in B_F\}$$

is a Banach sequence space (symmetric provided F and E are). If E is maximal, then $M(F,E)$ is maximal, and moreover

$$M(F,E) = M((F^\times)^\times, E) = M(E^\times, F^\times). \tag{2.1}$$

For a (symmetric) Banach sequence space F denote by F_a the order continuous part of F, i.e., the (symmetric) separable Banach sequence space formed by the closure of the span of all basis sequences e_k. It is well-known that $F_a^\times = F^\times$, and hence for any maximal Banach sequence space E by (2.1)

$$M(F_a,E) = M((F_a^\times)^\times, E) = M(F,E). \tag{2.2}$$

For information on Banach operator ideals, Riesz operators and their eigenvalue distribution, and s-numbers we refer to [18], [3], [7], [10] and [16]. Recall the definition of the k-th approximation number $a_k(T)$ of an operator $T : X \to Y$ between Banach spaces X and Y,

$$a_k(T) := \inf\{\|T - S\|;\, S \in \mathcal{L}(X,Y),\, \mathrm{rank}(S) < k\},$$

and its k-th Weyl number $x_k(T)$,

$$x_k(T) := \sup\{a_k(TS);\, \|S : \ell_2 \to X\| \le 1\}.$$

For basic results and notation from interpolation theory we refer e.g. to [1]. We recall that a mapping $\mathcal{F}$ from the category of all couples of Banach spaces into the category of all Banach spaces is said to be an interpolation functor if for any couple (X_0, X_1), the Banach space $\mathcal{F}(X_0, X_1)$ is intermediate with respect to (X_0, X_1) (i.e., $X_0 \cap X_1 \hookrightarrow \mathcal{F}(X_0, X_1) \hookrightarrow X_0 + X_1$), and $T : \mathcal{F}(X_0, X_1) \to \mathcal{F}(Y_0, Y_1)$ for all $T : (X_0, X_1) \to (Y_0, Y_1)$. Here as usual the notation $T : (X_0, X_1) \to (Y_0, Y_1)$ means that $T : X_0 + X_1 \to Y_0 + Y_1$ is a linear operator such that for $j = 0, 1$ the restriction of T to the space X_j is a bounded operator from X_j into Y_j. If additionally

$$\|T : \mathcal{F}(X_0, X_1) \to \mathcal{F}(Y_0, Y_1)\| \le \max\{\|T : X_0 \to Y_0\|,\, \|T : X_1 \to Y_1\|\}$$

holds, then $\mathcal{F}$ is called an exact interpolation functor. An intermediate space X of the couple $\bar{X} = (X_0, X_1)$ is called an interpolation space if $T : \bar{X} \to \bar{X}$ implies $T : X \to X$. In what follows, we will use the well-known fact that for any interpolation space X with respect to $\bar{X}$ there is an exact interpolation functor $\mathcal{F}$ such that $X = \mathcal{F}(\bar{X})$ up to equivalent norms.

3. Summing identity maps

Here we collect and improve some results from [5] needed for the proofs of the main results in Section 4. The first result is from [5, 6.1].

Proposition 3.1. *Let $E \hookrightarrow \ell_2$ be a 2-concave symmetric Banach sequence space which does not coincide with ℓ_2, and $T \in \mathcal{L}(\ell_2(\mathbb{C}))$ be an operator with values in $E(\mathbb{C})$. Then $T \in \mathcal{S}_{M(\ell_2,E)}$. In particular, $(|\lambda_n(T)|) \in M(\ell_2, E)$.*

The following definition is a natural extension of the notion of absolutely (r,p)-summing operators: For $1 \le p < \infty$ let E be a Banach sequence space such that $\ell_p \hookrightarrow E$. Then an operator $T : X \to Y$ between Banach spaces X and Y is called (E,p)-summing (for short, $T \in \Pi_{E,p}$) if there exists a constant $C > 0$ such that for all $x_1, \dots, x_n \in X$

$$\|(\|Tx_i\|_Y)_{i=1}^n\|_E \le C\, c_p^E \sup_{x' \in B_{X'}} \left(\sum_{i=1}^n |\langle x', x_i\rangle|^p \right)^{1/p},$$

where $c_p^E = \|\ell_p \hookrightarrow E\|$. We write $\pi_{E,p}(T)$ for the smallest constant C with the above property. If $\|e_n\|_E = 1$ for all n, then we obtain the Banach operator ideal $(\Pi_{E,p}, \pi_{E,p})$ and, for $E = \ell_r$ $(r \ge p)$, the well-known ideal $(\Pi_{r,p}, \pi_{r,p})$ of all absolutely (r,p)-summing operators.

The second proposition is a straightforward extension of a well-known result due to König [10, 2.a.3] and can be found in [6, Proposition 2].

Proposition 3.2. *Let E be a symmetric Banach sequence space such that $\ell_2 \hookrightarrow E$. Then for each $T \in \Pi_{E,2}$ and all k,*

$$x_k(T) \le \lambda_E(k)^{-1} \pi_{E,2}(T).$$

The last proposition needed extends the classical result for ℓ_p-spaces due to Bennett and Carl as well as the main result in [5], where the case $p = 2$ is proven.

Proposition 3.3. *For $1 \le p \le 2$ let E be a p-concave symmetric Banach sequence space. Then the identity map* $\mathrm{id} : E \hookrightarrow \ell_p$ *is $(M(\ell_p, E), 2)$-summing. Moreover, the inclusion* $\mathrm{id} : E(\mathbb{C}) \hookrightarrow \ell_p(\mathbb{C})$ *is also $(M(\ell_p, E), 2)$-summing.*

The proof of this result is based on abstract interpolation theory and follows closely the proof for the case $p = 2$ given in [5, Section 4]. For the convenience of the reader we sketch some relevant details. A minor modification of the proof of [5, 4.3] yields the following generalization thereof.

Lemma 3.4. *For $1 < p < \infty$ let E be a p-concave symmetric Banach sequence space. Then $M(\ell_p, E)$ is an interpolation space with respect to $(\ell_{p'}, \ell_\infty)$.*

The second lemma extends [5, 4.5].

Lemma 3.5. *For $1 < p \le 2$ and a p-concave symmetric Banach sequence space E let $\mathcal{F}$ be an exact interpolation functor such that $M(\ell_p, E) \hookrightarrow \mathcal{F}(\ell_{p'}, \ell_\infty)$. Then for some $c > 0$,*

$$\sup_{m,n} \|\mathrm{id} : \mathcal{L}(\ell_2^m, E_n) \hookrightarrow \mathcal{F}(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_p^n))\| \le \sqrt{2}\, c\, \mathbf{M}_{(\mathbf{p})}(E). \tag{3.1}$$

Again, the proof follows by an analysis of the proof of [5, 4.5]: Let $T \in \mathcal{L}(\ell_2^m, E_n)$. By a variant of the Maurey–Rosenthal Factorization Theorem (see [2, 4.2] and also

[15]) there exist an operator $R \in \mathcal{L}(\ell_2^m, \ell_p^n)$ and $\lambda \in \mathbb{R}^n$ such that $T = M_\lambda \circ R$ and $\|R\| \, \|\lambda\|_{M(\ell_p^n, E_n)} \leq \sqrt{2}\, \mathbf{M}_{(\mathbf{p})}(E) \, \|T\|$. Proceed as in the case $p = 2$ to obtain with $c := \|\mathrm{id} : M(\ell_p, E) \hookrightarrow \mathcal{F}(\ell_2, \ell_\infty)\|$ for all n, m

$$\|T\|_{\mathcal{F}(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_p^n))} \leq \sqrt{2}\, c\, \mathbf{M}_{(\mathbf{p})}(E) \, \|T\|_{\mathcal{L}(\ell_2^m, E_n)},$$

the required inequality. □

Proof of Proposition 3.3. According to Lemma 3.4 let $\mathcal{F}$ be an interpolation functor with $M(\ell_p, E) = \mathcal{F}(\ell_{p'}, \ell_\infty)$. We consider the mapping

$$\Phi^{m,n} : (\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_p^n)) \to (\ell_{p'}^m(\ell_p^n), \ell_\infty^m(\ell_p^n))$$

defined by $\Phi^{m,n}(S) := (Se_i)_1^m$. By Kwapień [12] the identity map $\mathrm{id} : \ell_1 \hookrightarrow \ell_p$ is absolutely $(p', 2)$-summing, hence there exists a constant $C > 0$ independent of n such that

$$\sup_m \|\Phi^{m,n} : \mathcal{L}(\ell_2^m, \ell_1^n) \to \ell_{p'}^m(\ell_p^n)\| = \pi_{p',2}(\mathrm{id} : \ell_1^n \hookrightarrow \ell_p^n) \leq C,$$

and trivially

$$\sup_m \|\Phi^{m,n} : \mathcal{L}(\ell_2^m, \ell_p^n) \to \ell_\infty^m(\ell_p^n)\| = \|\mathrm{id} : \ell_p^n \hookrightarrow \ell_p^n\| = 1.$$

Then by the interpolation property we obtain that

$$\Phi^{m,n} : \mathcal{F}(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_p^n)) \to \mathcal{F}(\ell_{p'}^m(\ell_p^n), \ell_\infty^m(\ell_p^n))$$

has norm $\leq C$. Now by the preceding lemma and the fact that with constants independent of n, m

$$M(\ell_p^m, E_m)(\ell_p^n) = \mathcal{F}(\ell_{p'}^m, \ell_\infty^m)(\ell_p^n) = \mathcal{F}(\ell_{p'}^m(\ell_p^n), \ell_\infty^m(\ell_p^n))$$

(for the latter equality, see e.g. [5, 4.4]), for some $c > 0$ and all n, m

$$\|\Phi^{m,n} : \mathcal{L}(\ell_2^m, E_n) \to M(\ell_p^m, E_m)(\ell_p^n)\| \leq \sqrt{2}\, C\, c\, \mathbf{M}_{(\mathbf{p})}(E).$$

Hence, since $\sup_m \|\Phi^{m,n}\| = \pi_{M(\ell_p, E),2}(\mathrm{id} : E_n \hookrightarrow \ell_p^n)$, we see that

$$\pi_{M(\ell_p, E),2}(\mathrm{id} : E_n \hookrightarrow \ell_p^n) \leq \sqrt{2}\, C\, c\, \mathbf{M}_{(\mathbf{p})}(E),$$

and since $\bigcup_n E_n$ is dense in E, this implies $(\mathrm{id} : E \hookrightarrow \ell_p) \in \Pi_{M(\ell_p, E),2}$. The second statement in Proposition 3.3 is then a straightforward consequence of the first one. □

4. Estimation of eigenvalues

In this section we present the main results of our paper.

Theorem 4.1. *Let E be a symmetric Banach sequence space.*

(i) *If E is p-concave, $1 < p < 2$, then for every Riesz operator T on $\ell_p(\mathbb{C})$ with values in $E(\mathbb{C})$,*

$$|\lambda_n(T)| \prec \frac{n^{1/p}}{\lambda_E(n)}.$$

(ii) *If E is p-convex, $2 < p < \infty$, then for every Riesz operator T on $E(\mathbb{C})$ with values in $\ell_p(\mathbb{C})$,*

$$|\lambda_n(T)| \prec \frac{\lambda_E(n)}{n^{1/p}}.$$

Proof. (i) Without loss of generality assume that $\mathbf{M}_{(\mathbf{p})}(E) = 1$ (see [14, 1.d.8]), and let $T \in \mathcal{L}(\ell_p(\mathbb{C}))$ be a Riesz operator with values in $E(\mathbb{C})$, E a p-concave symmetric Banach sequence space with $1 < p < 2$. Theorem 3.3 and factorization through $E(\mathbb{C})$ show that $T \in \Pi_{M(\ell_p,E),2}(\ell_p(\mathbb{C}))$. Hence by Lemma 3.2, for all k

$$x_k(T) \leq \pi_{M(\ell_p,E),2}(T)\, \lambda^{-1}_{M(\ell_p,E)}(k). \tag{4.1}$$

By [4, 3.5] we have $\lambda_{M(\ell_p,E)}(k) = \lambda_E(k)/k^{1/p}$. Define the sequence $\mu = (\mu_n)$ by

$$\mu_n := \Big(\big(\frac{n^{1/p}}{\lambda_E(n)}\big)^{**}\Big)^{-1} = \Big(\frac{1}{n}\sum_{k=1}^{n}\frac{k^{1/p}}{\lambda_E(k)}\Big)^{-1},$$

and let m_μ be the associated maximal and symmetric Marcinkiewicz sequence space of all real sequences y such that

$$\|y\|_{m_\mu} := \sup_{n\geq 1} \mu_n\, y_n^{**} < \infty.$$

Then (4.1) implies that $(x_n(T)) \in m_\mu$, and hence by Weyl's inequality [10, 2.a.8] we finally obtain $(|\lambda_n(T)|) \in m_\mu$, i.e.,

$$|\lambda_n(T)| \prec \frac{1}{n}\sum_{k=1}^{n}\frac{k^{1/p}}{\lambda_E(k)}.$$

Since the function $k \mapsto k/\lambda_E(k)$ is almost non-decreasing (see [13, 3.a.6]) and $\sum_{k=1}^{n} k^\alpha \asymp n^{\alpha+1}$ for $\alpha > -1$, we easily get that

$$\frac{1}{n}\sum_{k=1}^{n}\frac{k^{1/p}}{\lambda_E(k)} \asymp \frac{n^{1/p}}{\lambda_E(n)}.$$

(ii) follows by duality from (i) and the fact (see, again, [13, 3.a.6]) that $\lambda_{E^\times}(n) = n/\lambda_E(n)$. □

Note that the above statements hold true for any bounded operator T whenever E is even q-concave for some $q < p$ (resp., q-convex for some $q > p$). Indeed, for (i), this is a consequence of Pitt's Theorem and the fact that a q-concave symmetric Banach sequence space is continuously embedded into ℓ_q; hence, every operator on $\ell_p(\mathbb{C})$ with values in $E(\mathbb{C})$, E a q-concave Banach sequence space with $1 \leq q < p < \infty$, is compact, so is a Riesz operator. Thus (i) applies, since q-concavity implies p-concavity (see [14, 1.d.5]). For (ii), use duality arguments again.

Apparently, the preceding proof also works for the case $p = 2$. However, in this case, we are able to prove the following, somewhat stronger statement:

Theorem 4.2. *Let E and F be symmetric Banach sequence spaces, $E \neq F$, such that E is 2-concave and F is 2-convex. Then every operator T on $F(\mathbb{C})$ with values in $E(\mathbb{C})$ is a Riesz operator and $(|\lambda_n(T)|) \in M(F,E)$.*

Proof. Assume first that additionally F is separable. For the case $F = \ell_2$ see Proposition 3.1, and for the proof of the case $E = \ell_2$ denote the inclusion $\ell_2(\mathbb{C}) \hookrightarrow F(\mathbb{C})$ by I_2. Then if T_0 denotes the operator T considered as an operator with range space $\ell_2(\mathbb{C})$, we have $T = I_2T_0$. Hence by Pietsch's principle of related operators [16, 3.3.4], we see that $R := T_0I_2$ and T have the same nonzero eigenvalues with the same multiplicities. But then the conclusion follows by duality from the first case (use (2.1) and the fact that $F^\times$ is 2-concave, [14, 1.d.4]):

$$(|\lambda_n(T)|) = (|\lambda_n(R)|) = (|\lambda_n(R')|) \in M(\ell_2, F^\times) = M(F, \ell_2).$$

Now we assume that both spaces are different from ℓ_2. Denote by I_1 the inclusion $E(\mathbb{C}) \hookrightarrow \ell_2(\mathbb{C})$ and again by I_2 the inclusion $\ell_2(\mathbb{C}) \hookrightarrow F(\mathbb{C})$. Define $T_0 : F(\mathbb{C}) \to E(\mathbb{C})$ by $T_0 x := Tx$ for $x \in F(\mathbb{C})$. Again we conclude from the Pietsch principle of related operators that the operator $T : F(\mathbb{C}) \to F(\mathbb{C})$ is Riesz if and only if $I_1 T_0 I_2$ is Riesz, and both operators have the same eigenvalue sequences. Since E is a 2-concave symmetric Banach sequence space, it is separable, therefore the e_k's form a basis which, in particular, implies that $E(\mathbb{C})$ has the bounded approximation property. By Pisier's factorization theorem [17, 4.1], the operator T_0 factors through $\ell_2(\mathbb{C})$, say $T_0 = SR$. Then by Proposition 3.1 we have $I_1 S \in \mathcal{S}_{M(\ell_2,E)}$ and $I_2' R' \in \mathcal{S}_{M(\ell_2,F^\times)}$, which implies $RI_2 \in \mathcal{S}_{M(F,\ell_2)}$. Now the general fact that $BA \in \mathcal{S}_{M(F,E)}$ whenever $B \in \mathcal{S}_{M(\ell_2,E)}$ and $A \in \mathcal{S}_{M(F,\ell_2)}$ (simply imitate the first part of the proof of [7, 6.3]) implies that $I_1 T_0 I_2 \in \mathcal{S}_{M(F,E)}$. Finally, since $M(F,E)$ is symmetric and maximal, we conclude by Weyl's inequality [10, 2.a.8] that $(|\lambda_n(T)|) \in M(F,E)$. This finishes the proof for separable F. Let us finally assume that F is an arbitrary symmetric 2-convex Banach sequence space. As in the preliminaries let F_a be the order continuous part of F. Clearly, we have continuous inclusions $E(\mathbb{C}) \hookrightarrow F_a(\mathbb{C}) \hookrightarrow F(\mathbb{C})$. Denote the first inclusion by i, the second one by j, and as above let T_0 be the restriction of T to $E(\mathbb{C})$. Then $iT_0 j$ and T are related operators, so that by (2.2) and the separable case

$$(|\lambda_n(T)|) = (|\lambda_n(iT_0 j)|) \in M(F_a, E) = M(F, E).$$

This completes the proof of the theorem. □

Now we show that the convexity and concavity assumptions in the preceding two theorems are in fact essential. To see this, fix $1 < p < \infty$ and take any symmetric Banach sequence space E strictly contained in ℓ_p, but such that the inclusion map $E \hookrightarrow \ell_p$ is not strictly singular, i.e., it is an isomorphism on some infinite-dimensional subspace of E (for Orlicz sequence spaces E of this type we refer to [8] and [9]; see also [14, 4.c.3]). Recall the well-known fact that any separable ℓ_p-space is complementably minimal (i.e., each of its infinite-dimensional subspaces contains a subspace which is isomorphic to ℓ_p and complemented in ℓ_p). Since the inclusion map $I : E(\mathbb{C}) \hookrightarrow \ell_p(\mathbb{C})$ is not strictly singular, there exists an infinite-dimensional subspace $X \subset E$ isomorphic to $\ell_p(\mathbb{C})$ which is closed in both spaces $\ell_p(\mathbb{C})$ and $E(\mathbb{C})$, and which is moreover complemented in $\ell_p(\mathbb{C})$. Let $P : \ell_p(\mathbb{C}) \to X$ be a projection, $U : X \to \ell_p(\mathbb{C})$ an isomorphism, and $S : \ell_p(\mathbb{C}) \to \ell_p(\mathbb{C})$ any operator. Then we conclude from the Pietsch principle of related operators that the operator $T : \ell_p(\mathbb{C}) \to \ell_p(\mathbb{C})$ defined by $T := IU^{-1}SUP$ is Riesz if and only if S is Riesz, and both operators have the same eigenvalue sequences. Clearly, T takes values in $E(\mathbb{C})$. Now observe that if we take for S a diagonal operator induced by a sequence $(\sigma_n) \in c_0(\mathbb{C})$ with $(|\sigma_n|)$ being strictly decreasing, then we have $(\lambda_n(T)) = (\sigma_n)$. Hence, in this situation, there are Riesz operators $T : \ell_p(\mathbb{C}) \to \ell_p(\mathbb{C})$ which take their values in $E(\mathbb{C})$, but which have a sequence of eigenvalues tending to zero as slowly as wanted. Taking $1 < p \le 2$, we easily conclude by the above argument that the assumptions on concavity and convexity (by duality) in Theorem 4.1 and

Theorem 4.2 are essential in general. Furthermore, taking diagonal operators, one can see that the space of multipliers $M(F,E)$ in Theorem 4.2 is best possible.

Clearly, both theorems apply to operators acting on classical complex Banach symmetric sequence spaces including Orlicz sequence spaces ℓ_φ, or Lorentz sequence spaces $d(p,w)$, and in particular $\ell_{p,q}$. In fact, for these spaces there is a nice description of p-convexity and q-concavity in terms of the parameters generating these spaces (for details see, e.g., [4] and the references therein). To give an example, we present the following statement where part (i) is the analogue of König's result (ii) from the introduction for Lorentz sequence spaces.

Corollary 4.3. (i) *Assume that* $1 < q < 2 < p < \infty$ *and* $1 \le w \le 2 \le v \le \infty$, *or* $p = v = 2$ *(resp.,* $q = w = 2$*) and* q, w *(resp.,* p, v*) as before. Then every operator* T *on* $\ell_{p,v}(\mathbb{C})$ *with values in* $\ell_{q,w}(\mathbb{C})$*, is a Riesz operator for which* $(|\lambda_n(T)|) \in \ell_{r,s}$*, where* $1/r = 1/q - 1/p$ *and* $1/s = 1/w - 1/v$.

(ii) *Let* E *be a* 2*-convex (resp.,* 2*-concave) symmetric Banach sequence space. Then every operator* T *on* $E(\mathbb{C})$ *(resp.,* $\ell_\infty(\mathbb{C})$*) with values in* $\ell_1(\mathbb{C})$ *(resp.,* $E(\mathbb{C})$*) is a Riesz operator, and* $(|\lambda_n(T)|) \in E^\times$ *(resp.,* $(|\lambda_n(T)|) \in E$*).*

Both results are immediate consequences of Theorem 4.2: note that $\ell_{p,v}$ and $\ell_{q,w}$ in (i) are 2-convex and 2-concave, respectively. Moreover, an easy calculation in the latter case shows that $M(\ell_{p,v}, \ell_{q,w}) = \ell_{r,s}$. Since ℓ_∞ and ℓ_1 are 2-convex and 2-concave, respectively, the proof is complete.

We note that in both cases of (ii), T is a 2-summing operator. Thus, it follows by a remarkable result of Pietsch that T is a Riesz operator and $(|\lambda_n(T)|) \in \ell_2$. However, by (ii) in the case when $E \ne \ell_2$, this sequence belongs to the space $E^\times$ (resp., E) which is properly contained in ℓ_2.

To conclude the paper, we give an analogue of König's result (i) from the introduction for Lorentz sequence spaces, using his result for ℓ_p-spaces together with an extrapolation method. A family of Banach spaces $(X_\alpha, \|\cdot\|_{X_\alpha})$, $\alpha \in A$, is called a Banach family if there exists a Hausdorff topological vector space W such that $X_\alpha \hookrightarrow W$, for all $\alpha \in A$. If $\mathcal{X} := \{X_\alpha\}_{\alpha \in A}$ is a Banach family, its intersection is the Banach space $(\Delta(\mathcal{X}), \|\cdot\|_{\Delta(\mathcal{X})})$ consisting of all $x \in \bigcap_{\alpha \in A} X_\alpha$ for which

$$\|x\|_{\Delta(\mathcal{X})} := \sup\{\|x\|_{X_\alpha};\, \alpha \in A\} < \infty.$$

Note that any family $\mathcal{X}$ of symmetric Banach sequence spaces is a Banach family, and in this case, $\Delta(\mathcal{X})$ is also a symmetric Banach sequence space. For simplicity of formulation, we include in the following statement the cases of p, q, v, w already covered in Corollary 4.3, although the result here for those cases is slightly weaker.

Proposition 4.4. *Let* $1 < q < p < \infty$ *and* $1 \le v, w \le \infty$. *Then every operator* T *on* $\ell_{p,v}(\mathbb{C})$ *with values in* $\ell_{q,w}(\mathbb{C})$ *is a Riesz operator for which* $(|\lambda_n(T)|) \in \ell_{r,\infty}$*, where* $1/r = 1/q - 1/p$.

Proof. Let $T : \ell_{p,v}(\mathbb{C}) \to \ell_{p,v}(\mathbb{C})$ be an operator with values in $\ell_{q,w}(\mathbb{C})$ and $T_0 : \ell_{p,v}(\mathbb{C}) \to \ell_{q,w}(\mathbb{C})$ its restriction. Fix s and t such that $q < t < s < p$, let $i : \ell_s(\mathbb{C}) \hookrightarrow \ell_{p,v}(\mathbb{C})$ and $j : \ell_t(\mathbb{C}) \hookrightarrow \ell_s(\mathbb{C})$ be the canonical inclusions, and the operator $S : \ell_{p,v}(\mathbb{C}) \to \ell_t(\mathbb{C})$ be the restriction of T_0 to $\ell_t(\mathbb{C})$. Then the operator $jSi : \ell_s(\mathbb{C}) \to \ell_s(\mathbb{C})$ takes values in $\ell_t(\mathbb{C})$; hence by König's results (i) and (ii) from the Introduction, it is a Riesz operator for which $(|\lambda_n(jSi)|) \in \ell_{u,\infty}$, where $1/u = 1/t - 1/s$. Clearly, jSi and $T = ijS$ are related operators; hence T is also a Riesz operator for which $(|\lambda_n(T)|) \in \ell_{u,\infty}$. Running over all possible choices of

s and t as above, we get that $(|\lambda_n(T)|) \in \Delta := \Delta(\mathcal{X})$, where $\mathcal{X} := \{\ell_{u,\infty}\}_{u>r}$. Clearly, the fundamental function of Δ satisfies $\lambda_\Delta(n) = n^{1/r}$ for all $n \in \mathbb{N}$. This yields $\Delta \hookrightarrow \ell_{r,\infty}$, and thus $(|\lambda_n(T)|) \in \ell_{r,\infty}$. □

Note that the parameters r and s in Corollary 4.3(i), as well as r and ∞ in Proposition 4.4 in the case $1 < q < p < \infty$ and $1 \leq v \leq w \leq \infty$, are optimal (to see this, again take diagonal operators). However, for the remaining cases in Propostion 4.4, except for those already covered by Corollary 4.3, nothing is known about their optimality. It is still an open problem whether König's result (i) from the introduction is optimal.

References

[1] J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, 1978. MR **58:**2349

[2] A. Defant, *Variants of the Maurey–Rosenthal theorem for quasi Köthe function spaces*, Positivity **5** (2001), 153–175. MR **2002a:**46031

[3] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland, 1993. MR **94e:**46130

[4] A. Defant, M. Mastyło and C. Michels, *Summing inclusion maps between symmetric sequence spaces, a survey*, Recent progress in functional analysis (Valencia, 2000), 43–60, North-Holland Math. Stud., 189, North-Holland, Amsterdam, 2001. MR **2002h:**47027

[5] ———, *Summing inclusion maps between symmetric sequence spaces*, Trans. Amer. Math. Soc. **354** (2002), 4473–4492.

[6] ———, *Orlicz norm estimates for eigenvalues of matrices*, Israel J. Math. **132** (2002), 45–59.

[7] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Univ. Press, 1995. MR **96i:**46001

[8] F. L. Hernández and B. Rodriguez-Salinas, *On ℓ^p-complemented copies in Orlicz spaces II*, Israel J. Math. **68** (1989), 27–55. MR **91c:**46041

[9] N. J. Kalton, *Orlicz sequence spaces without local convexity*, Math. Proc. Cambridge Philos. Soc. **81** (1977), 253–277. MR **55:**6173

[10] H. König, *Eigenvalue distributions of compact operators*, Birkhäuser, 1986. MR **88j:**47021

[11] ———, *Eigenvalues of operators and applications*, Handbook of the geometry of Banach spaces, Vol. I, 941–974, North-Holland, Amsterdam, 2001.

[12] S. Kwapień, *Some remarks on (p,q)-absolutely summing operators in ℓ_p-spaces*, Studia Math. **29** (1968), 327–337. MR **37:**6767

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: Sequence spaces*, Springer-Verlag, 1977. MR **58:**17766

[14] ———, *Classical Banach spaces II: Function spaces*, Springer-Verlag, 1979. MR **81c:**46001

[15] F. Lust-Piquard and G. Pisier, *Noncommutative Khintchine and Paley inequalities*, Arkiv för Mat. **29** (1991), 241–260. MR **94b:**46011

[16] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, 1987. MR **88j:**47022b

[17] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS Regional Conference Series in Mathematics 60, Providence, RI, 1986. MR **88a:**47020

[18] N. Tomczak-Jaegermann, *Banach–Mazur distances and finite-dimensional operator ideals*, Longman Scientific & Technical, 1989. MR **90k:**46039

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