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ON SCHOTTKY GROUPS ARISING FROM THE HYPERGEOMETRIC EQUATION WITH IMAGINARY EXPONENTS

TAKASHI ICHIKAWA AND MASAAKI YOSHIDA

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ABSTRACT. In an article by Sasaki and Yoshida (2000), we encountered Schottky groups of genus 2 as monodromy groups of the hypergeometric equation with purely imaginary exponents. In this paper we study automorphic functions for these Schottky groups, and give a conjectural infinite product formula for the elliptic modular function λ.

1. INTRODUCTION

When the three exponents of the hypergeometric differential equation are purely imaginary, its monodromy group is a Schottky group of genus 2. We give a set of generators of the monodromy group in Proposition 1; these are chosen so that they reflect the symmetry of the hypergeometric equation with respect to the three exponents. The main result of this paper is Proposition 2, which gives an automorphic function with respect to the Schottky group as an absolutely convergent infinite product. This automorphic function maps the Riemann surface of genus 2 to $\mathbf{P}^1$. Its inverse map has an interesting property as stated in Proposition 3. By letting the three purely imaginary exponents go to zero in the formula in Proposition 2, we are led to an infinite product which would hopefully converge in some sense and represent the elliptic modular function λ; this conjectural formula is given in the last section.

2. SCHWARZ MAP FOR THE HYPERGEOMETRIC EQUATIONS

Let us consider the hypergeometric differential equation

$$x(1-x)\frac{d^2u}{dx^2} + \{c-(a+b+1)x\}\frac{du}{dx} - abu = 0$$

with purely imaginary exponents

$$1-c = i\theta_0, \quad c-a-b = i\theta_1, \quad a-b = i\theta_2,$$

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where we assume $\theta_0, \theta_1, \theta_2 > 0$ for simplicity. For (any) two linearly independent solutions u_1 and u_2, the (multi-valued) map

$$s : \mathbf{C} - \{0,1\} \ni x \longmapsto u_1(x) : u_2(x) \in \mathbf{P}^1 := \mathbf{C} \cup \{\infty\}$$

is often called a Schwarz map (or Schwarz's s-map).

3. The fundamental domains

We found in [SY] a domain F_x in the x-plane and a domain F_s in the s-plane so that the restricted map

$$s|_{F_x} : F_x \to F_s$$

is conformally isomorphic and the whole s can be recovered by $s|_{F_x}$ through Schwarz's reflection principle; they are called *fundamental domains* for the Schwarz map.

Let us denote by $C(c,r)$ the circle on the s-plane with center c and radius r, and consider the three disjoint circles on the s-plane

$$C_1 = C(0,1), \quad C_2 = C(0,T), \quad C_3 = C(-\mathtt{C}, \mathtt{R}),$$

where $T = e^{\theta_1 \pi}, r = e^{-\theta_0 \pi}$,

$$\mathtt{C} = \frac{\xi(1-r^2)}{\xi^2 - r^2}, \quad \mathtt{R} = \frac{r(1-\xi^2)}{\xi^2 - r^2}, \quad \xi = \left(\frac{\cosh \theta_2 \pi + \cosh(\theta_0 - \theta_1)\pi}{\cosh \theta_2 \pi + \cosh(\theta_0 + \theta_1)\pi} \right)^{\frac{1}{2}}.$$

Since ξ, as a function of $\theta_2 \geq 0$, increases monotonically to 1, and

$$1 > \xi|_{\theta_2 = 0} = \frac{Tr+1}{T+r} > r,$$

we have

$$\begin{aligned} \mathtt{C} - \mathtt{R} - 1 &= (1-r)(1-\xi)/(\xi + r) > 0, \\ T - \mathtt{C} - \mathtt{R} &= \frac{(T+r)\xi - (Tr+1)}{\xi - r} > 0, \end{aligned}$$

and so

$$-T < -\mathtt{C} - \mathtt{R} < -\mathtt{C} + \mathtt{R} < -1 < 1 < T.$$

The domain, in the upper half-plane, bounded by C_1, C_2, C_3 and the real axis can serve as a fundamental domain F_s, and has the shape of a two-arched bridge as in Figure 1. The fundamental domain F_x also has the shape of a two-arched bridge as in Figure 1, and is bounded by three real segments and three curves, which are *not* (part of) circles.

4. The monodromy group

Thanks to these fundamental domains and Schwarz's reflection principle applied along their sides, the monodromy group of the differential equation can be described as follows.

The reflection with respect to the circle $C(c,r)$ (c : real) is given by

$$\varphi(c,r) : s \longmapsto \frac{r^2}{\bar{s} - c} + c.$$

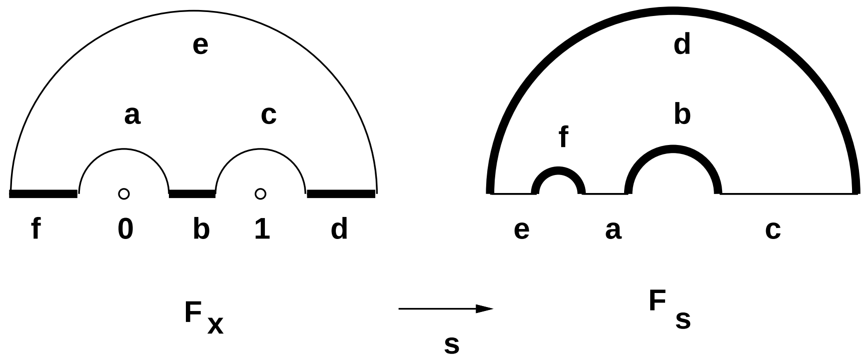

FIGURE 1. The fundamental domains F_x and F_s

Let $\tilde{\Lambda}$ be the group generated by the three reflections $\varphi_1, \varphi_2, \varphi_3$ with respect to the circles C_1, C_2, C_3, respectively. The monodromy group Λ_θ of the hypergeometric equation is the subgroup of $\tilde{\Lambda}$, of index 2, consisting of the even words of $\varphi_1, \varphi_2, \varphi_3$.

On the other hand, for the circle $C(c, r)$, we define the fractional linear transformation of order 2 which fixes the two intersection points of the circle and the real axis:

$$\gamma(c, r) : s \longmapsto \frac{r^2}{s-c} + c.$$

Let Γ_θ ($\theta = (\theta_0, \theta_1, \theta_2)$) be the group generated by the three involutions $\gamma_1, \gamma_2, \gamma_3$ with respect to the circles C_1, C_2, C_3, respectively. The monodromy group Λ_θ is the subgroup of Γ_θ, of index 2, consisting of the even words of $\gamma_1, \gamma_2, \gamma_3$. Let $\Omega(\subset \mathbf{P}^1)$ be the domain of discontinuity of Γ_θ and the Schottky group Λ_θ.

The presentation above has some problems. Though the hypergeometric differential equation is symmetric with respect to $\theta_0, \theta_1, \theta_2$, the three circles C_1, C_2, C_3 are not so; for example, if we let $\theta_2 \to 0$, then the circles C_2 and C_3 kiss, and if we let $\theta_1 \to 0$, then C_3 tends to a point and C_1 and C_2 coincide. Moreover, since the two circles C_1 and C_2 are concentric, the infinite product in the next section does not converge. So we make a linear fractional change of the coordinate s as

$$s \longmapsto \frac{(3+T^2)s + 1 + 3T^2}{4(s+T^2)}.$$

Then the diameters of the three circles on the real axis are given as

$$C_1 : [s_4, s_5], \quad C_2 : [s_1, s_6], \quad C_3 : [s_2, s_3],$$

(see Figure 2) where

$$\begin{aligned}
s_1 &= -\frac{(1-T)^2}{4T}, \quad s_2 = -\frac{(T-1)^3 - (3+T^2)(T-\mathtt{C}-\mathtt{R})}{4(T^2 - T + T - \mathtt{C} - \mathtt{R})}, \\
s_3 &= -\frac{(1+T^2)(\mathtt{C}-\mathtt{R}-1)}{4(T^2-1-(\mathtt{C}-\mathtt{R}-1))} + \frac{1}{2}, \quad s_4 = \frac{1}{2}, \\
s_5 &= 1, \quad s_6 = \frac{(1+T)^2}{4T}.
\end{aligned}$$

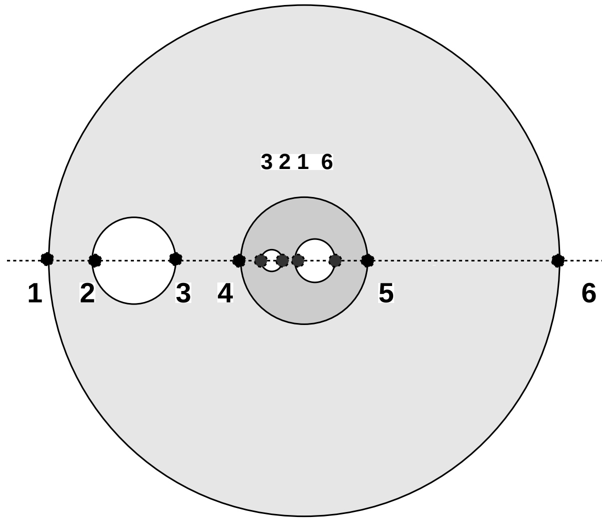

Figure 2. The circles C_1, C_2, C_3

Note that we have $s_1 < \cdots < s_6$. Now it is easy to see

Proposition 1. *If $\theta_1 = 0$, then C_1 and C_2 kiss; if $\theta_2 = 0$, then C_2 and C_3 kiss; and if $\theta_0 = 0$, then C_3 and C_1 kiss.*

Proof. If $\theta_1 = 0$, then $T = 1$, and so $s_5 = s_6$; if $\theta_2 = 0$, then $T - \mathtt{C} - \mathtt{R} = 0$, and so $s_1 = s_2$; if $\theta_0 = 0$, then $r = 1$, $\mathtt{C} - \mathtt{R} - 1 = 0$, and so $s_3 = s_4$. □

5. Schottky automorphic functions

From now on, we regard that the groups defined above are represented with respect to this new coordinate s. The following proposition and Corollary 1 are shown in [GP], IX, §2, for Schottky groups over nonarchimedean local fields which are called Whittaker groups.

Proposition 2. *For $p, q \in \Omega$ with $\Gamma_\theta \cdot p \neq \Gamma_\theta \cdot q$, the infinite product*

$$f_\theta(p, q; s) := \prod_{\gamma \in \Gamma_\theta} \frac{s - \gamma(p)}{s - \gamma(q)}$$

converges uniformly on any compact subset of $\Omega - \Gamma_\theta \cdot q$ and defines a Γ_θ-automorphic function, which induces an isomorphism

$$f_\theta(p, q) : \Omega/\Gamma_\theta \to \mathbf{P}^1,\ p \mapsto 0,\ q \mapsto \infty,\ \infty \mapsto 1.$$

Proof. The Schottky group Λ_θ of rank 2 has a fundamental domain bounded by the circles C_2, C_3 and their reflections with respect to the circle C_1; hence Λ_θ is circle decomposable in the sense of [BBEIM], 5.2. Then by a result of Schottky [S], $\sum_{\gamma \in \Lambda_\theta} |\gamma'(z)|$ is convergent for any $z \in \Omega$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{C})$, we have

$$|g(p) - g(q)| = \left| \frac{p - q}{(cp + d)(cq + d)} \right| \leq \frac{|p - q|}{2} \left(|g'(p)| + |g'(q)| \right);$$

hence if $p, q \in \Omega$, then the infinite products

$$\prod_{\gamma\in\Lambda_\theta} \frac{s-\gamma(p)}{s-\gamma(q)} = \prod_{\gamma\in\Lambda_\theta} \left(1 - \frac{\gamma(p)-\gamma(q)}{s-\gamma(q)}\right),$$

$$f_\theta(p,q;s) = \prod_{\gamma\in\Lambda_\theta} \left(\frac{s-\gamma(p)}{s-\gamma(q)} \cdot \frac{s-\gamma(\gamma_1(p))}{s-\gamma(\gamma_1(q))}\right)$$

are convergent absolutely and uniformly on any compact subset of $\Omega - \Gamma_\theta \cdot q$. Since

$$\frac{\rho(s)-\rho(\gamma(p))}{\rho(s)-\rho(\gamma(q))} \left(\frac{s-\gamma(p)}{s-\gamma(q)}\right)^{-1}$$

is independent of s, there is a map $\psi : \Gamma_\theta \to \mathbf{C}^\times$ such that

$$f_\theta(p,q;\rho(s)) = \psi(\rho) f_\theta(p,q;s) \ (\rho \in \Gamma_\theta).$$

Then from the fact that ψ is a group homomorphism and that Γ_θ is generated by the elements γ_i of order 2, $\mathrm{Im}(\psi)$ is contained in $\{\pm 1\}$, hence is independent of $p, q \in \Omega$. Since Ω is connected, $\mathrm{Im}(\psi)$ is, in fact, $\{1\}$ which implies that $f_\theta(p,q;s)$ gives a meromorphic function on Ω/Γ_θ having only one pole at $\Gamma_\theta \cdot q$ and this pole is of order 1. Therefore, $f_\theta(p,q;s)$ induces an isomorphism $\Omega/\Gamma_\theta \cong \mathbf{P}^1$. □

Corollary 1. *The curve Ω/Λ_θ of genus two is represented as the double cover of the line branching at the six points $f_\theta(p,q;s_j)$, where $s_1, \ldots, s_6$ are fixed points of $\gamma_1, \gamma_2, \gamma_3$, which are the intersection points of the circles C_1, C_2, C_3 and the real axis.*

Remark 1. If we choose other p', q', then $f_\theta(p',q';s)$ and $f_\theta(p,q;s)$ are related linear fractionally with coefficients independent of s.

Corollary 2. *If we take p and q reals, then $f_\theta(p,q)$ maps the fundamental domain F_s conformally onto the upper half-plane.*

Proof. Since $f = f_\theta(p,q)$ is real, the three real segments on the boundary ∂F_s of F_s are mapped on the real axis. Let us see that the hemicircles on ∂F_s are also mapped on the real axis. Suppose f is invariant under the involution $s \mapsto r^2/(s-c)+c$. The image of a point $s = c + re^{i\phi}$ on the circle $C(c,r)$ is given as follows:

$$f(s) = f(r^2/(s-c)+c) = f(c+re^{-i\phi}) = \overline{f(s)}.$$

□

6. Fuchsian equations with six singular points

For real p, q, put

$$t_1 = f_\theta(p,q;s_1) < \cdots < t_6 = f_\theta(p,q;s_6),$$

where $s_1 < \cdots < s_6$ are as above. According to the theory of Schwarzian derivatives there is a unique second-order linear differential equation

$$E_\theta : \frac{d^2v}{dt^2} + R_\theta(t)v = 0$$

with regular singular points at $t = t_j$ of exponent $1/2$ such that the ratio of two suitable linearly independent solutions (the Schwarz's s-map for E_θ) is the inverse

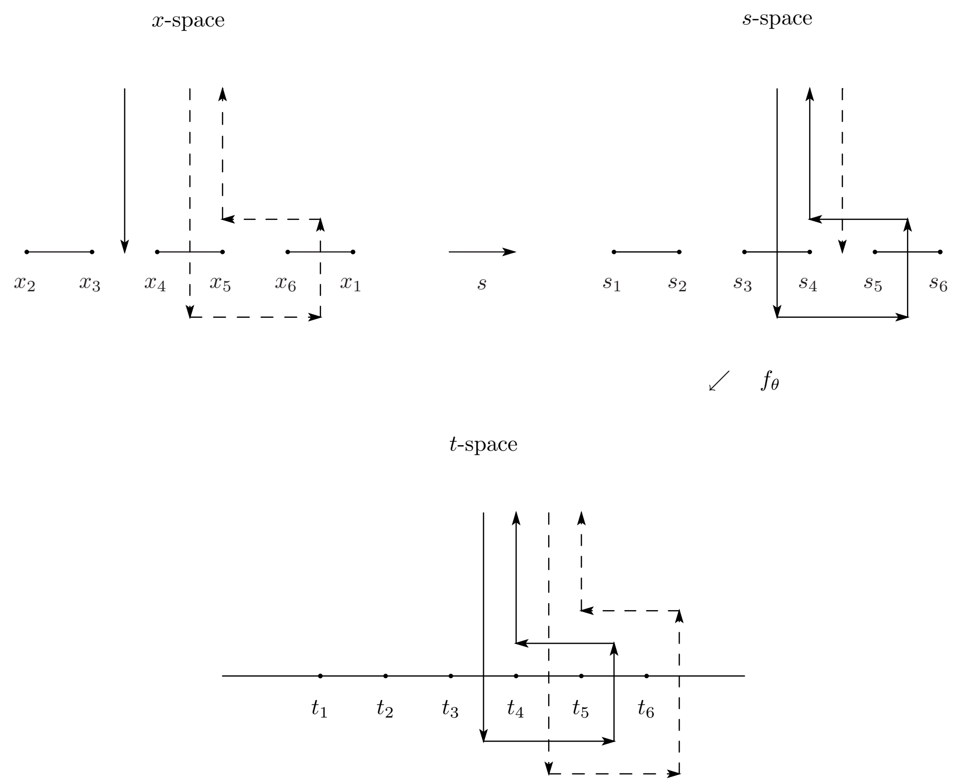

Figure 3. Local monodromies

function of $t = f_\theta(p, q; s)$. The coefficient $R_\theta(t)$ can be, assuming $s_6 = \infty$, expressed as

$$R_\theta(t) = P_\theta(t) / \prod_1^5 (t - t_j)^2,$$

where $P_\theta(t)$ is a polynomial of degree eight. Among the nine coefficients of $P_\theta(t)$, six are determined by the local condition (exponent is $1/2$). The remaining three are not determined by local data (in this sense, these are classically called the *accessory parameters*), are functions of $\theta = (\theta_1, \theta_2, \theta_3)$. Though the authors have no idea what kind of functions they are, we can tell the very specific monodromy behavior of this equation. Let us take a point p on the upper half t-space, and a path ρ_j starting from p, going straight near to t_j, turning once around t_j, and traveling straight back to p.

Proposition 3. *Let M_j be the projective local monodromy of the equation E_θ along the loop ρ_j $(j = 1, \dots, 6)$. Then we have*

$$M_3 \circ M_2 = M_5 \circ M_4 = M_1 \circ M_6 = id.$$

This proposition can be readily shown if we trace these loops and their inverse images under f_θ in Figure 3.

7. A conjectural formula for the λ function

We recall Jacobi's theta functions

$$\begin{aligned}\vartheta_{00}(v,\tau) &= \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}z^2)(1-q^{2n-1}z^{-2}),\\ \vartheta_{01}(v,\tau) &= \vartheta_{00}(v+\frac{1}{2},\tau),\end{aligned}$$

and theta constants

$$\vartheta_{00}(\tau) = \vartheta_{00}(0,\tau), \quad \vartheta_{01}(\tau) = \vartheta_{01}(0,\tau),$$

where $q = e^{\pi i \tau}, z = e^{\pi i v}$, and the elliptic modular function, the λ function

$$\lambda(\tau) = \left(\frac{\vartheta_{01}(\tau)}{\vartheta_{00}(\tau)}\right)^4, \quad \tau \in \mathbf{H},$$

which gives an isomorphism

$$\mathbf{H}/\Gamma(2) \longrightarrow \mathbf{P}^1 - \{0,1,\infty\}, \quad 0 \mapsto 0,\ 1 \mapsto \infty,\ \infty \mapsto 1,$$

where $\mathbf{H}$ is the upper half-plane $\{\tau \in \mathbf{C} \mid \Im(\tau) > 0\}$, and $\Gamma(2)$ is the principal congruence subgroup of level 2 of the elliptic modular group.

Now we go back to the situation of §4 and §5. Letting $\theta_1, \theta_2, \theta_3 \to 0$, the generating involutions $\gamma_1, \gamma_2, \gamma_3$ of Γ_θ tend to the three involutions with respect to the three kissing circles

$$C(3/4, 1/4),\ C(1/2, 1/2),\ C(1/4, 1/4),$$

respectively; we denote by Γ_0 the group generated by these three involutions. Accordingly, the group Λ_θ tends to the modular group $\Gamma(2)$. Proposition 2 suggests the following conjecture.

Conjecture. *As $\theta_1, \theta_2, \theta_3 \to 0$, the function $f_\theta(0, 1; \tau)$ converges uniformly on any compact set of $\mathbf{H}$ to $\lambda(\tau)$. The infinite product*

$$\prod_{\gamma \in \Gamma_0} \frac{\tau - \gamma(0)}{\tau - \gamma(1)}$$

converges, in some sense, to $\lambda(\tau)$.

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Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502 Japan

E-mail address: ichikawa@ms.saga-u.ac.jp

Department of Mathematics, Kyushu University, Fukuoka 810-8560 Japan

E-mail address: myoshida@math.kyushu-u.ac.jp