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A PHILOSOPHY FOR THE MODELLING OF REALISTIC NONLINEAR SYSTEMS

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Abstract. A nonlinear dynamical system is modelled as a nonlinear mapping from a set of input signals into a corresponding set of output signals. Each signal is specified by a set of real number parameters, but such sets may be uncountably infinite. For numerical simulation of the system each signal must be represented by a finite parameter set and the mapping must be defined by a finite arithmetical process. Nevertheless the numerical simulation should be a good approximation to the mathematical model. We discuss the representation of *realistic* dynamical systems and establish a stable approximation theorem for numerical simulation of such systems.

1. Introduction

To construct a mathematical model of a *realistic* dynamical system it is necessary to formalize definitions of such crucial physical properties as *causality*, *finite memory* and *stationarity*. The philosophy of realistic systems has been considered by many authors including Russell [1], Paley and Wiener [2], Foures and Segal [3], Falb and Freedman [4], Willems [5], Gohberg [6] and Sandberg and Xu [7]. We propose a generic topological structure to describe *realistic* nonlinear systems and extend the methods of Torokhti and Howlett [8], [9], [10] to prove *stable* approximation theorems for numerical simulation of these systems. We define a class of $\mathcal{R}$-operators and prove that an $\mathcal{R}$-continuous operator F can be approximated by an $\mathcal{R}$-continuous operator S constructed from an algebra of elementary functions by a finite arithmetic process. The approximation is *stable* to small disturbances. Our theorem is a generalization of the Stone-Weierstrass theorem. Theorems of this type were extended to operators on topological vector spaces by Prenter [11] and Bruno [12]. A Stone-Weierstrass theory for approximation of continuous functions by superpositions of a sigmoidal function was given by Cybenko [13]. Daugavet [14] considered nonlinear operator approximation by generalized causal operators. We provide a substantial extension of this work and show that our definition of the $\mathcal{R}$-continuous operator includes the accepted notions of causality [1] - [7], [14] and other fundamental realistic properties as special cases. Several key results on operator approximation [11], [12], [14] also follow from particular applications of our main

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theorems. In future work we intend to show that certain specific approximation problems [15], [16] can be formulated and solved for $\mathcal{R}$-operators.

2. Representation of realistic dynamical systems

We define a class of *realistic* systems. The fundamental idea is that each input or output is uniquely defined by a corresponding legend of historical information. We pay particular attention to systems in which the output history depends continuously on the input history.

2.1. $\mathcal{R}$-spaces.

Definition 2.1. [14] Let X and A be Banach spaces and let $\mathcal{L}(X, A)$ be the set of continuous linear operators from X into A. Let $T = (T, \rho)$ be a compact metric space and let $\mathcal{M} = \{M_t\}_{t \in T}$ be a family of operators $M_t \in \mathcal{L}(X, A)$ with norm $||M_t|| \leq 1$ for each $t \in T$ and such that $M_s[u] \to M_t[u]$ as $\rho(s,t) \to 0$ for each $u \in X$. The space X equipped with the family of operators $\mathcal{M}$ is called an $\mathcal{R}$-space and is denoted by $X_{\mathcal{R}} = (X, A, T, \mathcal{M})$.

The family $\mathcal{M}$ provides a mechanism for storing and manipulating information about elements in X.

Definition 2.2. For each $x \in X$ the collection of elements $\mathcal{M}[x] = \{M_t[x] \mid t \in T\} \subseteq A$ is called the legend or the complete history of the element x. For each $t \in T$ the element $M_t[x] \in A$ represents the current history of x.

We assume that each element $x \in X$ is uniquely defined by specifying the legend $\mathcal{M}[x]$ of the element.[1]

Lemma 2.3. $\mathcal{M}[x] = \{0\} \Leftrightarrow x = 0$.

If we define $\mathcal{M}[x] + \mathcal{M}[y] = \mathcal{M}[x+y]$ and $\alpha\mathcal{M}[x] = \mathcal{M}[\alpha x]$ for each $\alpha \in \mathbb{C}$, then the set $\mathcal{X} = \mathcal{M}[X] = \{\mathcal{M}[x] \mid x \in X\}$ of all legends is a linear space over $\mathbb{C}$ with zero element $\mathcal{M}[0]$. If we further define $\|\mathcal{M}[x]\| = \sup_{t \in T} \|M_t[x]\|$, then $\mathcal{X}$ is a normed linear space.

Definition 2.4. The archival function $\mathcal{H} : \mathcal{X} \mapsto X$ is a well-defined linear function given by the formula $\mathcal{H}(\mathcal{M}[x]) = x$ for all $\mathcal{M}[x] \in \mathcal{X}$.

Definition 2.5. The family $\mathcal{M}$ is said to be pointwise normally extreme on X if, for each $x \in X$, there exists $t = t_x \in T$ such that $\|M_t[x]\| = \|x\|$.

Lemma 2.6. *If the family $\mathcal{M}$ is pointwise normally extreme on X, then the normed linear spaces $\mathcal{X}$ and X are isometrically isomorphic under the archival mapping $\mathcal{H} : \mathcal{X} \mapsto X$.*

Corollary 2.7. *If the family $\mathcal{M}$ is pointwise normally extreme on X, then $\mathcal{X}$ is a Banach space and $\mathcal{H} \in \mathcal{L}(\mathcal{X}, X)$ with $\|\mathcal{H}\| = 1$.*

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[1] This is an adaption of the idea that a function is defined by specifying the complete set of function values.

2.2. $\mathcal{R}$-operators and $\mathcal{R}$-continuous operators.

Definition 2.8. [14] Let $X_{\mathcal{R}} = (X, A, T, \mathcal{M})$ and $Y_{\mathcal{R}} = (Y, B, T, \mathcal{N})$ be $\mathcal{R}$-spaces, and let the closed set $E \subseteq T \times T$ be an equivalence relation. Let $K \subseteq X$ be a compact set, and let $x \in K$ and $t \in T$. The operator $F : K \to Y$ is an $\mathcal{R}$-operator at $M_t[x] \in A$ if $M_s[v] = M_t[x] \Rightarrow N_s[F(v)] = N_t[F(x)]$ whenever $(s,t) \in E$ and $v \in K$. If $F : K \mapsto Y$ is an $\mathcal{R}$-operator at $M_t[x] \in A$ for all $x \in K$ and $t \in T$, then we say that $F : K \mapsto Y$ is an $\mathcal{R}$-operator.

A dynamical system defined by an $\mathcal{R}$-operator $F : K \mapsto Y$ has the following interpretation. For each $x \in K$ and $t \in T$ the current history $N_t[F(x)]$ of the output depends only on the current history $M_t[x]$ of the input. For a theory of constructive approximation we require the dependence to be continuous.

Definition 2.9. Let $X_{\mathcal{R}} = (X, A, T, \mathcal{M})$ and $Y_{\mathcal{R}} = (Y, B, T, \mathcal{N})$ be $\mathcal{R}$-spaces, and let the closed set $E \subseteq T \times T$ be an equivalence relation. Let $K \subseteq X$ be compact, and let $x \in K$ and $t \in T$. The operator $F : K \to Y$ is $\mathcal{R}$-continuous at $M_t[x] \in A$ if, for each open neighbourhood of zero $H \subseteq B$, there is an open neighbourhood of zero $G = G(x,t,H) \subseteq A$ such that $M_s[v] \in M_t[x] + G \Rightarrow N_s[F(v)] \in N_t[F(x)] + H$ when $(s,t) \in E$ and $v \in K$. If $F : K \mapsto Y$ is $\mathcal{R}$-continuous at $M_t[x] \in A$ for all $x \in K$ and $t \in T$, then we say that F is $\mathcal{R}$-continuous.

Lemma 2.10. *If $F : K \mapsto Y$ is $\mathcal{R}$-continuous, then F is also an $\mathcal{R}$-operator.*

Lemma 2.11. *For each $t \in T$ the set $M_t[K] = \{M_t[x] \mid x \in K\} \subseteq A$ is compact.*

Proof. If $\{G_\gamma\}_{\gamma\in\Gamma}$ is a collection of open sets, then $M_t[K] \subseteq \bigcup_{\gamma\in\Gamma} G_\gamma \Rightarrow K \subseteq \bigcup_{\gamma\in\Gamma} U_\gamma$ where each $U_\gamma = M_t^{-1}[G_\gamma]$ is also open. Since K is compact there is a finite subcollection $U_{\gamma_1}, \ldots, U_{\gamma_r}$ such that $K \subseteq \bigcup_{i=1}^r U_{\gamma_i} \Rightarrow M_t[K] \subseteq \bigcup_{i=1}^r G_{\gamma_i}$. □

For each $t \in T$ let $E_t = \{s \mid (s,t) \in E\} \subseteq T$. Note that E_t is compact. We wish to show that the set $\mathcal{M}_t[K] = \{M_s[K] \mid s \in E_t\}$ is also compact.

Lemma 2.12. *Let $s \in T$. If $M_s[K] \subseteq G$ where G is an open set, then we can find $\delta = \delta(s,G) > 0$ such that $M_r[K] \subseteq G$ when $\rho(r,s) < \delta$.*

Proof. If not $\exists$ sequences $\{r_i\} \subseteq T$ with $\rho(r_i,s) \to 0$ and $\{u_i\} \subseteq K$ such that $M_{r_i}[u_i] \notin G$ for each i. We can assume $u_i \to v$ for some $v \in K$. Choose $\alpha > 0$ and $G_\alpha = \{a \mid \|a\| < \alpha\} \subseteq A$ so that $M_s[v] + G_\alpha \subseteq G$. If $U_\alpha = \{u \mid \|u\| < \alpha\} \subseteq X$, then $u \in U_\alpha/2 \Rightarrow M_r[u] \in G_\alpha/2$. If i is so large that $u_i - v \in U_\alpha/2$ and $M_{r_i}[v] \in M_s[v] + G_\alpha/2$, then $M_{r_i}[u_i] \in M_s[v] + G_\alpha \subseteq G$. This is a contradiction. □

Lemma 2.13. *For each $t \in T$ the set $\mathcal{M}_t[K]$ is a compact subset of A.*

Proof. Let $t \in T$ and $s \in E_t$ and suppose that $\{G_\gamma\}_{\gamma\in\Gamma}$ is a collection of open sets with $\mathcal{M}_t[K] \subseteq \bigcup_{\gamma\in\Gamma} G_\gamma$. Since $M_s[K]$ is compact and $M_s[K] \subseteq \mathcal{M}_t[K]$ for each $s \in E_t$ there is a finite subset $\Gamma(s) \subseteq \Gamma$ with $M_s[K] \subseteq \bigcup_{\gamma\in\Gamma(s)} G_\gamma = G(s)$. Choose $\delta(s) > 0$ such that $M_r[K] \subseteq G(s)$ whenever $\rho(r,s) < \delta(s)$, and define the open sets $R(s) = \{r \mid \rho(r,s) < \delta(s)\} \subseteq T$ for each $s \in T$. Since E_t is compact we know that $E_t \subseteq \bigcup_{s\in E_t} R(s) \Rightarrow E_t \subseteq \bigcup_{j=1}^q R(s_j)$ for some finite subcollection $\{R(s_j)\}_{j=1,2,\ldots,q}$ and since $\bigcup_{r\in R(s_j)} M_r[K] \subseteq G(s_j)$ for each $j = 1, \ldots, q$ we have

$$\mathcal{M}_t[K] = \bigcup_{r\in E_t} M_r[K] = \bigcup_{j=1}^q \left[\bigcup_{r\in R(s_j)} M_r[K] \right] \subseteq \bigcup_{j=1}^q G(s_j) = \bigcup_{j=1}^q \left[\bigcup_{\gamma\in\Gamma(s_j)} G_\gamma \right].$$

Therefore $\mathcal{M}_t[K]$ is compact. $\square$

Lemma 2.14. *Let $F : K \mapsto Y$ be continuous and $\mathcal{R}$-continuous. For each neighbourhood of zero, $H \subseteq B$ there exists a neighbourhood of zero $G = G(H)$ such that $M_r[u] - M_s[v] \in G \Rightarrow N_r[F(u)] - N_s[F(v)] \in H$ whenever $(r, s) \in E$ and $u, v \in K$. Hence F is uniformly $\mathcal{R}$-continuous.*

Proof. If not then for some $\beta > 0$ there exist neighbourhoods of zero $H_\beta = \{b \mid \|b\| < \beta\} \subseteq B$ and $G_{1/n} = \{a \mid \|a\| < 1/n\} \subseteq A$ for each $n = 1, 2, \ldots$ and $u_n, v_n \in K$ and $r(n), s(n), t(n) \in T$ with $r(n), s(n) \in E_{t(n)}$ for each $n = 1, 2, \ldots$ such that $M_{r(n)}[u_n] - M_{s(n)}[v_n] \in G_{1/n}$ and $N_{r(n)}[F(u_n)] - N_{s(n)}[F(v_n)] \notin H_\beta$. We suppose, without loss of generality, that there exist $u, v \in K$ with $u_n \to u$ and $v_n \to v$ as $n \to \infty$ and points $r, s, t \in T$ with $\rho(r(n), r) \to 0$, $\rho(s(n), s) \to 0$ and $\rho(t(n), t) \to 0$ as $n \to \infty$. Since $(r(n), t(n)) \in E$ and $(s(n), t(n)) \in E$ and since E is closed, it follows that $(r, t) \in E$ and $(s, t) \in E$. Hence $r, s \in E_t$. Choose $\alpha > 0$ and define $G_\alpha = \{a \mid \|a\| < \alpha\} \subseteq A$. We have $M_r[x] \in G_\alpha/5$, whenever $x \in U_\alpha/5$ where $U_\alpha = \{x \mid \|x\| < \alpha\} \subseteq X$. If we take n so large that $u - u_n, v - v_n \in U_\alpha/5$, $M_{r(n)}[u] - M_r[u], M_{s(n)}[v] - M_s[v] \in G_\alpha/5$ and $G_{1/n} \subseteq G_\alpha/5$ then $M_r[u] - M_s[v] \in G_\alpha$. Since α is arbitrary it follows that $M_r[u] - M_s[v] = 0$ and since $r, s \in E_t$ the $\mathcal{R}$-continuity of F implies that $N_r[F(u)] - N_s[F(v)] = 0$. Define $V_\beta = \{y \mid \|y\| < \beta\} \subseteq Y$. Note that $N_r[y] \in H_\beta/4$ whenever $y \in V_\beta/4$. Choose n so large that $F(u_n) - F(u), F(v_n) - F(v) \in V_\beta/4$ and $N_{r(n)}[F(u)] - N_r[F(u)], N_{s(n)}[F(v)] - N_s[F(v)] \in H_\beta/4$. Hence $N_{r(n)}[F(u_n)] - N_{s(n)}[F(v_n)] \in H_\beta$, which is a contradiction. $\square$

2.3. **The collection of auxiliary mappings.** To establish a constructive approximation for $\mathcal{R}$-continuous mappings we define a collection of auxiliary mappings.

Definition 2.15. Let $F : K \mapsto Y$ be $\mathcal{R}$-continuous. For each $t \in T$ define the auxiliary mapping $f_t : \mathcal{M}_t[K] \mapsto B$ by setting $f_t(M_s[v]) = N_s[F(v)]$ for each $s \in E_t$ and $v \in K$.

This is a good definition because $M_r[u] = M_s[v] \Rightarrow N_r[F(u)] = N_s[F(v)]$ for each $r, s \in E_t$ and each $u, v \in K$. The mapping $f_t : \mathcal{M}_t[K] \mapsto B$ is continuous at each point $M_s[v] \in \mathcal{M}_t[K]$ because, for each open neighbourhood of zero $H \subseteq B$, there is a corresponding open neighbourhood of zero $G = G_t(v, s, H) = G(v, s, H) \cap \mathcal{M}_t[K] \subseteq A$ such that $M_r[u] - M_s[v] \in G \Rightarrow f_t(M_r[u]) - f_t(M_s[v]) = N_r[F(u)] - N_s[F(v)] \in H$ whenever $r \in E_t$ and $u \in K$. Because $\mathcal{M}_t[K]$ is compact the mapping $f_t : \mathcal{M}_t[K] \mapsto B$ is uniformly continuous and for each neighbourhood of zero $H \subseteq B$, there is a neighbourhood of zero $G = G_t(H) \subseteq A$ such that $M_r[u] - M_s[v] \in G \Rightarrow f_t(M_r[u]) - f_t(M_s[v]) \in H$ whenever $r, s \in E_t$ and $u, v \in K$. Lemma 2.14 shows that when $F : K \mapsto Y$ is continuous the collection $\{f_t\}_{t \in T}$ is uniformly equi-continuous. That is, for each neighbourhood of zero $H \subseteq B$ there is a neighbourhood of zero $G = G(H) \subseteq A$ such that for all $t \in T$ we have $M_r[u] - M_s[v] \in G \Rightarrow f_t(M_r[u]) - f_t(M_s[v]) \in H$ whenever $r, s \in E_t$ and $u, v \in K$.

2.4. **Some examples of $\mathcal{R}$-continuous operators.** The following theorem of M. Riesz is used in the examples to justify compactness of the set K.

Theorem 2.16. *Let $K \subseteq L^p([0, 1])$ and write $\mathcal{T}_h x(r) = x(r + h)$ $\forall\, x \in K$; $r, r + h \in [0, 1]$. The set K is compact if and only if $\exists\, M > 0$ with $\|x\|_p \leq M$ and $\delta = \delta(\epsilon)$ such that $\|\mathcal{T}_h x - x\|_p < \epsilon$ whenever $|h| < \delta$ for all $x \in K$.*

2.4.1. *A causal operator.* Let $X = L^1([0,1])$, $K = \{x \mid |x(s)-x(t)| \le |s-t| \; \forall\, s,t \in [0,1]\} \subseteq X$ and $Y = C([0,1])$. The operator $F : K \mapsto Y$ is a $\mathcal{C}$-operator on the time interval $T = [0,1]$ if, for all $t \in T$ and $u, x \in K$, $\{u(s) = x(s) \; \forall\; s \in [0,t]\} \Rightarrow \{[F(u)](s) = [F(x)](s) \; \forall\; s \in [0,t]\}$. Note that the output at time t depends only on the input prior to time t. The operator F is uniformly $\mathcal{C}$-continuous if, for all $t \in T$ and all $u, x \in K$ and for each $\beta > 0$, we can find $\alpha = \alpha(\beta) > 0$ such that $\{|\int_{[0,s]} u(r)dr - \int_{[0,s]} x(r)dr| < \alpha \; \forall\; s \in [0,t]\} \Rightarrow \{|[F(u)](s) - [F(x)](s)| < \beta \; \forall\, s \in [0,t]\}$. To show that a uniformly $\mathcal{C}$-continuous operator is a special case of a uniformly $\mathcal{R}$-continuous operator, set $A = B = C(T)$ and $\tau = \min(s,t)$ and define $M_t[x](s) = \int_{[0,\tau]} x(r)dr$ and $N_t[y](s) = y(\tau)$ for $x \in X$, $y \in Y$ and $s, t \in T$. Let $E = \{(t,t) \mid t \in T\}$. In this notation F is a uniformly $\mathcal{C}$-continuous operator on T if and only if for all $t \in T$, and all $u, x \in K$ and for each $\beta > 0$, we can find $\alpha = \alpha(\beta)$ such that $\{\|M_t[u] - M_t[x]\| < \alpha\} \Rightarrow \{\|[F(u)] - [F(x)]\| < \beta\}$, which, in turn, is equivalent to saying that F is a uniformly $\mathcal{R}$-continuous operator on T. For a particular instance we note that the operator $F_C : K \mapsto Y$ defined by $[F_C(x)](t) = e^{-t} \int_{[0,t]} e^s x(s) ds$ for each $t \in T$ is a uniformly $\mathcal{C}$-continuous operator.

2.4.2. *A stationary operator with finite memory.* Let $X = L^\infty(\mathbb{R})$, $K = \{x \mid x(t) = 0 \text{ for } t \notin [0,1] \text{ and } |x(s)-x(t)| \le |s-t| \; \forall\, s,t \in \mathbb{R}\} \subseteq X$ and $Y = C(\mathbb{R})$. The operator $F : K \mapsto Y$ is a stationary operator with finite memory $\Delta > 0$ on the time interval $T = [0, 1+\Delta]$ if, for all $u, x \in K$ and all $s, t \in T$, $\{u(s+r-\Delta) = x(t+r-\Delta) \; \forall\, r \in [0,\Delta]\} \Rightarrow \{[F(u)](s) = [F(x)](t)\}$. The output at time t depends only on the inputs at times $s \in [t-\Delta, t]$. We say that F is an $\mathcal{S}$-operator. The operator F is uniformly $\mathcal{S}$-continuous on T if, $\forall\, \{u, x \in K;\; s,t \in T;\; \beta > 0\}$, we can find $\alpha = \alpha(\beta) > 0$ such that $\{|u(s+r-\Delta) - x(t+r-\Delta)| < \alpha \; \forall\, r \in [0,\Delta]\} \Rightarrow \{|[F(u)](s) - [F(x)](t)| < \beta\}$. To show that a uniformly $\mathcal{S}$-continuous operator is a special case of a uniformly $\mathcal{R}$-continuous operator, set $A = L^\infty([0,\Delta])$ and $B = C([0, 1+\Delta])$. For each $t \in T$ define $M_t : X \mapsto A$ by $M_t[x](r) = x(r + t - \Delta) \; \forall\; r \in [0,\Delta]$ and $N_t : Y \mapsto C([0, 1+\Delta])$ by $N_t[y](r) = y(t) \; \forall\, r \in [0, 1+\Delta]$. Let $E = T \times T$. The operator F is uniformly $\mathcal{S}$-continuous on T if and only if for all $u, x \in K$ and all $s, t \in T$ and for each $\beta > 0$, we can find $\alpha = \alpha(\beta) > 0$ such that $\{\|M_s[u] - M_t[x]\| < \alpha\} \Rightarrow \{\|N_s[F(u)] - N_t[F(x)]\| < \beta\}$, which is equivalent to saying that F is a uniformly $\mathcal{R}$-continuous operator on T. In particular, the mapping $F_\Delta : K \mapsto Y$ defined by $[F_\Delta(x)](t) = \frac{1}{\Delta} \int_{[t-\Delta,t]} x(r)dr$ for each $x \in X$ and $t \in \mathbb{R}$ is a uniformly $\mathcal{S}$-continuous operator.

3. The modulus of continuity

Definition 3.1. Let X and Y be separable Banach spaces. Let $K \subseteq X$ be a compact set, and let $F : K \to Y$ be a continuous map. The modulus of continuity $\omega = \omega[F] : \mathbb{R}_+ \to \mathbb{R}_+$ is given by the formula

$$\omega(\delta) = \sup_{x_1, x_2 \in K,\; \|x_1 - x_2\| \le \delta} \|F(x_1) - F(x_2)\|.$$

Note that $\omega(0) = 0$ and $\omega(\delta) \le \omega(\delta')$ whenever $\delta \le \delta'$. We will show that ω is a uniformly continuous function.

Lemma 3.2. *Let X and Y be separable Banach spaces. Let $K \subseteq X$ be a compact set and $F : K \to Y$ a continuous map. Let $\omega = \omega[F] : \mathbb{R}_+ \to \mathbb{R}_+$ be the corresponding* *modulus of continuity. Then for each $\tau > 0$ we can find $\sigma = \sigma(\tau) > 0$ such that $0 \leq \omega(\delta') - \omega(\delta) \leq \tau$ whenever $0 \leq \delta' - \delta \leq \sigma$.*

Proof. Define $\Delta F : K \times K \mapsto Y$ by setting $\Delta F(x) = F(x_2) - F(x_1)$ for each $x = (x_1, x_2) \in K$. Clearly ΔF is continuous with respect to the norm $\|x\|_{K\times K} = \|x_1\| + \|x_2\|$ and hence, since $K \times K$ is compact, ΔF is uniformly continuous. If we define $D_\delta = \{x \mid \|x_2 - x_1\| \leq \delta\}$, then $D_\delta \subseteq K \times K$ is compact and $\omega(\delta) = \sup_{x \in D_\delta} \|\Delta F(x)\|$ for each $\delta \geq 0$. Fix $\tau > 0$ and choose $\sigma = \sigma(\tau) > 0$ such that $\|\Delta F(x') - \Delta F(x)\| < \tau$ whenever $\|x' - x\|_{K\times K} < \sigma$. Now suppose that $0 \leq \delta' - \delta \leq \sigma$. Find $x' \in D_{\delta'}$ with $x'_2 \neq x'_1$ and $\omega(\delta') = \|\Delta F(x')\|$, and define $\theta \in [0, 1]$ so that $\theta\|x'_2 - x'_1\| = \delta$. Let $x'' = (x'_2, x'_1)$ and define $x = \theta x' + (1-\theta)(x' + x'')/2$. It is easy to see that $\|x_2 - x_1\| = \delta$ and that $\|x' - x\|_{K\times K} \leq \sigma$. It follows that $\omega(\delta') = \|\Delta F(x')\| \leq \|\Delta F(x)\| + \tau \leq \omega(\delta) + \tau$. Thus $0 \leq \omega(\delta') - \omega(\delta) \leq \tau$ when $0 \leq \delta' - \delta \leq \sigma$ and hence ω is uniformly continuous on $\mathbb{R}_+$. □

3.1. **The $\mathcal{R}$-modulus of continuity.** The $\mathcal{R}$-modulus of continuity will be used to characterize our constructive approximation theorems for $\mathcal{R}$-continuous operators.

Definition 3.3. [14] Let $X_\mathcal{R} = \{X, A, T, \mathcal{M}\}$ and $Y_\mathcal{R} = \{Y, B, T, \mathcal{N}\}$ be $\mathcal{R}$-spaces, and let $E \subseteq T \times T$ be the given equivalence relation. Let $K \subseteq X$ be a compact set, and suppose that the map $F : K \to Y$ is $\mathcal{R}$-continuous. The function $\omega_\mathcal{R} = \omega_\mathcal{R}[F] : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\omega_\mathcal{R}(\delta) = \sup_{\substack{u,v \in K;\ (r,s)\in E: \\ \|M_r[u] - M_s[v]\| \leq \delta}} \|N_r[F(u)] - N_s[F(v)]\|$$

is called the $\mathcal{R}$-modulus of continuity of the operator F.

Definition 3.4. We say that $(X_\mathcal{R}, Y_\mathcal{R})$ is a complete $\mathcal{R}$-pair if $E = T \times T$ and an incomplete $\mathcal{R}$-pair if $E \neq T \times T$.

Lemma 3.5. *Let $(X_\mathcal{R}, Y_\mathcal{R})$ be a complete $\mathcal{R}$-pair and suppose that $F : K \mapsto Y$ is $\mathcal{R}$-continuous. Then the $\mathcal{R}$-modulus of continuity $\omega_\mathcal{R} = \omega_\mathcal{R}[F] : \mathbb{R}_+ \to \mathbb{R}_+$ is uniformly continuous with $\omega_\mathcal{R}(0) = 0$.*

Proof. Since $E_t = T$ for all $t \in T$ it follows that $\mathcal{M}[K] = \mathcal{M}_t[K] = \{M_s[x] \mid x \in K \text{ and } s \in T\} \subseteq A$ for all $t \in T$. Define an auxiliary mapping $f : \mathcal{M}[K] \mapsto B$ by setting $f(M_t[x]) = N_t[Fx]$ for each $x \in K$ and $t \in T$. Recall from our earlier remarks that the mapping $f : \mathcal{M}[K] \mapsto B$ is uniformly continuous. The function $\omega_f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is the modulus of continuity of f. Lemma 3.2 shows that ω_f is uniformly continuous. Since $\omega_f(\delta) = \omega_\mathcal{R}(\delta)$ we obtain the desired result. □

Lemma 3.6. *Let $(X_\mathcal{R}, Y_\mathcal{R})$ be an incomplete $\mathcal{R}$-pair and suppose that $F : K \mapsto Y$ is both continuous and $\mathcal{R}$-continuous. Then the $\mathcal{R}$-modulus of continuity $\omega_\mathcal{R} = \omega_\mathcal{R}[F] : \mathbb{R}_+ \to \mathbb{R}_+$ is uniformly continuous with $\omega_\mathcal{R}(0) = 0$.*

Proof. Since $(X_\mathcal{R}, Y_\mathcal{R})$ is an incomplete $\mathcal{R}$-pair we consider the various equivalence classes E_t for each $t \in T$. We saw earlier that for each $t \in T$ there is an auxiliary mapping $f_t : M_t[K] \mapsto B$ defined by $f_t(M_t[x]) = N_t[F(x)]$ for all $x \in K$. Let $\omega[f_t] : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the modulus of continuity for the map f_t, and consider the argument used in Lemma 3.2. Define $\Delta f_t : M_t[K] \times M_t[K]$ by the formula $\Delta f_t(p, q) = \|f_t(p) - f_t(q)\|$ for each $(p, q) \in M_t[K] \times M_t[K]$. Choose $\tau > 0$. From our earlier remarks about the uniform equi-continuity of the family of auxiliary mappings $\{f_t\}_{t\in T}$, we can choose $\sigma = \sigma(\tau) > 0$ such that for all $t \in T$ we have $\|\Delta f_t(p', q') - \Delta f_t(p, q)\| < \tau$

whenever $\|(p',q')-(p,q)\|<\sigma$. Now it is clear from Lemma 3.2 that for all $t\in T$ we have $0\le \omega[f_t](\delta')-\omega[f_t](\delta)\le\tau$ whenever $0\le\delta'-\delta\le\sigma$. Thus the family $\{\omega[f_t]\}_{t\in T}$ is also uniformly equi-continuous. Since $\omega_{\mathcal{R}}(\delta)=\sup_{t\in T}\omega[f_t](\delta)$, it follows that $0\le\omega_{\mathcal{R}}(\delta')-\omega_{\mathcal{R}}(\delta)\le\tau$ whenever $0\le\delta'-\delta\le\sigma$. □

4. Approximation of nonlinear operators on compact sets

We describe briefly the recent work by Torokhti and Howlett [8]. Let X, Y be locally convex topological vector spaces and let $K\subseteq X$ be a compact subset. Let $F: K\subseteq X\to Y$ be a continuous map. If F is known only on K, then for some suitable neighbourhood ϵ of zero in X the construction of an extended operator $S: K+\epsilon\subseteq X\to Y$ is an important ingredient in the approximation procedure. The extension of the domain allows consideration of a small disturbance in the input signal. Such disturbances are unavoidable in the modelling process. The main result is formulated as follows. Let X, Y be topological vector spaces with the Grothendieck property of approximation[2] and with approximating sequences $\{G_m\}_{m=1,2,\ldots}\subseteq\mathcal{L}(X,X_m)$, $\{H_n\}_{n=1,2,\ldots}\subseteq\mathcal{L}(Y,Y_n)$ of continuous linear operators, where $X_m\subseteq X, Y_n\subseteq Y$ are subspaces of dimension m,n. Write $X_m=\{x_m\in X \mid x_m=\sum_{j=1}^m a_j u_j\}$ and $Y_n=\{y_n\in Y\mid y_n=\sum_{k=1}^n b_k v_k\}$, where $a\in\mathbb{R}^m, b\in\mathbb{R}^n$ and $\{u_j\}_{j=1,2,\ldots,m}, \{v_k\}_{k=1,2,\ldots,n}$ are bases in X_m, Y_n respectively. Let $\mathcal{G}=\{g\}$ be an algebra of continuous functions $g:\mathbb{R}^m\to\mathbb{R}$ that satisfies the conditions of Stone's algebra. Define the operators $Q\in\mathcal{L}(X_m,\mathbb{R}^m)$, $Z:\mathbb{R}^m\to\mathbb{R}^n$ and $W\in\mathcal{L}(\mathbb{R}^n,Y_n)$ by $Q(x_m)=a,\quad Z(a)=(g_1(a),g_2(a),\ldots,g_n(a))$ and $W(z)=\sum_{k=1}^n z_k v_k$ where each $g_k\in\mathcal{G}$ and $z_k=g_k(a)$. Subject to an appropriate choice of the functions $\{g_k\}\in\mathcal{G}$, so that z_k provides a sufficiently good approximation to b_k, the following *stable* approximation theorems can be established.

Theorem 4.1. *Let X, Y be locally convex topological vector spaces as above, and let X be normal. Let $K\subseteq X$ be a compact set and $F: K\to Y$ a continuous map. For a given convex neighbourhood of zero $\tau\subseteq Y$ there exists a neighbourhood of zero $\sigma\subseteq X$, an associated continuous operator $S: X\to Y_n$ defined by finite arithmetic in the form $S=S_\sigma=WZQG_m$ and a neighbourhood of zero $\epsilon\subseteq X$ such that for all $x\in K$ and all $x'\in X$ with $x'-x\in\epsilon$ we have $F(x)-S(x')\in\tau$.*

Theorem 4.2. *Let X and Y be separable Banach spaces. Let $K\subseteq X$ be a compact set and $F: K\to Y$ a continuous map. For any given numbers $\delta>0$ and $\tau>0$ and for all $x\in K$ and all $x'\in X$ with $\|x'-x\|\le\delta$, there exists an operator $S=WZQG_m: X\mapsto Y$ defined by finite arithmetic such that $\|F(x)-S(x')\|\le \frac{1}{2}\omega[F](2\delta)+\tau$.*

Proof. The proof of the latter result uses an argument proposed by Daugavet [14]. Since [14] is difficult to obtain, the proof is given in Appendix A.

4.1. **A model for constructive approximation in the class of $\mathcal{R}$-continuous operators.** When F is an $\mathcal{R}$-continuous operator we prove the existence of an approximating $\mathcal{R}$-continuous operator S that is *stable* to small disturbances. The operator S defines a model of the real system and is constructed from an algebra of elementary continuous functions by a process of finite arithmetic.

[2]The space X possesses the Grothendieck property of approximation if there is a sequence $\{G_m\}_{m\in\mathbb{N}}\subseteq\mathcal{L}(X,X_m)$ where $X_m\subseteq X$ is a subspace of dimension m and the operators G_m are equi-continuous on compacta and uniformly convergent to unit operators on those compacta.

Theorem 4.3. *Let A and B be Banach spaces with the Grothendieck property of approximation, and let $X_\mathcal{R} = (X, A, T, \mathcal{M})$ and $Y_\mathcal{R} = (Y, B, T, \mathcal{N})$ be $\mathcal{R}$-spaces. Suppose that $(X_\mathcal{R}, Y_\mathcal{R})$ is a complete $\mathcal{R}$-pair and that $\mathcal{N}$ is pointwise normally extreme on Y. Let $K \subseteq X$ be a compact set, and let the map $F : K \mapsto Y$ be an $\mathcal{R}$-continuous operator. Then for any fixed real numbers $\delta > 0$ and $\tau > 0$ there exists an associated $\mathcal{R}$-continuous operator S defined by finite arithmetic in the form $S = WZQG : X \mapsto Y$ such that for all $x \in K$ and $x' \in X$ with $\|x - x'\| \leq \delta$ we have $\|F(x) - S(x')\| \leq \frac{1}{2}\omega_\mathcal{R}(2\delta) + \tau$.*

Proof. We recall from Lemma 3.5 that the auxiliary mapping $f : \mathcal{M}[K] \mapsto B$ is uniformly continuous. We will construct a mapping $\sigma : A \to B$ in the form $\sigma = \pi\nu\lambda\theta$ where $A_m \subseteq A$ is a subspace of dimension m and $B_n \subseteq B$ is a subspace of dimension n, and where $\theta \in \mathcal{L}(A, A_m)$ and $\lambda \in \mathcal{L}(A_m, \mathbb{R}^m)$, where $\nu : \mathbb{R}^m \mapsto \mathbb{R}^n$ is continuous and where $\pi \in \mathcal{L}(\mathbb{R}^n, B_n)$. By Theorem 4.2 there exists a continuous mapping $\sigma : A \mapsto B$ in the above form such that for all $w \in \mathcal{M}[K]$ and all w' with $\|w - w'\| < \delta$ we have $\|f(w) - \sigma(w')\| \leq \frac{1}{2}\omega_\mathcal{R}(2\delta) + \tau$, where we have used the fact that the modulus of continuity of f satisfies $\omega_f(\alpha) = \omega_\mathcal{R}(\alpha)$ for all $\alpha \in \mathbb{R}_+$. Now define $S : X \mapsto Y$ by setting $N_t[Sx] = \sigma(M_t[x])$ for each $x \in X$ and each $t \in T$. Our indirect definition assumes that if $N_t[y] \in B$ is known for all $t \in T$, then $y \in Y$ is also known. We will follow our earlier notation and write $y = \mathcal{K}(\mathcal{N}[y])$ where $\mathcal{K} : \mathcal{Y} \mapsto Y$ is the appropriate archival function. The mapping $\sigma : A \mapsto B$ is continuous and hence $S : X \mapsto Y$ is an $\mathcal{R}$-continuous operator. Since $\|M_t[x - x']\| \leq \|x - x'\|$, it follows that $\|N_t[Fx - Sx']\| = \|f(M_t[x]) - \sigma(M_t[x'])\| < \frac{1}{2}\omega_\mathcal{R}(2\delta) + \tau$ for all $t \in T$ whenever $x \in K$ and $\|x - x'\| < \delta$. But we can choose $t = t_{[F(x) - S(x')]} \in T$ such that $\|N_t[Fx - Sx']\| = \|F(x) - S(x')\|$ and so $\|F(x) - S(x')\| < \frac{1}{2}\omega_\mathcal{R}(2\delta) + \tau$ whenever $x \in K$ and $\|x - x'\| < \delta$. Since we defined $N_t[Sx] = \pi\nu\lambda\theta M_t[x]$ we can now write $\mathcal{N}[Sx] = \pi\nu\lambda\theta\mathcal{M}[x]$ or, equivalently, $S(x) = \mathcal{K}\pi\nu\lambda\theta\mathcal{H}^{-1}(x)$ for each $x \in X$. Note that $\|\mathcal{H}^{-1}\| \leq 1$ and that $\|\mathcal{K}\| = 1$. If we define $G = \theta\mathcal{H}^{-1}$, $Q = \lambda$, $Z = \nu$ and $W = \mathcal{K}\pi$, then we can see that S has the desired form. We assume that G and W can be defined by finite arithmetic or replaced by suitable approximations. □

Lemma 4.4. *Let $K \subseteq X$ be a compact set. Then for each $\epsilon > 0$ we can find $\delta > 0$ such that $\|M_s[x] - M_t[x]\| < \epsilon$ for all $x \in K$ whenever $s, t \in T$ and $\rho(s, t) < \delta$.*

Theorem 4.5. *Let A and B be Banach spaces with the Grothendieck property of approximation. Let $X_\mathcal{R} = (X, A, T, \mathcal{M})$ and $Y_\mathcal{R} = (Y, B, T, \mathcal{N})$ be $\mathcal{R}$-spaces and suppose that $(X_\mathcal{R}, Y_\mathcal{R})$ is an incomplete $\mathcal{R}$-pair and that $\mathcal{N}$ is pointwise normally extreme on Y. Let $K \subseteq X$ be a compact set and let the map $F : K \mapsto Y$ be continuous and $\mathcal{R}$-continuous. Then for any fixed real numbers $\delta > 0$ and $\tau > 0$ there exists an associated operator $S : X \mapsto Y$ defined by $N_t[Su] = \sum_{j=1}^N \psi_j(t) N_t[S_j u]$ where $\psi_j : T \mapsto \mathbb{R}$ for each $j = 1, 2, \ldots, N$ and $\{\psi_1, \ldots, \psi_N\}$ is a partition of unity and where $S_j = W_j Z_j Q_j G_j : X \mapsto Y$ for each $j = 1, 2, \ldots, N$ and each $u \in K$ and $t \in T$. The mapping S is continuous and $\mathcal{R}$-continuous and is defined by a process of finite arithmetic in such a way that for all $x \in K$ and $x' \in X$ with $\|x - x'\| \leq \delta$ we have $\|F(x) - S(x')\| \leq \frac{1}{2}\omega_\mathcal{R}(2\delta) + \tau$.*

Proof. Let $t \in T$ and consider the auxiliary mappings $f_t : M_t[K] \mapsto B$ and the associated moduli of continuity $\omega[f_t] : \mathbb{R}_+ \mapsto \mathbb{R}_+$. We recall from Lemmas 2.14 and 3.6 that the families $\{f_t\}_{t \in T}$ and $\{\omega[f_t]\}_{t \in T}$ are each uniformly equi-continuous. Hence, for the given $\tau > 0$, it is possible to choose $\epsilon = \epsilon(\tau) > 0$ so small that

$\lambda \leq \delta + \epsilon \Rightarrow \omega_{\mathcal{R}}(2\lambda) \leq \omega_{\mathcal{R}}(2\delta) + \tau$ and $\|M_r[u] - M_s[v]\| < \epsilon \Rightarrow \|N_r[Fu] - N_s[Fv]\| < \tau/12$ whenever $(r,s) \in E$ and $u, v \in K$. Since K and $F(K)$ are both compact, we can use Lemma 4.4 to find $\gamma > 0$ so that both $\|M_s[x] - M_t[x]\| < \epsilon$ and $\|N_s[Fx] - N_t[Fx]\| < \tau/4$ for all $x \in K$ when $\rho(s,t) < \gamma$. Choose a γ-net $\{t_1, \ldots, t_N\} \subseteq T$ such that whenever $t \in T$ we can always find some $j = j(t)$ with $\|t - t_j\| < \gamma$ and let $\{\psi_1(t), \ldots, \psi_N(t)\}$, where $\psi_j : T \mapsto \mathbb{R}$ for each $j = 1, 2, \ldots, N$, be a partition of unity on T such that $\psi_1, \ldots, \psi_N \in C(T)$, $\psi_j(t) \geq 0$ for all $t \in T$, $\sum_{j=1}^N \psi_j(t) = 1$ for all $t \in T$, and $\psi_j(t) = 0$ whenever $\rho(t, t_j) \geq \gamma$. Let $x \in K$ and choose $u \in X$ with $\|u - x\| \leq \delta$. If $\rho(t, t_j) < \gamma$, then $\|M_t[u] - M_{t_j}[x]\| \leq \|M_t[u - x]\| + \|M_t[x] - M_{t_j}[x]\| \leq \|u - x\| + \epsilon = \lambda \leq \delta + \epsilon$. By applying Theorem 4.2 we can define a function $\sigma_j : A \to B$ in the form $\sigma_j = \pi_j \nu_j \lambda_j \theta_j$ such that for all $w \in M_{t_j}[K]$ and w' with $\|w' - w\| < \lambda$ we have $\|f_j(w) - \sigma_j(w')\| < \frac{1}{2}\omega[f_j](2\lambda) + \frac{\tau}{4}$. Define $S_j : X \to Y$ by setting $N_t[S_j u] = \sigma_j(M_t[u])$ and $S : X \mapsto Y$ by the formula $N_t[Su] = \sum_{j=1}^N \psi_j(t)\sigma_j(M_t[u])$ for all $u \in X$ and $t \in T$. Now for $x \in K$, $u \in X$ with $\|x - u\| < \delta$ and all $t \in T$ we have

$$\|N_t[Fx] - N_t[Su]\| = \| \sum_{\rho(t,t_j) < r} \psi_j(t) \left[N_t[Fx] - \sigma_j(M_t[u]) \right] \|.$$

We make two observations. Firstly, for $\rho(t, t_j) < r$ we have

$$\begin{aligned} \|N_t[Fx] - \sigma_j(M_t[u])\| &\leq \|N_t[Fx] - N_{t_j}[Fx]\| + \|N_{t_j}[Fx] - \sigma_j(M_t[u])\| \\ &\leq \|f_j(M_{t_j}[x]) - \sigma_j(M_t u)\| + \frac{\tau}{4}. \end{aligned}$$

Secondly, since $\|M_{t_j}[x] - M_t[u]\| \leq \lambda$, it follows that $\|f_j(M_{t_j}[x]) - \sigma_j(M_t[u])\| \leq \frac{1}{2}\omega_{\mathcal{R}}(2\lambda) + \frac{\tau}{4}$. The desired result can now be established. □

Appendix A. Proof of Theorem 4.2

It is well known that any separable Banach space is isometric and isomorphic to a subspace of the space $C([0,1])$ of continuous functions on the interval $[0,1]$. Thus, without loss of generality, we assume $X = Y = C([0,1])$. Define $\varphi : K \times [0,1] \to \mathbb{R}$ by setting $\varphi(x,t) = F[x](t)$ for all $t \in [0,1]$. Fix $\delta > 0$ and $t \in [0,1]$. For each $u \in K_\delta = \{u \mid \|u - x\| \leq \delta \text{ for some } x \in K\}$ choose $x^+[u] = x^+_{\delta,t}[u], x^-[u] = x^-_{\delta,t}[u] \in K$ so that

$$\varphi^+_\delta(u,t) = \varphi(x^+[u], t) = \max_{x \in K, \|x-u\| \leq \delta} \varphi(x,t)$$

and

$$\varphi^-_\delta(u,t) = \varphi(x^-[u], t) = \min_{x \in K, \|x-u\| \leq \delta} \varphi(x,t)$$

and set $\varphi_\delta(u,t) = \frac{1}{2}[\varphi^+_\delta(u,t) + \varphi^-_\delta(u,t)]$. Define $F_\delta : K_\delta \to C([0,1])$ by setting $F_\delta[u](t) = \varphi_\delta(u,t)$ for all $\delta > 0$ and each $t \in [0,1]$. If $u \in K_\delta$ and $x \in K$ with $\|u - x\| \leq \delta$, then $|\varphi(x,t) - \varphi_\delta(u,t)| \leq \omega(2\delta)/2$ for all $t \in [0,1]$, and hence it follows that $\|F(x) - F_\delta(u)\| \leq \frac{1}{2}\omega(2\delta)$. However, F_δ may not be continuous. Therefore for fixed $t \in [0,1]$ and each pair of positive real numbers λ and μ we define

$$\varphi_{\lambda,\mu}(u,t) = \frac{1}{2\mu} \int_{[\lambda, \lambda+\mu]} [\varphi^+_\xi(u,t) + \varphi^-_\xi(u,t)] d\xi$$

and $F_{\lambda,\mu} : K_\lambda \to C([0,1])$ by setting $F_{\lambda,\mu}[u](t) = \varphi_{\lambda,\mu}(u,t)$ for all $t \in [0,1]$. If $\|u - v\| < \rho$, then it can be shown that

$$\|F_{\lambda,\mu}[u] - F_{\lambda,\mu}[v]\| \leq \frac{2\rho F_K}{\mu}$$

where $F_K = \max_{x \in K} \|F(x)\|$. This shows that the operator $F_{\lambda,\mu}$ is continuous. If $x \in K$ and $\|x - u\| < \lambda$, then it follows that $\|F(x) - F_{\lambda,\mu}(u)\| \leq \frac{1}{2}\omega(2\nu)$ where $\nu = \lambda + \mu$. To prove the desired result we take $\tau > 0$ and choose $\epsilon > 0$ so that $\omega(2\delta+\epsilon) \leq \omega(2\delta)+\tau$ for all $\delta > 0$. Now we set $\lambda = \delta+\epsilon/2$ and $\mu = \epsilon/2$ and note that if $\|x-u\| \leq \lambda$, then $\|F(x)-F_{\lambda,\mu}(u)\| \leq \frac{1}{2}\omega(2\delta)+\frac{\tau}{2}$. Let $0 = t_0 < \cdots < t_N = 1$ be a partition of the interval $[0,1]$, and define the operator $P_N \in \mathcal{L}(C([0,1]), PL([0,1]))$, where $PL([0,1]) \subseteq C([0,1])$ is the subspace of piecewise linear functions defined by setting $P_N[x](t_k) = x(t_k)$ for each $k = 0, \ldots, N$ with the partition sufficiently fine to ensure that $\|x - P_N(x)\| \leq \epsilon/4$ for all $x \in K$. Let L_δ denote the closure of the set $P_N(K_\delta)$. Since L_δ lies in an $(N+1)$-dimensional subspace and is bounded and closed, it follows that L_δ is compact. It can be shown that $L_\delta \subseteq K_\lambda$, and hence $F_{\lambda,\mu}$ is well defined on L_δ. By Theorem 4.1 for all $v \in L_\delta$ there exists an operator $S_{\lambda,\mu} : X \to C(T)$ in the form $S_{\lambda,\mu} = WZQG_m^\star$ such that $\|F_{\lambda,\mu}(v) - S_{\lambda,\mu}(v)\| \leq \frac{\tau}{2}$. We can now define the operator $S : X \to C(T)$ in the form $S = WZQG_m$, where $G_m = G_m^\star P_N$, by the equality $S(u) = S_{\lambda,\mu}(P_N[u])$ for each $u \in K_\delta$. □

References

[1] B. Russell, *On the notion of cause*, Proc. Aristotelian Soc. **13** (1913), 1–25.

[2] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, reprint of the 1934 original, Amer. Math. Soc. Colloq. Publ. **19**, Providence, RI, 1987. MR **98a:**01023

[3] Y. Foures and I. E. Segal, *Causality and analyticity*, Trans. Amer. Math. Soc. **78** (1955), 385–405. MR **16:**1032d

[4] P. L. Falb and M. I. Freedman, *A generalized transform theory for causal operators*, SIAM J. Control **7** (1969), 452–471. MR **58:**33192a

[5] J. C. Willems, *Stability, instability, invertibility and causality*, SIAM J. Control **7** no. 4 (1969), 645–671. MR **43:**1689

[6] I. Z. Gohberg and M. G. Kreĭn, *Theory and Applications of Volterra Operators in Hilbert Space*, Transl. of Math. Monographs, Vol. 24, Amer. Math. Soc., Providence, RI, 1970. MR **41:**9041

[7] I. W. Sandberg and L. Xu, *Uniform approximation of multidimensional myopic maps*, IEEE Trans. on Circuits and Systems–1: Fund. Theory and Appl. **44** no. 6 (1997), 477–485. MR **97m:**93024

[8] A. Torokhti and P. G. Howlett, *On the constructive approximation of nonlinear operators in the modelling of dynamical systems*, J. Austral. Math. Soc. Ser. B. **39** (1997), 1–27. MR **98c:**41039

[9] P. G. Howlett and A. Torokhti, *A methodology for the numerical representation of nonlinear operators defined on noncompact sets*, Numer. Funct. Anal. and Optimiz. **18**, no. 3-4 (1997), 343–365. MR **98i:**47065

[10] P. G. Howlett and A. P. Torokhti, *Weak interpolation and approximation of nonlinear operators on the space* $\mathcal{C}([0,1])$, Numer. Funct. Anal. and Optimiz. **19**, no. 9-10 (1998), 1025–1043. MR MR **99j:**41003

[11] P. M. Prenter, *A Weierstrass theorem for real, separable Hilbert spaces*, J. Approx. Theory **3** (1970), 341–351. MR **55:**6193

[12] V. J. Bruno, *A Weierstrass approximation theorem for topological vector spaces*, J. Approx. Theory **42** (1984), 1–3. MR **85j:**41047

[13] G. Cybenko, *Approximation by superpositions of a sigmoidal function*, Math. Control Signals Systems **2** (1989), 303–314. MR **90m:**41033.

[14] I. K. Daugavet, *On operator approximation by causal operators and their generalizations. II: Nonlinear case (Russian)*, Methods of Optimiz. and their Applic., Irkutsk. Sib. Energ. Institut (1988), 166–178.

[15] A. Torokhti and P. Howlett, *On the Best Quadratic Approximation of Nonlinear Systems*, IEEE Trans. on Circuits and Systems. Part I, Fundamental theory and applications, **48**, No. 5 (2001), 595–602. MR **2003e:**93015

[16] A. Torokhti and P. Howlett, *Optimal Fixed Rank Transform of the Second Degree*, IEEE Trans. on Circuits and Systems. Part II, Analog and Digital Signal Processing, **48**, No. 3 (2001), 309–315.

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