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SPECTRALLY BOUNDED OPERATORS ON SIMPLE C^*-ALGEBRAS

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Abstract. A linear mapping T from a subspace E of a Banach algebra into another Banach algebra is called spectrally bounded if there is a constant $M \geq 0$ such that $r(Tx) \leq M\,r(x)$ for all $x \in E$, where $r(\,\cdot\,)$ denotes the spectral radius. We prove that every spectrally bounded unital operator from a unital purely infinite simple C^*-algebra onto a unital semisimple Banach algebra is a Jordan epimorphism.

1. Introduction and main result

A simple C^*-algebra A is said to be *purely infinite* if every nonzero hereditary C^*-subalgebra is infinite. In particular, every nonzero projection has to be infinite. Zhang showed in [13] that every purely infinite simple C^*-algebra has real rank zero, that is, the selfadjoint elements with finite spectrum are dense in the selfadjoint part of the C^*-algebra. Whether the converse holds, i.e., an infinite simple C^*-algebra with real rank zero has to be purely infinite, remains an open problem. By Rørdam's example [12], the assumption of real rank zero is crucial here.

Let A be a unital C^*-algebra, and let B be a unital semisimple Banach algebra. A linear mapping $T\colon A \to B$ is called a *Jordan epimorphism* if it is surjective and $T(x^2) = (Tx)^2$ for all $x \in A$. It is well known that every Jordan epimorphism is unital, that is, $T1 = 1$, and preserves invertibility. Denoting by $r(x)$ the spectral radius of an element x, it thus follows that T satisfies the estimate $r(Tx) \leq r(x)$ for all $x \in A$. More generally, let $E \subseteq A$ be a subspace of A. Then $T\colon E \to B$ is said to be *spectrally bounded* if there exists a constant $M \geq 0$ such that $r(Tx) \leq M\,r(x)$ for all $x \in A$. This concept has proven to be very useful in automatic continuity theory; see, for instance, [1]. A number of basic properties of spectrally bounded operators are established in [8].

The following spectral characterization of Jordan epimorphisms was obtained in [9, Theorem 3.6].

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Theorem A. *Let $T\colon A \to B$ be a unital surjective spectrally bounded operator from a properly infinite von Neumann algebra A onto a unital semisimple Banach algebra B. Then T is a Jordan epimorphism.*

The hypothesis on the domain turns out to be essential, since on a commutative von Neumann algebra *every* bounded operator is spectrally bounded; hence the theorem is bound to fail in the finite case.

In the present note we take a first step to extend Theorem A appropriately to the setting of C^*-algebras. Since the center is not easy to control, we restrict our attention to simple C^*-algebras. Since an abundance of projections is needed (see Lemma 1 below), we assume that the C^*-algebra has real rank zero. Also, since the existence of traces provides an obstruction, we confine ourselves to infinite C^*-algebras. That is why, in this paper, the framework of unital purely infinite simple C^*-algebras is chosen.

With this background in mind, we will prove the following theorem, which is the exact analogue of Theorem A.

Theorem B. *Let $T\colon A \to B$ be a unital surjective spectrally bounded operator from a unital purely infinite simple C^*-algebra A onto a unital semisimple Banach algebra B. Then T is a Jordan epimorphism.*

2. Proof of the main result

The first lemma will allow us to do the final step in the proof of Theorem B. It was obtained for von Neumann algebras in [9, Lemma 2.1]. Since the argument relies only on the fact that every selfadjoint element can be approximated by finite linear combinations of orthogonal projections, and this holds in every C^*-algebra of real rank zero by [2], the proof takes over without problems and is hence omitted here.

Lemma 1. *Let $T\colon A \to B$ be a bounded linear operator from a C^*-algebra A of real rank zero into a Banach algebra B sending projections in A to idempotents in B. Then T is a Jordan homomorphism.*

The next lemma is a special case of [9, Lemma 3.1]. From a purely spectral hypothesis we derive a strong algebraic property of the operator T.

Lemma 2. *Let $T\colon A \to B$ be a spectrally bounded operator from a C^*-algebra A onto a semisimple Banach algebra B. Suppose that $x \in A$ satisfies $x^2 = 0$. Then $(Tx)^2 = 0$.*

In order to put Lemma 2 into action we need a good supply with elements of square zero in the domain. This is provided by the following result.

Lemma 3. *Let A be a unital purely infinite simple C^*-algebra, and let p be a nonzero projection in A. Then each element in pAp is a finite sum of elements in pAp with square zero.*

Proof. Since every infinite projection in A is properly infinite by [3, Proposition 2.2], there exist orthogonal subprojections p_1, p_2 of p such that $p_1 \sim p_2 \sim p$. Thus [5, Theorem 2.1] entails that every element in pAp is the sum of 10 commutators (see also [11]). Put $p_3 = p - (p_1 + p_2)$ and note that p_3 is a proper projection in pAp and orthogonal to p_1 and p_2. Since $p - p_3$ is infinite and A is simple, it follows that $p_3 \precsim p - p_3$ [4, Lemma V.5.4], and it is clear that $p_i \precsim p - p_i$ for $i = 1, 2$.

Therefore, the assumptions of [6, Theorem 3.5] are satisfied, and it follows that every commutator in pAp is the sum of 13 elements in pAp of square zero. As a result, for each $x \in pAp$, there are at most 130 elements in pAp with square zero whose sum is x. □

To complete the preparations for the proof of Theorem B, we note the following well-known automatic continuity result, which is a direct consequence of, e.g., [1, Lemma A].

Lemma 4. *Let $T\colon A \to B$ be a spectrally bounded operator from a C^*-algebra A onto a semisimple Banach algebra B. Then T is bounded.*

Proof of Theorem B. By [13], A has real rank zero; so in view of Lemmas 1 and 4 we only need to show that T maps every projection in A onto an idempotent in B.

Let p be a nonzero projection in A such that $q = 1 - p$ is nonzero as well. If $a \in pAp$ and $b \in qAq$, by Lemma 3, there are finitely many $a_i \in pAp$, $b_j \in qAq$ such that $a = \sum_i a_i$, $b = \sum_j b_j$, and $a_i^2 = b_j^2 = 0$ for all i, j. We claim that

$$(Ta)(Tb) + (Tb)(Ta) = 0. \tag{1}$$

Since $(a_i + b_j)^2 = 0$ for all i, j, Lemma 2 entails that $(T(a_i + b_j))^2 = 0$ for all i, j. On the other hand,

$$(T(a_i+b_j))^2 = (Ta_i)^2+(Ta_i)(Tb_j)+(Tb_j)(Ta_i)+(Tb_j)^2 = (Ta_i)(Tb_j)+(Tb_j)(Ta_i),$$

wherefore $(Ta_i)(Tb_j) + (Tb_j)(Ta_i) = 0$ for all i, j. Summing over all indices yields the claim.

Applying (1) to $a = p$ and $b = 1 - p$ yields

$$2\,(Tp - (Tp)^2) = (Tp)(1 - Tp) + (1 - Tp)(Tp) = 0,$$

since $T1 = 1$. Consequently, Tp is idempotent. □

3. An outlook

It appears that the main obstruction to an extension of Theorem B to finite (simple) C^*-algebras is the existence of bounded traces. Indeed, if A is a finite-dimensional simple C^*-algebra, every surjective unital spectrally bounded operator from A into itself is a linear combination of the canonical trace on A and a Jordan automorphism of A. In [7] we showed that a unital spectrally bounded operator from a unital C^*-algebra into its center necessarily is a bounded trace (though possibly not positive). Therefore it seems to be conceivable that a splitting of a spectrally bounded operator whose values lie in a C^*-algebra that has a nonzero finite trace into a superposition of a bounded trace and a Jordan homomorphism is possible. We have not been able to establish this yet.

Call a linear mapping T between C^*-algebras a *spectral isometry* if $r(Tx) = r(x)$ for all x in the domain. In [8] we raised the following question: *Is every unital surjective spectral isometry between unital C^*-algebras necessarily a Jordan isomorphism?* We summarize the results known until now including the consequences of Theorems A and B above in the following result.

Theorem C. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C^*-algebras A and B. Then T is a Jordan isomorphism provided A or B belong to one of the following classes of C^*-algebras:*

(1) *properly infinite von Neumann algebras;*
(2) *unital purely infinite simple C^*-algebras;*
(3) *commutative C^*-algebras;*
(4) *finite-dimensional C^*-algebras.*

Proof. Note at first that T is injective, by [8, Proposition 4.2]. Therefore, $T^{-1}\colon B \to A$ is a unital spectral isometry as well. Statements (1) and (2) thus follow by applying Theorem A and Theorem B, respectively, to T or to T^{-1}.

Suppose that A is commutative. By [8, Corollary 4.4], T is an algebra isomorphism from A onto $Z(B)$, the center of B. Let $S = T \circ T^{-1}$; this is a unital bijective spectral isometry from B onto $Z(B)$. By [7, Lemma 2.1], $S(xy) = S(yx)$ for all $x, y \in B$. Since S is injective, B is commutative and the result follows. The other case in statement (3) is obtained by considering T^{-1} instead of T.

Statement (4) is a consequence of joint work with A. R. Sourour, which appears in [10]. □

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