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THE ENERGY OF SIGNED MEASURES

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ABSTRACT. We generalize the concept of energy to complex measures of finite variation. We show that although the energy dimension of a measure can exceed that of its total variation, it is always less than the Hausdorff dimension of the measure. As an application we prove a variant of the uncertainty principle.

1. INTRODUCTION

The Riesz energy of a finite, positive measure on $\mathbb{R}^d$ is defined as

$$I_t(\mu) = \int \int |x-y|^{-t} d\mu(y) d\mu(x)$$

and is an important concept which has found many interesting applications (cf. [1], [10], [11] and [12]). The finiteness versus non-finiteness of the energy determines the energy dimension of the measure

$$\dim_e(\mu) = \sup\{t : I_t(\mu) < \infty\}.$$

In this article we extend the definition of the energy dimension to complex measures and give an application to a variation of the uncertainty principle.

Of course, it is natural to attempt to extend the definition by using the linearity of the integral and the decomposition of the complex measure as a linear combination of (four) finite, positive measures. This approach was successfully studied by J. Doob in [2, XIII] for complex measures that were linear combinations of positive measures of finite energy. However, this natural approach can fail when (some of) these positive measures have infinite energy. Moreover, even if the natural extension is well defined, it is not obvious that this energy integral will be real, much less positive.

We introduce a modification of the energy integral which is defined for *all* finite, complex measures and is always positive. The exponent at which our modified energy formula changes from finite to infinite coincides with the energy dimension for positive measures and hence provides a natural generalization of the definition of energy dimension. A different approach to this problem, valid for the one-dimensional torus, can be found in [8].

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Any finite measure and its total variation will have the same Hausdorff dimension; this is not true for their energy dimensions. In section 3 we present an example of a signed measure on the one-dimensional torus that has energy dimension one, but whose total variation measure has energy dimension zero.

However, as is the case for positive measures, the Hausdorff dimension of any measure is at least as great as its energy dimension. This is established in section 4 and improves upon a classical result of J. P. Kahane and R. Salem [8].

There is a Fourier transform formula for our energy integral (as is known for positive measures), and this allows us to show that if two measures are concentrated on sets of small Hausdorff dimension and the difference of their Fourier transforms belongs to a (suitable) weighted l^2-space, then the two measures coincide. The precise statement and proof of this variant of the uncertainty principle can also be found in section 4. For further discussion and other illustrations of this principle the reader is referred to [6].

The main results of this article are obtained by using harmonic analysis techniques and are valid for both $\mathbb{R}^d$ and the d-dimensional torus, $\mathbb{T}^d$. In each case we give the proof for only one of these cases; the extension to the other is an exercise which can be done using the methods of [7].

2. Definition of general energy

By a measure we mean a complex, regular, Borel measure of finite variation on $\mathbb{R}^d$ or $\mathbb{T}^d$.

2.1. Measures on $\mathbb{R}^d$.

For a measure μ on $\mathbb{R}^d$ let us define

$$I^*_{t,\varphi_\tau}(\mu) = \int\int (|.|^{-t} * \varphi_\tau(x-y))d\mu(x)d\overline{\mu(y)} \tag{2.1}$$

where $\varphi_\tau(x) = \tau^{-d}\varphi(x/\tau)$ is an approximation of the identity based on a positive, C^∞ function φ, supported on the unit ball, with positive Fourier transform. Since $|\cdot|^{-t} * \varphi_\tau$ is a continuous, bounded function, the integrals $I^*_{t,\varphi_\tau}(\mu)$ are well defined. Applying Parseval's formula gives the identity

$$I^*_{t,\varphi_\tau}(\mu) = c_{t,d}\int |\xi|^{t-d}|\widehat{\mu}(\xi)|^2\widehat{\varphi_\tau}(\xi)d\xi.$$

Since the integrand is positive and $\widehat{\varphi_\tau}$ converges from below to 1, the limit of $I^*_{t,\varphi_\tau}(\mu)$ as τ tends to zero exists and is independent of the choice of φ. Consequently, we can make the following definitions.

Definition 2.1. Define the *general energy of order t* of a measure μ on $\mathbb{R}^d$ by

$$I^*_t(\mu) \equiv \lim_{\tau\to 0} I^*_{t,\varphi_\tau}(\mu) = c_{t,d}\int |\xi|^{t-d}|\widehat{\mu}(\xi)|^2 d\xi.$$

Notice that the general energy of any nonzero measure is positive.

Definition 2.2. Define the *energy dimension* of a measure μ on $\mathbb{R}^d$ by

$$\dim_e(\mu) = \sup\{t < d : I^*_t(\mu) < \infty\} = \sup\{t < d : \int |\xi|^{t-d}|\widehat{\mu}(\xi)|^2 d\xi < \infty\}.$$

Since the t-energy of a positive measure satisfies the same Fourier transform formula,

$$I_t(\mu) = c_{t,d}\int |\xi|^{t-d}|\widehat{\mu}(\xi)|^2 d\xi$$

(cf. [9, p. 162]), our definitions of the general energy and energy dimension coincide with the classical definitions of the Riesz energy and energy dimension when the measure is positive.

We also remark that if $\mu = \nu_1 + i\nu_2$ where ν_1 and ν_2 are real-valued measures, then because $\widehat{\nu}(\xi) = \overline{\widehat{\nu}(-\xi)}$ for any real-valued measure ν it follows that

$$|\widehat{\mu}(\xi)|^2 + |\widehat{\mu}(-\xi)|^2 = 2|\widehat{\nu_1}(\xi)|^2 + 2|\widehat{\nu_2}(\xi)|^2.$$

Thus

$$\begin{aligned} I_t^*(\mu) &= c_{t,d}\int |\xi|^{t-d}|\widehat{\mu}(\xi)|^2 d\xi = c_{t,d}\int |\xi|^{t-d}\frac{1}{2}(|\widehat{\mu}(\xi)|^2 + |\widehat{\mu}(-\xi)|^2)d\xi \\ &= c_{t,d}\int |\xi|^{t-d}\left(|\widehat{\nu_1}(\xi)|^2 + |\widehat{\nu_2}(\xi)|^2\right)d\xi = I_t^*(\mathrm{Re}(\mu)) + I_t^*(\mathrm{Im}(\mu)). \end{aligned}$$

2.2. **Measures on $\mathbb{T}^d$.** In [7] it was shown that for a positive measure μ on the d-dimensional torus the classical energy integral given by

$$I_t(\mu) = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \mathrm{dist}(x,y)^{-t} d\mu(x)d\mu(y)$$

(where $\mathrm{dist}(\cdot,\cdot)$ denotes the usual metric on the torus) is comparable to

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{t-d}\,|\widehat{\mu}(n)|^2 + |\widehat{\mu}(0)|^2\,.$$

This was done by establishing the existence of a function F_t defined on the torus, which is positive, integrable, satisfies $\widehat{F_t}(n) \sim |n|^{t-d}$ for $n \neq 0$, is comparable to $|x|^{-t}$ near the origin and has the property that for positive measures μ,

$$I_t(\mu) \sim \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} F_t(x-y)d\mu(x)d\mu(y).$$

Motivated by this, for a complex measure μ on the d-dimensional torus we define

$$I_{t,\varphi_\tau}^*(\mu) = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} F_t * \varphi_\tau(x-y)d\mu(x)d\overline{\mu(y)}$$

where F_t is the function found in [7] and φ_τ is as in the previous section. Since $F_t * \varphi_\tau$ is a continuous function, Parseval's theorem implies that for all complex measures μ,

$$I_{t,\varphi_\tau}^*(\mu) \sim \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{t-d}\,\widehat{\varphi_\tau}(n)\,|\widehat{\mu}(n)|^2 + \widehat{F_t}(0)\,\|\mu\|^2\,.$$

Since $\widehat{\varphi_\tau}(n)$ tends to 1 from below, the finiteness/non-finiteness of $\limsup_\tau I_{t,\varphi_\tau}^*(\mu)$ is independent of the choice of φ (and F_t), and is determined, as in the classical case for $I_t(\mu)$, by the finiteness/non-finiteness of

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{t-d}\,|\widehat{\mu}(n)|^2\,.$$

Thus we can similarly generalize the energy dimension to complex measures on $\mathbb{T}^d$ by defining

$$\begin{aligned} \dim_e(\mu) &= \sup\{t<d : \limsup_\tau I_{t,\varphi_\tau}^*(\mu) < \infty\} \\ &= \sup\{t<d : \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{t-d}\,|\widehat{\mu}(n)|^2 < \infty\} \end{aligned}$$

where φ can be any function as described in 2.1.

As with measures on $\mathbb{R}^d$, the energy dimension of a measure is the minimum of the energy dimensions of its real and imaginary parts.

It is shown in [7, Sec. 3.3] that any positive measure on $\mathbb{T}^d$ can be lifted to a measure on $\mathbb{R}^d$ with the same Hausdorff and energy dimensions. Similar arguments apply to complex measures.

3. The counterexample

Since $I^*_{t,\varphi_\tau}(\mu) \leq I^*_{t,\varphi_\tau}(|\mu|)$, the energy dimension of μ is always at least as great as the energy dimension of $|\mu|$. The next example shows that the energy dimension of μ can be strictly larger.

Although our example of a measure μ with $\dim_e \mu = 1$ and $\dim_e |\mu| = 0$ is on the torus $\mathbb{T}$, the corresponding example on $\mathbb{R}$ can be easily obtained by the lifting method mentioned above. By taking the product of our measure on $\mathbb{T}$ with Lebesgue measure on $\mathbb{T}^{d-1}$ one can obtain a similar example on $\mathbb{T}^d$.

The measure μ we construct was motivated by an example given in [4] and will be of the form

$$\mu = \sum_{m=1}^{\infty} 2^{-m}(f_m\mu_m - f_m\lambda_{\mathbb{T}})$$

where f_m are positive, trigonometric polynomials, μ_m are Riesz product measures that are singular and mutually singular, and $\lambda_{\mathbb{T}}$ is Lebesgue measure on the torus.

We begin by choosing a suitable Féjer kernel f_m, for each $m = 1, 2, \ldots$, so that $\sum_{n\neq 0} \left|\widehat{f_m}(n)\right|^2 |n|^{-1} > 8^m$. Assume that $\operatorname{supp}\widehat{f_m} = \{-N_m, \ldots, N_m\}$.

Choose disjoint infinite subsets Φ_m of $\{3^{3^n}\}_{n=1}^{\infty}$ such that for each m the minimum element of Φ_m exceeds $2N_m$. We let μ_m be the Riesz product based on Φ_m and the constant sequence $1/(2N_m + 1)$, i.e.,

$$\mu_m = \prod_{n\in\Phi_m} \left(1 + \frac{2\cos nx}{2N_m + 1}\right).$$

Such measures are known to be singular and mutually singular (cf. [5, 7.2.1]).

(a) *Correctness.* We start by checking that the sum defining the measure μ is convergent.

Given $\Phi \subseteq \mathbb{N}$ we let $\Omega(\Phi)$ denote the set of words

$$\left\{\sum_{j=1}^{N} \varepsilon_j n_j : n_j \in \Phi, \varepsilon_j = 0, \pm 1 \text{ and } N \in \mathbb{N}\right\}.$$

Since $\Phi_m \subset \{3^{3^n}\}_{n=1}^{\infty}$ and the minimal element of Φ_m is more than $2N_m$,

$$\Omega(\Phi_m) \cap (\Omega(\Phi_m) + \{-N_m, \ldots, -1, 1, \ldots, N_m\})$$

is empty.

This ensures that for each integer k and m there is at most one choice of j with $\widehat{f_m}(k-j)\ \widehat{\mu_m}(j) \neq 0$. Thus if $\widehat{f_m\mu_m}(k) \neq 0$, then there is a unique choice of j with

$$\widehat{f_m\mu_m}(k) = \sum_i \widehat{f_m}(k-i)\widehat{\mu_m}(i) = \widehat{f_m}(k-j)\widehat{\mu_m}(j) \neq 0.$$

In particular, if $k \in \operatorname{supp}\widehat{f_m}$, then $\widehat{f_m\mu_m}(k) = \widehat{f_m}(k)\widehat{\mu_m}(0) = \widehat{f_m}(k)$.

Of course, this implies that the measure norm of the positive measure, $f_m\mu_m$, is $\widehat{f_m\mu_m}(0) = \widehat{f_m}(0) = 1$. Since the measure norm of $f_m\lambda_{\mathbb{T}}$ is 1 as well (being the L^1 norm of the Féjer kernel f_m) $\mu = \sum 2^{-m}(f_m(\mu_m - \lambda_{\mathbb{T}}))$ is a finite measure.

(b) $\dim_e(|\mu|) = 0$. The singularity and mutual singularity of the measures μ_m imply that $|\mu|$ is equal to $\sum 2^{-m}(f_m(\mu_m + \lambda_{\mathbb{T}}))$.

Notice that if ν and ν' are positive measures, then $I_t(\nu + \nu') \geq I_t(\nu)$. Thus for any $0 < t < 1$, the Fourier transform formula for energy implies that

$$I_t(|\mu|) \geq I_t(2^{-m} f_m \lambda_{\mathbb{T}}) \geq c2^{-2m} \sum_{n \neq 0} \left| \widehat{f_m}(n) \right|^2 |n|^{t-1} \geq c2^m.$$

Hence the energy dimension of $|\mu|$ is zero.

(c) $\dim_e(\mu) = 1$. Fix $\varepsilon > 0$. We have already observed that if $k \in \operatorname{supp}\widehat{f_m}$, then $\widehat{f_m\mu_m}(k) = \widehat{f_m}(k)$, and if $k \notin \operatorname{supp}\widehat{f_m}$, then there is a unique $j = j_k$ with $\widehat{f_m\mu_m}(k) = \widehat{f_m}(k - j_k)\widehat{\mu_m}(j_k)$. Thus

$$\sum_{k \neq 0} |k|^{-\varepsilon} \left| \Big(f_m(\mu_m - \lambda_{\mathbb{T}}) \Big)^{\wedge}(k) \right|^2 = \sum_{k \notin \operatorname{supp}\widehat{f_m}} |k|^{-\varepsilon} \left| \widehat{f_m}(k - j_k)\widehat{\mu_m}(j_k) \right|^2. \tag{3.1}$$

Since $k - j_k \in \operatorname{supp}\widehat{f_m}$ and $j_k \in \Omega(\Phi_m)$ for any nonzero term in the sum above, the definition of $\Omega(\Phi_m)$ ensures that $|k| \geq |j_k|/2$. Since any $j \in \Omega(\Phi_m)$ can occur as j_k for at most $\left|\operatorname{supp}\widehat{f_m}\right| = 2N_m + 1$ choices of k, it follows that (3.1) is bounded by

$$(2N_m + 1) \sum_{j \in \Omega(\Phi_m) \setminus \{0\}} |j/2|^{-\varepsilon} \left|\widehat{\mu_m}(j)\right|^2.$$

Assume that $\Phi_m = \{n_j^{(m)}\}_{j=1}^{\infty}$. The structure of the Riesz product shows that the expression above is majorized by

$$4(2N_m + 1) \sum_j \left| \frac{n_j^{(m)}}{4} \right|^{-\varepsilon} \left| \widehat{\mu_m}(n_j^{(m)}) \right|^2 \prod_{k=1}^{j-1} \left(1 + 2 \left| \widehat{\mu_m}(n_k^{(m)}) \right|^2 \right).$$

Since $\widehat{\mu_m}(n_k^{(m)}) = (2N_m + 1)^{-1}$ and $n_j^{(m)} \geq 3^{3^j}$, this is easily seen to be bounded by a constant $C(\varepsilon)$ that is independent of m.

To complete the argument we use the elementary inequality $\left|\sum a_j\right|^2 \leq \sum 2^j |a_j|^2$. Thus

$$\begin{aligned}
\sum_{k \neq 0} |k|^{-\varepsilon} |\widehat{\mu}(k)|^2 &= \sum_{k \neq 0} |k|^{-\varepsilon} \left| \sum_{m=1}^{\infty} 2^{-m} \Big(f_m(\mu_m - \lambda_{\mathbb{T}}) \Big)^{\wedge}(k) \right|^2 \\
&\leq \sum_{k \neq 0} |k|^{-\varepsilon} \sum_{m=1}^{\infty} 2^m 2^{-2m} \left| \Big(f_m(\mu_m - \lambda_{\mathbb{T}}) \Big)^{\wedge}(k) \right|^2 \\
&\leq \sum_{m=1}^{\infty} 2^{-m} C(\varepsilon) < \infty.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the energy dimension of μ is one.

4. Energy and Hausdorff dimension

The Hausdorff dimension of a complex measure μ, defined as

$$\dim_H(\mu) = \inf\{\dim_H(E) : \mu(E) \neq 0\},$$

coincides with the Hausdorff dimension of its total variation. It is known that for a nonzero, positive measure μ, $\dim_H \mu \geq \dim_e \mu$ ([3, 4.3]). In this section we show that this relationship is true for complex measures as well. The proof is presented for the $\mathbb{R}^d$ case. The technique of 3.3 in [7] again allows us to obtain the same result for measures on $\mathbb{T}^d$.

One can compare our result with a classical result of Kahane and Salem ([8, III.V]) which states that the support of a distribution on the circle has Hausdorff dimension not less than the energy dimension of the distribution. Theorem 4.1 improves this classical result for the case of measures, even on the one-dimensional torus, since the Hausdorff dimension of the support of a measure is always at least as great as the Hausdorff dimension of the measure.

Theorem 4.1. *Suppose μ is a nonzero measure on $\mathbb{R}^d$ and $I_t^*\mu < \infty$. Then $\dim_H(\mu) \geq t$.*

For the proof we need the following lemmas. We use the notation $B(\xi,\tau)$ to denote the closed ball of radius τ, centered at ξ.

Lemma 4.2. *Let μ_1 and μ_2 be two finite, positive, mutually singular measures in $\mathbb{R}^d$. For any constants $C, c, \varepsilon > 0$ there exists a Borel set $K = K(C, c, \varepsilon)$ such that $\mu_1(\mathbb{R}^d \setminus K) < \varepsilon$ and $c\mu_1(B(\xi,\tau)) \geq C\mu_2(B(\xi,\tau))$ for all $\xi \in K$ and $\tau \leq \rho = \rho(\varepsilon)$.*

Proof. Since the measures are mutually singular, we can choose two disjoint sets A_1 and A_2 such that $\mu_j(\mathbb{R}^d \setminus A_j) = 0$ for $j = 1, 2$. Choose two compact sets K_1 and K_2 such that $K_j \subset A_j$ and $\mu_j(\mathbb{R}^d \setminus K_j) < c'\varepsilon$, where the constant c' depends on c, C and d and will be specified later. Let $\rho = \frac{1}{2}\operatorname{dist}(K_1, K_2)$.

Let us denote by K' the Borel set

$$K' = \{\xi \in K_1 : c\mu_1(B(\xi,\tau)) < C\mu_2(B(\xi,\tau)) \text{ for some } \tau \leq \rho\}.$$

We wish to estimate $\mu_1(K')$. By definition, for each point $x \in K'$ there exists a ball B_x centered at x that does not intersect K_2 and for which $c\mu_1(B_x) < C\mu_2(B_x)$. By the Besicovitch covering theorem we can choose a covering $\{B_k\}$ of K' by such balls, with the property that each point of K' belongs to at most $b(d)$ balls. Then,

$$\mu_1(K') \leq \sum \mu_1(B_k) \leq \sum c^{-1}C\mu_2(B_k) \leq c^{-1}Cb(d)\mu_2(\bigcup B_k).$$

Since the balls B_k are disjoint from K_2, we obtain

$$\mu_1(K') \leq c^{-1}Cb(d)\mu_2(\mathbb{R}^d \setminus K_2) < c^{-1}Cb(d)c'\varepsilon.$$

If we choose $c' = \min(1/2, c/(2Cb(d)))$, then the set $K = K_1 \setminus K'$ satisfies the required conditions. □

Lemma 4.3. *Let φ be a positive function supported by the unit ball $B(0,1)$ and having positive Fourier transform. There exist constants $A, B > 0$ such that the functions $\psi_\tau = |.|^{-t} * \varphi_\tau$ satisfy the estimates*

$$A\psi_\tau(x) < \min\{|x|^{-t}, \tau^{-t}\} < B\psi_\tau(x)$$

for all τ.

Proof. First, we prove that the constants A and B exist for $\tau = 1$ ($\varphi_1 = \varphi$), i.e., we want to show

$$A|.|^{-t} * \varphi(x) \leq \min\{|x|^{-t}, 1\} \leq B|.|^{-t} * \varphi(x)$$

for all $x \in \mathbb{R}^n$.

We consider two cases: $|x| \leq 2$ and $|x| > 2$. It is enough to find that suitable constants exist for each case separately.

Case $|x| \leq 2$. Then $2^{-t} \leq \min\{|x|^{-t}, 1\} \leq 1$. The function $|.|^{-t} * \varphi(x)$ is continuous being a convolution of a test function and a locally summable one, and is strictly positive being the convolution of two positive functions, one of which is strictly positive. Hence on the compact set $B(0, 2)$, $|.|^{-t} * \varphi(x)$ is bounded above and below from zero by, say, C and c respectively. We can take $A = C^{-1}2^{-t}$ and $B = c^{-1}$.

Case $|x| > 2$. Here we will use the fact that $\min\{|x|^{-t}, 1\} = |x|^{-t}$. Since φ is supported on the unit ball

$$|.|^{-t} * \varphi(x) = \int_{\mathbb{R}^n} |x - y|^{-t}\varphi(y)dy = \int_{B(0,1)} |x - y|^{-t}\varphi(y)dy.$$

When $|x| > 2$, then $\frac{1}{2}|x| < |x - y| < 2|x|$ for any $y \in B(0, 1)$, and since $\int \varphi = 1$,

$$2^{-t}|x|^{-t} \leq |.|^{-t} * \varphi(x) \leq 2^t|x|^{-t}.$$

Thus we can choose $A = 2^{-t}$ and $B = 2^t$.

The claim can be proved for arbitrary τ by noting that

$$|.|^{-t} * \varphi_\tau(x) = \tau^{-t}|.|^{-t} * \varphi(\frac{x}{\tau})$$

and

$$\min\{|x|^{-t}, \tau^{-t}\} = \tau^{-t} \min\{|\frac{x}{\tau}|^{-t}, 1\}.$$

□

Proof of the Theorem. First, observe that there is no loss of generality in assuming that the measure μ is real-valued.

It is enough to prove that under the given condition $|\mu|$ can be approximated in the strong sense by positive measures of finite t-energy (note that an approximation in the weak sense is not enough). Let us decompose the measure as $\mu_+ - \mu_-$, where μ_+ and μ_- are two positive, mutually singular measures. We can assume that μ_+, μ_- are both nonzero measures; for otherwise we can use the classical result. We will use Lemma 4.2 to prove that an approximation exists for μ_+. The approximation for μ_- can be constructed in the same way and together they give the approximation for $|\mu|$.

Let $\psi_\tau = |.|^{-t} * \varphi_\tau$ where φ is as in section 2.1 and let $K = K(C, c, \varepsilon)$ be the set given by Lemma 4.2 for $c = B^{-1}/2, C = 2A^{-1}$ (A, B as in Lemma 4.3) and arbitrary $\varepsilon > 0$. Let $\rho = \rho(\varepsilon)$. Denote $\mu_+|_K$ by μ_ρ and let $\mu_s = \mu_+ - \mu_\rho$.

Since $I_t^*(\mu)<\infty$, for some fixed number M we have

$$\begin{aligned} M &\geq \int \psi_\tau(x-y)d\mu(x)d\mu(y) \\ &= \int \psi_\tau(x-y)\left(d\mu_+(x)d\mu_+(y)+d\mu_-(x)d\mu_-(y)-2d\mu_+(x)d\mu_-(y)\right) \\ &\geq \int \psi_\tau(x-y)\left(d\mu_\rho(x)d\mu_+(y)-2d\mu_\rho(x)d\mu_-(y)\right) \\ &\quad + \int \psi_\tau(x-y)\left(d\mu_s(x)d\mu_s(y)+d\mu_-(x)d\mu_-(y)-2d\mu_s(x)d\mu_-(y)\right). \end{aligned}$$

Observe that the final integral in the expression above is positive since by Parseval's formula it equals

$$c_{t,d}\int |\xi|^{t-d}|(\mu_s-\mu_-)^\wedge(\xi)|^2\widehat{\varphi_\tau}(\xi)d\xi.$$

Thus

$$\begin{aligned} M &\geq \int \psi_\tau(x-y)\left(d\mu_\rho(x)d\mu_+(y)-2d\mu_\rho(x)d\mu_-(y)\right) \\ &= \int\limits_{|x-y|\geq\rho} \psi_\tau(x-y)((d\mu_+(y)-2d\mu_-(y))d\mu_\rho(x)) \\ &\quad + \int\limits_{|x-y|<\rho} \psi_\tau(x-y)\left(d\mu_+(y)-2d\mu_-(y)\right)d\mu_\rho(x). \end{aligned}$$

Lemma 4.3 implies that for $\tau\leq\rho$ the integral over the region $\{(x,y):|x-y|\geq\rho\}$ dominates

$$\int\limits_{|x-y|\geq\rho} \psi_\tau(x-y)d\mu_+(y)d\mu_\rho(x)-2A^{-1}\|\mu_-\|\|\mu_\rho\|\rho^{-t}.$$

The integral over $\{(x,y):|x-y|<\rho\}$ can be estimated from below as follows: Lemma 4.3 again shows that

$$\begin{aligned} \int\limits_{|x-y|<\rho} \psi_\tau(x-y)(d\mu_+(y)-2d\mu_-(y)) &\geq \int\limits_{|x-y|<\rho} B^{-1}\min\{|x-y|^{-t},\tau^{-t}\}d\mu_+(y) \\ &\quad - \int\limits_{|x-y|<\rho} 2A^{-1}\min\{|x-y|^{-t},\tau^{-t}\}d\mu_-(y), \end{aligned}$$

which after passing to polar coordinates and integrating simplifies to

$$\begin{aligned} &B^{-1}\left(\rho^{-t}\mu_+(B(x,\rho))+t\int_\tau^\rho \omega^{-t-1}\mu_+(B(x,\omega))d\omega\right) \\ &-2A^{-1}\left(\rho^{-t}\mu_-(B(x,\rho))+t\int_\tau^\rho \omega^{-t-1}\mu_-(B(x,\omega))d\omega\right). \end{aligned} \tag{4.1}$$

But the choice of K and ρ ensures that

$$\frac{B^{-1}}{2}\mu_+(B(x,\omega))\geq 2A^{-1}\mu_-(B(x,\omega))$$

for any $\omega \leq \rho$ and $x \in K$. Thus for μ_ρ a.e. x expression (4.1) dominates

$$\begin{aligned}
& \frac{1}{2}B^{-1}\left(\rho^{-t}\mu_+(B(x,\rho)) + t\int_\tau^\rho \omega^{-t-1}\mu_+(B(x,\omega))d\omega\right) \\
= \; & \frac{1}{2}B^{-1}\int_{|x-y|<\rho} \min\{|x-y|^{-t}, \tau^{-t}\}d\mu_+(y) \\
\geq \; & \frac{1}{2}AB^{-1}\int_{|x-y|<\rho} \psi_\tau(x-y)d\mu_+(y).
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \int\int_{|x-y|<\rho} \psi_\tau(x-y)\left(d\mu_+(y) - 2d\mu_-(y)\right)d\mu_\rho(x) \\
\geq \; & \frac{1}{2}AB^{-1}\int\int_{|x-y|<\rho} \psi_\tau(x-y)d\mu_+(y)d\mu_\rho(x).
\end{aligned}$$

Since all the estimates are independent of τ and $d\mu_+ \geq d\mu_\rho$, these arguments imply

$$M \geq \min(1, \tfrac{1}{2}AB^{-1})I_t^*(\mu_\rho) - 2A^{-1}\|\mu_-\|\|\mu_\rho\|\rho^{-t},$$

and therefore $I_t^*(\mu_\rho) < \infty$. To conclude, note that by construction $\mu_\rho \to \mu_+$ in the strong sense as $\varepsilon \to 0$. $\square$

The Fourier transform formula for the energy dimension and an application of Hölder's inequality gives the following corollary, which was previously obtained for positive measures in [7].

Corollary 4.4. *If μ is a nonzero measure on $\mathbb{T}^d$ and $\widehat{\mu} \in l^p(\mathbb{Z}^d)$ for some $p > 2$, then* $\dim_H \mu \geq 2d/p$.

Remark 4.5. The capacity dimension of a Borel set A is defined as

$$\dim_c(A) = \sup\{t : \exists \mu \in M_+(A) \text{ such that } I_t(\mu) < \infty\}$$

and is known to equal the Hausdorff dimension of A ([10, 8.9] or [3, 4.3]). In [8, p. 40] Kahane and Salem showed that for a compact subset of $\mathbb{T}$ to have capacity dimension of order at least α it is sufficient for the set to support a distribution of finite α-energy. Our result shows that, in fact, for any Borel set A we have

$$\dim_c(A) = \sup\{t : \exists \mu \in M(\mathbb{R}^n) \text{ with } |\mu|(A) \neq 0 \text{ and } I_t(\mu) < \infty\}.$$

Theorem 4.1 also allows us to prove the variant of the uncertainty principle mentioned in the introduction.

Proposition 4.6. *If two measures μ_1 and μ_2 are concentrated on sets of Hausdorff dimension less than t and*

$$\sum_{n \in \mathbb{Z}^d \setminus \{0\}} |n|^{t-d}|\widehat{\mu_1}(n) - \widehat{\mu_2}(n)|^2 < \infty,$$

then the two measures coincide.

Proof. The assumption on the Fourier transforms implies that the energy dimension of $\mu_1 - \mu_2$ is at least t. Hence if $\mu_1 - \mu_2$ is nonzero, then its Hausdorff dimension is at least t. But since μ_1 and μ_2 are concentrated on sets of Hausdorff dimension less than t, so is their difference and this is clearly a contradiction. □

Remark 4.7. For measures on $\mathbb{R}^d$ the corresponding result states:

If measures μ_1 and μ_2 are concentrated on sets of Hausdorff dimension less than t and

$$\int |\xi|^{t-d} \left|\widehat{\mu_1}(\xi) - \widehat{\mu_2}(\xi)\right|^2 < \infty,$$

then the two measures coincide.

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