$2S_4 * Q_8$ -EXTENSIONS IN CHARACTERISTIC 3

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ABSTRACT. We present an explicit construction of the complete family of Galois extensions of a field K of characteristic 3 with Galois group the central product $2S_4 * Q_8$ of a double cover $2S_4$ of the symmetric group S_4 and the quaternion group Q_8 , containing a given S_4 -extension of the field K .

Let $2S_4 * Q_8$ be the central product of a double cover $2S_4$ of the symmetric group S_4 and the quaternion group Q_8 . In [2], we gave a method of construction of $2S_4 * Q_8$ -extensions of a field K of characteristic different from 2. In this paper we examine more closely the characteristic 3 case. Given an S_4 -extension L_1 of a field K containing the finite field $\mathbb{F}_9$ of 9 elements, we give explicitly all $2S_4 * Q_8$ -extensions of K containing L_1 . In the case when $K = k((Y_1, Y_2, Y_3))$, the quotient field of the formal power series ring $R = k[[Y_1, Y_2, Y_3]]$ in 3 variables over an algebraically closed field k of characteristic 3, the determination of all $2S_4 * Q_8$ -extensions of K is interesting in the framework of Abhyankar's Normal Crossings Local Conjecture (see [1]). As mentioned by Abhyankar in [1], the groups $2S_4 * Q_8$ are examples of groups G such that the quotient $G/3(G)$ of G by the subgroup $3(G)$ generated by all of its 3-Sylow subgroups is abelian, generated by 3 generators, but $3(G)$ does not have an abelian supplement in G , i.e., an abelian subgroup of G generating G together with $3(G)$. By a result of Harbater et al. (see [4]), this property makes them the "simplest" groups not appearing as Galois groups of a Galois extension L of K such that RY_1, RY_2, RY_3 are the only height-one primes in R that are possibly ramified in L . We note that Harbater and Lefcourt (see [6]) have shown that every finite group can be obtained as a ramified extension of $k((Y_1, Y_2, Y_3))$. The results in this paper provide an explicit construction of $2S_4 * Q_8$ -extensions in the characteristic 3 case. As a first step towards the determination of the ramification locus of the corresponding coverings, we give a formula for the discriminant.

Let us first recall the definitions and fix notation. We denote by $2S_n$ one of the two double covers of the symmetric group S_n reducing to the nontrivial double cover $2A_n$ of the alternating group A_n and by Q_8 the quaternion group, which is a double cover of the Klein group V_4 . The group $2S_4 * Q_8$ is the central product of $2S_4$ and Q_8 . Let $L_1|K$ be a Galois extension with Galois group the symmetric group S_4 , for K a field of characteristic different from 2. We assume that L_1 is given as the splitting field of a polynomial $P(X) \in K[X]$ of degree 4. We want to determine when L_1 is embeddable in a Galois extension of K with Galois group

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$2S_4 * Q_8$. This fact is equivalent to the existence of a Galois extension $L_2|K$ with Galois group V_4 , disjoint from L_1 , and such that, if L is the compositum of L_1 and L_2 , the Galois embedding problem

$$(1) \quad 2S_4 * Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K)$$

is solvable. We recall that a solution to this embedding problem is a quadratic extension field $\tilde{L}$ of the field L , which is a Galois extension of K with Galois group $2S_4 * Q_8$ and such that the restriction epimorphism between the Galois groups $\text{Gal}(\tilde{L}|K) \rightarrow \text{Gal}(L|K)$ agrees with the given epimorphism $2S_4 * Q_8 \rightarrow S_4 \times V_4$. If $\tilde{L} = L(\sqrt{\gamma})$ is a solution, then the general solution is $L(\sqrt{r\gamma})$, $r \in K^*$. Given a Galois extension $L_1|K$ with Galois group S_4 , in order to obtain all $2S_4 * Q_8$ -extensions of K containing L_1 , we have to determine all V_4 -extensions L_2 of K , disjoint from L_1 , and such that the embedding problem (1) is solvable.

Let us now specify notation by writing 2^+S_n or 2^-S_n depending on whether transpositions in S_n lift in the double cover to involutions or to elements of order 4. Let $E = K[X]/(P(X))$, for $P(X)$ the polynomial of degree 4 realizing L_1 , let Q_E denote the trace form of the extension $E|K$, i.e., $Q_E(x) = \text{Tr}_{E|K}(x^2)$, and let d be the discriminant of the polynomial $P(X)$. Let $L_2 = K(\sqrt{a}, \sqrt{b})$. We denote by w the Hasse-Witt invariant of a quadratic form and by $(\cdot, \cdot)$ a Hilbert symbol. In [2], we saw that the embedding problem $2^\pm S_4 * Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K)$ is solvable if and only if

$$(2) \quad w(Q_E) = (\pm 2, d) \cdot (a, b) \cdot (-1, ab) \text{ in } H^2(G_K, \{\pm 1\}).$$

From now on, we assume that K is a field of characteristic 3, containing $\mathbb{F}_9$. We write $P(X) = X^4 + s_2X^2 - s_3X + s_4$. By computation of the trace form Q_E , we obtain

$$w(Q_E) = (ds_2, (s_2^2 - s_4)s_4).$$

The equivalent condition (2) for the solvability of the embedding problem (1) then turns into

$$(3) \quad (ds_2, (s_2^2 - s_4)s_4) = (a, b),$$

that is, the equality of two Hilbert symbols.

Now, taking into account that K contains $\mathbb{F}_9$, the equality of Hilbert symbols (3) holds if and only if the two quadratic forms $\langle ds_2, ms_4, dms_2s_4 \rangle$ (where $m = s_2^2 - s_4$) and $\langle a, b, ab \rangle$ are K -equivalent (see [5, 3.2]).

Remark 1. We note that if $s_2 = 0$ or $s_2^2 - s_4 = 0$, the embedding problem $2S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$ is solvable. In this case, L_1 is embeddable in a $2S_4 * Q_8$ -extension of K if and only if there exists a V_4 -extension $L_2|K$ disjoint with L_1 and such that the embedding problem $Q_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$ is solvable. Any $2S_4 * Q_8$ -extension of K containing L_1 is then obtained as $L(\sqrt{\alpha\beta})$, for $L = L_1.L_2$, $L_1(\sqrt{\alpha})$ a solution to $2S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$ and $L_2(\sqrt{\beta})$ a solution to $Q_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$, with L_2 a V_4 -extension of K with the conditions above. The element α can be computed explicitly by the method obtained in [3], the element β is given by the formula of Witt given in [7].

Theorem 1. *Let K be a field of characteristic 3 and containing $\mathbb{F}_9$. Let $P(X) = X^4 + s_2X^2 - s_3X + s_4 \in K[X]$ be a polynomial with discriminant d and Galois group S_4 , and let L_1 be its splitting field. Let $A \in \text{GL}(3, K)$ be such that $D = A^t \langle ds_2, ms_4, dms_2s_4 \rangle A$ is a diagonal matrix of the form $\langle a, b, ab \rangle$ and such that*

$L_2 = K(\sqrt{a}, \sqrt{b})$ is a V_4 -extension of K , disjoint from L_1 . Let us choose the matrix A such that $\det A = ab/(dms_2s_4)$. Let L be the compositum of L_1 and L_2 . Let M be the matrix with entries in L given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{ad}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{bd}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{abd}} \end{pmatrix} \cdot \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

where (x_1, x_2, x_3, x_4) are the four roots of the polynomial $P(X)$ in L_1 . Let $B \in \text{GL}(4, K)$ be the matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Let P be the matrix in $\text{GL}(4, K)$ given by

$$P = \frac{1}{m} \begin{pmatrix} m & 0 & -s_2d + s_3^2s_2^4 & -s_2^3s_3d \\ 0 & dm & (s_2^2 + s_4)ms_2s_3 & (s_2^2 + s_4)dm \\ 0 & 0 & d - s_3^2s_2^3 & s_2^2s_3d \\ 0 & 0 & s_3s_2^2m & s_2dm \end{pmatrix}.$$

Let γ be the element in L^* given by

$$\gamma = \det(MPB + I).$$

Then the general solution to the embedding problem $2^-S_4 * Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K)$ is $L(\sqrt{r\gamma_1})$ for $r \in K^*$ and

$$\gamma_1 = (1 + \sqrt{a})^2(1 + \sqrt{b})^2(1 + \sqrt{d})^2\gamma.$$

The general solution to the embedding problem $2^+S_4 * Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K)$ is $L(\sqrt{r\gamma_2})$ for $r \in K^*$ and

$$\gamma_2 = \sqrt{d}\gamma_1.$$

Proof. We can check that the matrix P satisfies $P^tTP = \langle 1, ds_2, ms_4, dms_2s_4 \rangle$, where T is the matrix of the quadratic trace form Q_E in the basis $(1, x_1, x_1^2, x_1^3)$ and apply [[2], theorems 1 and 2]. In particular, the choice of the sign of the determinant of the matrix A assures that the element γ is nonzero. $\square$

Remark 2. For each of the two extensions $L(\sqrt{\gamma_i})|K$, $i = 1, 2$, with γ_1, γ_2 given by the theorem, we consider the discriminant Δ_i of the basis $\{x_1^{i_1}x_2^{i_2}x_3^{i_3}a^{\epsilon_1}b^{\epsilon_2}\gamma_i^{\epsilon_3} : 0 \leq i_1 \leq 3, 0 \leq i_2 \leq 2, i_3 = 0, 1, \epsilon_1, \epsilon_2, \epsilon_3 = 0, 1/2\}$, where x_1, x_2, x_3 are three distinct roots of the polynomial $P(X)$ realizing S_4 . We have

$$\Delta_1 = (dab)^{96}((1-a)(1-b)(1-d))^{24}N_{L|K}(\gamma) ; \quad \Delta_2 = d^{12}\Delta_1.$$

Corollary 1. Let K and L_1 be as in the theorem and assume that the extension $L_2 = K(\sqrt{a}, \sqrt{b})|K$, with $a = ds_2, b = ms_4$, has Galois group V_4 and is disjoint from L_1 . Then the element γ in the theorem corresponding to this field L_2 is given in terms of the coefficients s_2, s_3, s_4 of the polynomial $P(X)$ realising the extension $L_1|K$, of three distinct roots x_1, x_2, x_3 of this polynomial $P(X)$ and the square roots

of the elements a and b , by

$$\begin{aligned}
\gamma = & m^2 s_2 s_4 \sqrt{a} x_1^3 x_2^2 + 2m^2 s_2 s_4 \sqrt{a} x_1^3 x_2 x_3 + (2m s_2 d \sqrt{b} + (2s_3 s_4 m \\
& + 2s_3 m^2) \sqrt{ab}) x_1^3 x_2 + (s_4^2 m^2 \sqrt{a} + (2s_4 m^3 s_3 + ds_3 s_4 + s_4^2 s_2 s_3^3 + m^2 s_3^3 s_2 \\
& + ds_3 m + 2s_3 s_4^2 m^2 + 2m s_4 s_2 s_3^3) \sqrt{b} + (2s_4 m^2 + d + m s_2 s_3^2 \\
& + s_4 s_2 s_3^2) \sqrt{ab}) x_1^3 + (ds_2 + 2s_4 m s_3^2 + s_3^2 m^2 + s_3^2 s_4^2) \sqrt{a} x_1^2 x_2^2 x_3 + ((2s_3 m^2 \\
& + 2s_3 s_4 m) \sqrt{ab}) x_1^2 x_2^2 + (m s_2 d \sqrt{b} + (2s_3 m^2 + 2s_3 s_4 m) \sqrt{ab}) x_1^2 x_2 x_3 \\
& + ((2dm + 2s_4 m^3 + 2m^2 s_3^2 s_2 + 2s_4^2 m^2 + 2m s_4 s_2 s_3^2) \sqrt{a} + (2ds_3 m \\
& + 2ds_3 s_4) \sqrt{b}) x_1^2 x_2 + ((2s_2 s_4^2 s_3^2 + 2m s_4 s_2 s_3^2 + 2s_4^2 m^2 + 2ds_4) \sqrt{a} \\
& + (s_3 s_4^2 m^2 + s_4 m^3 s_3 + ds_3 s_4 + 2m^2 s_3^3 s_2 + 2s_4^2 s_2 s_3^3 + ds_3 m + m s_4 s_2 s_3^3) \sqrt{b} \\
& + (s_4 m^2 + m s_2 s_3^2 + s_4 s_2 s_3^2) \sqrt{ab}) x_1^2 x_3 + (2m^2 s_2 s_4 s_3 + s_4 m s_3^3 + 2s_3^3 m^2 \\
& + 2ds_2 s_3 + 2s_4^2 s_3^3) \sqrt{a} x_1^2 + (m s_2 d \sqrt{b} + (2s_3 m^2 + 2s_3 s_4 m) \sqrt{ab}) x_1 x_2^2 x_3 \\
& + ((2m s_4 s_2 s_3^2 + 2s_4^2 m^2 + 2m^2 s_3^2 s_2 + 2dm) \sqrt{a} + (2ds_3 m + s_3 s_4^2 m^2 \\
& + 2s_4^2 s_2 s_3^3 + 2m^2 s_3^3 s_2 + 2ds_3 s_4 + m s_4 s_2 s_3^3 + s_4 m^3 s_3) \sqrt{b} \\
& + (s_4 s_2 s_3^2 + m s_2 s_3^2 + s_4 m^2) \sqrt{ab}) x_1 x_2^2 \\
& + ((m^2 s_3^2 s_2 + s_4^2 m^2 + dm + m s_4 s_2 s_3^2) \sqrt{a} + (2m s_4 s_2 s_3^3 + m^2 s_3^3 s_2 \\
& + 2s_3 s_4^2 m^2 + s_4^2 s_2 s_3^3 + 2s_4 m^3 s_3) \sqrt{b} + (2d + 2s_4 m^2) \sqrt{ab}) x_1 x_2 x_3 \\
& + ((2s_4^2 s_3^3 + 2s_3^3 m^2 + 2ds_2 s_3 + s_4 m s_3^3) \sqrt{a} + (2ds_4 m + 2dm^2) \sqrt{b}) x_1 x_2 \\
& + ((s_3^3 m^2 + m^2 s_2 s_4 s_3 + 2s_4 m s_3^3 + ds_2 s_3 + s_4^2 s_3^3) \sqrt{a}) x_1 x_3 \\
& + ((2m^2 s_2 s_4^2) \sqrt{a} + (2m s_2 ds_3 + 2m^3 s_3 s_2 s_4 + 2m^2 s_2 s_4^2 s_3 + m^3 s_3^3 \\
& + s_4^3 s_3^3) \sqrt{b} + (s_4^2 s_3^2 + ds_2 + 2s_3^2 m^2 + 2m^2 s_2 s_4) \sqrt{ab}) x_1 + ((2ds_4 \\
& + 2m s_4 s_2 s_3^2 + 2s_2 s_4^2 s_3^2) \sqrt{a} + (2ds_3 m + 2ds_3 s_4) \sqrt{b} + (2m s_2 s_3^2 + d \\
& + 2s_4 s_2 s_3^2) \sqrt{ab}) x_2^2 x_3 + (2ds_2 s_3 + 2s_3^3 m^2 + 2s_4^2 s_3^3 + s_4 m s_3^3) \sqrt{a} x_2^2 \\
& + (2s_4 m s_3^3 + s_3^3 m^2 + ds_2 s_3 + s_4^2 s_3^3) \sqrt{a} x_2 x_3 + (m^2 s_2 s_4^2 \sqrt{a} + (ds_4 s_2 s_3 \\
& + m^3 s_3^3 + 2m^3 s_3 s_2 s_4 + 2m^2 s_2 s_4^2 s_3 + s_4^3 s_3^3) \sqrt{b} + (2m^2 s_2 s_4 + 2ds_2 \\
& + s_3^2 m^2 + s_4 m s_3^2) \sqrt{ab}) x_2 + ((s_3^2 s_4^3 + 2s_3^2 s_4^2 m + m^2 s_2 s_4^2 + s_4 m^2 s_3^2 \\
& + s_2 s_4 d) \sqrt{a} + (2s_4^3 s_3^3 + 2m^3 s_3^3 + m s_2 ds_3 + m^3 s_3 s_2 s_4 + m^2 s_2 s_4^2 s_3) \sqrt{b} \\
& + (m^2 s_2 s_4 + ds_2 + 2s_4 m s_3^2 + 2s_3^2 m^2) \sqrt{ab}) x_3 + m^2 s_2 s_4 d \\
& + (s_2 s_4^2 s_3^3 + ds_4 s_3 + m s_4 s_2 s_3^3) \sqrt{a} + (ds_3^2 s_4 + ds_3^2 m + m s_4 s_2 d) \sqrt{b} \\
& + (2ds_3 + m s_2 s_3^3 + s_2 s_3^3 s_4) \sqrt{ab}.
\end{aligned}$$

Proof. We apply the theorem with the matrix B equal to the identity matrix. The computation of the corresponding determinant is simplified by applying the Cramer identities. The stated expression for the element γ is obtained by using the symmetric functions in the four roots x_1, x_2, x_3, x_4 of the polynomial $P(X)$. The computations are carried out with Maple. $\square$

Example 1. The polynomial $P(X) = X^4 + (Y_1 + Y_2)X^2 + Y_2X + (Y_1 + Y_3)$ has Galois group S_4 over the field $K = \overline{\mathbb{F}}_3((Y_1, Y_2, Y_3))$. Let L_1 be the splitting field of $P(X)$. We write $s_2 = Y_1 + Y_2, s_3 = -Y_2, s_4 = Y_1 + Y_3$ and consider the elements $a = ds_2, b = (s_2^2 - s_4)s_4$. To assure that the extension $L_2 = K(\sqrt{a}, \sqrt{b})|K$ is a V_4 -extension disjoint from L_1 we check that the elements a, b, ab, s_2, db, s_2b are not squares in K . To this end we use the following lemma, which is a direct consequence of the Weierstrass preparation theorem (see e.g. [8]).

Lemma 1. *Let $F \in R = \overline{\mathbb{F}}_3[[Y_1, Y_2, Y_3]]$ be regular and polynomial in Y_i for some $i = 1, 2, 3$. If the discriminant $\Delta(F, Y_i)$ is nonzero, then F is not a square in K .*

Let L be the composition field of L_1 and L_2 . By applying the corollary, we obtain extensions $L(\sqrt{\gamma_1})|K$ and $L(\sqrt{\gamma_2})|K$ with Galois groups $2^-S_4 * Q_8$ and $2^+S_4 * Q_8$, respectively, with an explicit expression for the elements γ_1 and γ_2 .

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