ON RAKHMANOV'S THEOREM FOR JACOBI MATRICES

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(Communicated by Andreas Seeger)

ABSTRACT. We prove Rakhmanov's theorem for Jacobi matrices without the additional assumption that the number of bound states is finite. This result solves one of Nevai's open problems.

Consider a measure $d\mu$ with bounded support in $\mathbb{R}$. Assume that it has infinitely many growth points. Let $\{p_k\}$ ($k = 0, 1, \dots$) be the system of polynomials orthonormal with respect to that measure, i.e.,

$$\int_{-\infty}^{\infty} p_k(x)p_m(x)d\mu(x) = \delta_{k,m}, \quad \deg p_l(x) = l.$$

It is known [1] that the following recurrence relations hold:

$$(1) \quad \begin{aligned} b_{-1}p_0(x) &= 1, & v_0p_0(x) + b_0p_1(x) &= xp_0(x), \\ b_{k-1}p_{k-1}(x) + v_kp_k(x) + b_kp_{k+1}(x) &= xp_k(x), & k &= 1, 2, \dots, \end{aligned}$$

with $b_k > 0$, $k = -1, 0, \dots$, and v_k real. Thus, for any fixed x , p_k are generalized eigenvectors of the following Jacobi matrix [1]:

$$(2) \quad J = \begin{bmatrix} v_0 & b_0 & 0 & 0 & \dots \\ b_0 & v_1 & b_1 & 0 & \dots \\ 0 & b_1 & v_2 & b_2 & \dots \\ 0 & 0 & b_2 & v_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

The corresponding operator J is bounded and self-adjoint in the Hilbert space $\ell^2(\mathbb{Z}^+)$. And, vice-versa, we can associate to any real Jacobi matrix with positive off-diagonal entries a certain measure on the line (spectral measure). In a general situation, a limit circle case might occur such that this measure is non-unique. But in the limit point case, this measure is uniquely defined. If the corresponding self-adjoint operator J is bounded, then the measure has bounded support. In this paper, we consider only that case. We introduce the following notation: $\sigma(A)$ denotes the spectrum of the self-adjoint bounded operator A , $\sigma_{ess}(A)$ its essential spectrum, and $\|A\|$ the norm of the operator. The following theorem was proved by Nevai.

Received by the editors October 16, 2002 and, in revised form, November 8, 2002.
 2000 *Mathematics Subject Classification*. Primary 47B36.

Theorem 1 (Nevai [7], [6]). *If $\sigma_{\text{ess}}(J) = [-1, 1]$, where J has a finite number of eigenvalues outside the segment $[-1, 1]$ and, for the corresponding spectral measure μ , $\mu'(x) > 0$ for a.e. $x \in [-1, 1]$, then $v_k \rightarrow 0$, $b_k \rightarrow 1/2$ as $k \rightarrow \infty$.*

The following open question was posed in [6] (Conjecture 2.7):

Is the statement of the theorem true without the second assumption on J , i.e., for the case when one has an infinite number of eigenvalues (bound states) outside the segment $[-1, 1]$?

The main result of our short note is the affirmative answer to this question. To prove that, we will need one auxiliary lemma.

Lemma 1. *Consider a bounded self-adjoint operator A in the Hilbert space, such that $\sigma_{\text{ess}}(A) = [a, b]$. Let $\{e_j\}, j = 1, 2, \dots$, be the orthonormal basis, and A_n the self-adjoint bounded operator, generated by the quadratic form (Af, f) , where f is orthogonal to $\{e_1, \dots, e_n\}$. Then $\inf \sigma(A_n) \rightarrow a$, and $\sup \sigma(A_n) \rightarrow b$ as $n \rightarrow \infty$.*

Proof. We will only show that $\inf \sigma(A_n) \rightarrow a$ as $n \rightarrow \infty$. The other statement can be proved similarly. Let λ_j, ψ_j ($j = 1, 2, \dots$) denote eigenvalues and the corresponding normed eigenfunctions, such that $\lambda_1 \leq \lambda_2 \leq \dots < a, \|\psi_j\| = 1$. Consider any vector f from the Hilbert space. By the spectral theorem, $f = \sum_{j=1}^{\infty} (f, \psi_j) \psi_j + f_c$ and

$$(3) \quad \begin{aligned} (Af, f) &\geq a(f_c, f_c) + \sum_{j=1}^{\infty} \lambda_j |(f, \psi_j)|^2 = a \left[(f_c, f_c) + \sum_{j=1}^{\infty} |(f, \psi_j)|^2 \right] \\ &\quad + \sum_{j=1}^{\infty} (\lambda_j - a) |(f, \psi_j)|^2 = a \|f\|^2 + \sum_{j=1}^{\infty} (\lambda_j - a) |(f, \psi_j)|^2. \end{aligned}$$

Notice that for fixed j , $(f, \psi_j) \rightarrow 0$ uniformly in $f \in \{\|f\| \leq 1\}$ and f is orthogonal to $\{e_1, \dots, e_n\}$ as $n \rightarrow \infty$. Indeed,

$$(f, \psi_j) = \sum_{k=1}^{\infty} (f, e_k) \overline{(\psi_j, e_k)} = \sum_{k=n+1}^{\infty} (f, e_k) \overline{(\psi_j, e_k)}.$$

Consequently, by the Cauchy inequality,

$$|(f, \psi_j)| \leq \|f\| \left(\sum_{k=n+1}^{\infty} |(\psi_j, e_k)|^2 \right)^{1/2} \rightarrow 0$$

as $n \rightarrow \infty$. By the variational principle,

$$\inf \sigma(A_n) = \inf_{\|f\|=1, (f, e_j)=0, j=1, \dots, n} (Af, f).$$

By the Courant-Hilbert variational principle [11], $\inf \sigma(A_n) \leq \lambda_{n+1}$. We know that $\lambda_j \rightarrow a$ as $j \rightarrow \infty$. Consequently, from (3), we have $\inf \sigma(A_n) \rightarrow a$ as $n \rightarrow \infty$. $\square$

Let ψ ($x \in [-1, 1], \psi(-1) = 0$) be a non-decreasing bounded function with an infinite number of growth points. Consider the system of polynomials p_k ($k = 0, 1, \dots$) orthonormal with respect to the measure $d\psi$ on the segment $[-1, 1]$. Introduce the function

$$(4) \quad \sigma(\theta) = \begin{cases} -\psi(\cos \theta), & 0 \leq \theta \leq \pi, \\ \psi(\cos \theta), & \pi \leq \theta \leq 2\pi, \end{cases}$$

which is bounded and non-decreasing on $[0, 2\pi]$. Consider the polynomials $\phi_k(z) = \alpha_k z^k + \dots$ ($\alpha_k > 0$) orthonormal on the circle with respect to the measure $d\sigma$, i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} \phi_k(z) \overline{\phi_m(z)} d\sigma(\theta) = \delta_{k,m}, \quad z = e^{i\theta}.$$

ϕ_n are related to p_k by the formula

$$(5) \quad p_k(x) = \frac{\phi_{2k}(z) + \phi_{2k}^*(z)}{\sqrt{2\pi [1 + \alpha_{2k}^{-1} \phi_{2k}(0)]}} z^{-k}, \quad k = 0, 1, \dots,$$

where $x = (z + z^{-1})/2$ and $\phi_k^*(z) = z^k \overline{\phi_k(\bar{z}^{-1})}$ [12], [3]. Calculations yield the following identities [3], [4]:

$$(6) \quad b_k = \frac{1}{2} \sqrt{(1 - a_{2k-1})(1 - a_{2k}^2)(1 + a_{2k+1})},$$

$$(7) \quad v_k = -\frac{1}{2} [a_{2k-2}(1 + a_{2k-1}) - a_{2k}(1 - a_{2k-1})], \quad k \geq 1,$$

where $a_k = -\alpha_{k+1}^{-1} \phi_{k+1}(0)$ are the so-called reflection coefficients for $d\sigma$ (parameters of the system [3]). In our particular case, a_k are real numbers. Notice that multiplication of the function ψ by the positive constant leads to the multiplication of $p_n(x), \sigma(\theta), \phi_n(z)$ by certain constants, but it does not change b_n, v_n, a_n , i.e., parameters of the orthogonal systems.

The following remarkable theorem was first proved by Rakhmanov [10]. Later, various proofs were given by other authors.

Theorem 2 (Rakhmanov [10], [5]). *If $\sigma'(\theta) > 0$ for a.e. $\theta \in [0, 2\pi]$, then $a_k \rightarrow 0$ as $k \rightarrow \infty$.*

We will need the following generalization of this result.

Theorem 3. *If $\mathbb{K} \equiv \{\theta \in [0, 2\pi) : \sigma'(\theta) > 0\}$, then*

$$(8) \quad \limsup_{n \rightarrow \infty} |a_n|^2 \leq 8 \left(1 - \left(\frac{|\mathbb{K}|}{2\pi} \right)^{3/2} \right),$$

where $|\mathbb{K}|$ means the Lebesgue measure of a set $\mathbb{K}$.

The proof can be obtained by a slight modification of the arguments given in [5]. We will also need the following well-known result (see, e.g., [5]).

Lemma 2 ([5]). *If ν is a measure on $\mathbb{R}$ singular with respect to Lebesgue measure, then there exists a sequence of continuous functions $\{h_n\}$ such that $0 < h_n(x) \leq 1$ for all $x \in \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} h_n(x) = 1$$

for a.e. x , and

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} h_n(x) d\nu(x) = 0.$$

Proof of Theorem 3. From Theorem 3 of [8],

$$|a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2} - 1 \right| d\theta, \quad z = e^{i\theta}.$$

The Cauchy inequality yields

$$\left[\int_0^{2\pi} \left| \frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2} - 1 \right| d\theta \right]^2 \leq \int_0^{2\pi} \left(\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|} + 1 \right)^2 d\theta \int_0^{2\pi} \left(\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|} - 1 \right)^2 d\theta.$$

Due to the well-known identity [3]

$$\int_0^{2\pi} \frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2} d\theta = \int_0^{2\pi} |\phi_n(z)|^2 d\sigma(\theta) = 2\pi, \quad z = e^{i\theta},$$

we have an estimate

$$(9) \quad |a_n|^2 \leq 8 \left[1 - \frac{1}{2\pi} \int_0^{2\pi} \frac{|\phi_n(z)|}{|\phi_{n+1}(z)|} d\theta \right].$$

We show that

$$(10) \quad \left(\frac{|\mathbb{K}|}{2\pi} \right)^{3/2} \leq \liminf_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{|\phi_n(z)|}{|\phi_{n+1}(z)|} d\theta.$$

Then, the statement of the theorem will be immediate due to (9). Estimate (10), in turn, can be proved by the argument used in Theorem 3 of [5]. One needs to start with a fixed non-negative, continuous, 2π periodic function $f(x)$. Then, Hölder's inequality yields

$$\begin{aligned} & \left[\frac{1}{2\pi} \int_0^{2\pi} (\sigma'(\theta) f(\theta))^{1/4} d\theta \right]^4 \\ & \leq \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{|\phi_n(z)|}{|\phi_{n+1}(z)|} d\theta \right]^2 \left[\frac{1}{2\pi} \int_0^{2\pi} |\phi_{n+1}(z)|^2 \sigma'(\theta) d\theta \right] \left[\frac{1}{2\pi} \int_0^{2\pi} f(\theta) |\phi_n(z)|^{-2} d\theta \right]. \end{aligned}$$

The second factor is not greater than 1; measure $|\phi_n(\theta)|^{-2} d\theta$ converges weakly to $d\sigma$ [3]. Therefore,

$$(11) \quad \begin{aligned} & \left[\frac{1}{2\pi} \int_0^{2\pi} (\sigma'(\theta) f(\theta))^{1/4} d\theta \right]^4 \\ & \leq \left[\liminf_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{|\phi_n(z)|}{|\phi_{n+1}(z)|} d\theta \right]^2 \left[\frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\sigma \right]. \end{aligned}$$

Assume that σ has a singular component σ_s . Then, by Lemma 2, we can choose the sequence of 2π periodic functions $\{h_k\}$ with the properties $0 < h_k \leq 1$, $h_k(\theta) \rightarrow 1$ a.e.,

$$\int_0^{2\pi} h_k(\theta) d\sigma_s(\theta) \rightarrow 0$$

as $k \rightarrow \infty$. If σ is purely a.c., let $h_k(\theta) \equiv 1$ for all k . Fix any $\varepsilon > 0$, then we can find the sequence of continuous, non-negative, 2π periodic functions $g_l(\theta)$ such

that $g_l \leq \varepsilon^{-1}$, $g_l \rightarrow (\sigma' + \varepsilon)^{-1}$ a.e. For fixed $\varepsilon > 0$, choose f in (11) equal to $h_k g_l$. Taking $k \rightarrow \infty$, then $l \rightarrow \infty$, and, finally, $\varepsilon \rightarrow 0$, we get estimate (10). $\square$

Remark. That is not clear even for small values of $(2\pi - |\mathbb{K}|)$ whether the estimate (8) is sharp or not.

Consider a bounded operator in $\ell^2(\mathbb{Z}^+)$ given by a Jacobi matrix

$$(12) \quad J = \begin{bmatrix} v_0 & b_0 & 0 & 0 & \dots \\ b_0 & v_1 & b_1 & 0 & \dots \\ 0 & b_1 & v_2 & b_2 & \dots \\ 0 & 0 & b_2 & v_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Theorem 4. *Assume that $\sigma_{ess}(J) = [-1, 1]$, and for the corresponding spectral measure $\mu(x)$, we have $\mu'(x) > 0$ for a.e. $x \in (-1, 1)$. Then, $v_k \rightarrow 0$ and $b_k \rightarrow 1/2$ as $k \rightarrow \infty$.*

Proof. Consider the sequence of matrices

$$(13) \quad J_n = \begin{bmatrix} v_n & b_n & 0 & 0 & \dots \\ b_n & v_{n+1} & b_{n+1} & 0 & \dots \\ 0 & b_{n+1} & v_{n+2} & b_{n+2} & \dots \\ 0 & 0 & b_{n+2} & v_{n+3} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

These are matrices of the operators generated by the quadratic forms (Jx, x) , restricted to $x \in \{(x, e_j) = 0, j = 0, 1, \dots, n-1\}$, where $e_j = (0, \dots, 0, 1, 0, \dots)$ is the standard basis in $\ell^2(\mathbb{Z}^+)$. Matrix J can be represented as the rank-two perturbation of the following orthogonal sum $J_n \oplus Q_n$, where Q_n is the finite Jacobi matrix

$$(14) \quad Q_n = \begin{bmatrix} v_0 & b_0 & 0 & \dots & \dots \\ b_0 & v_1 & b_1 & \dots & \dots \\ 0 & b_1 & v_2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{n-2} & v_{n-1} \end{bmatrix}.$$

From Weyl's theorem, we infer that $\sigma_{ess}(J_n) = [-1, 1]$. The support of the a.c. component of the spectral measure is invariant under finite-rank perturbations. So, it is equal to $[-1, 1]$ for any J_n . Due to Lemma 1, we have $\inf \sigma(J_n) \rightarrow -1$, $\sup \sigma(J_n) \rightarrow 1$ as $n \rightarrow \infty$. That means $\|J_n\| \rightarrow 1$. Consider the matrix (operator) $\hat{J}_n = \|J_n\|^{-1} J_n$. Now, the spectral measure $d\psi^{(n)}$ of this operator is supported on the interval $[-1, 1]$. Therefore, we can consider the corresponding system of polynomials orthogonal on the unit circle with measure $d\sigma^{(n)}$, given by (4), and the sequence of reflection parameters $\{a_k^{(n)}\}$. The complement in $[0, 2\pi]$ to the support of the a.c. part of the measure $d\sigma^{(n)}$ is the union of three intervals: $(0, \kappa_1^{(n)})$, $(\pi - \kappa_2^{(n)}, \pi + \kappa_2^{(n)})$ and $(2\pi - \kappa_1^{(n)}, 2\pi)$, where $\kappa_{1(2)}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, Theorem 3 yields $\limsup_{k \rightarrow \infty} |a_k^{(n)}| \rightarrow 0$ as $n \rightarrow \infty$. Bearing in mind formulas (6), (7), and identity $\lim_{n \rightarrow \infty} \|J_n\| = 1$, we have

$$\limsup_{k \rightarrow \infty} |v_k| = 0 \quad \text{and} \quad \limsup_{k \rightarrow \infty} |b_k - 1/2| = 0.$$

$\square$

The idea of a truncation process was suggested by Yuditskii. Meanwhile, this approach is standard and was used before (see, e.g., [9]). The important analogs of Theorem 4 can be obtained for “continuous orthogonal systems” such as Krein systems, Dirac operators, one-dimensional Schrödinger operators. This is done in a separate publication [2].

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