A SYMMETRY COMPLETENESS CRITERION FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. A simple criterion for the completeness of an infinitesimal automorphism of a second-order differential equation is given.

Given a second-order differential equation Y on a connected manifold M , and a vector field X on M , we say that X is an infinitesimal automorphism of Y if $[\bar{X}, Y] = 0$, where $\bar{X}$ is the natural lift of X to TM . It is of interest to know when X has a complete flow, which is to say that X is the infinitesimal generator of a one-parameter group of diffeomorphisms of M , and hence generates a one-parameter group of automorphisms of Y .

Example 1. Consider the second-order equation Y on $M = \mathbb{R}$ with

$$Y = v\partial_x + (1 + 2x)(1 + x^2)^{-1}v^2\partial_v.$$

Then $X = (1 + x^2)\partial_x$ satisfies $[\bar{X}, Y] = 0$, but X is not complete.

In this example, the flow of Y is not complete. However, Y can be incomplete with X complete as in

Example 2. Let g be any left-invariant Lorentz metric on the affine group $G = A(1, \mathbb{R})$. Then the geodesic flow is incomplete, but each left-invariant vector field X on G generates a one-parameter group of symmetries of the geodesic vector field.

In light of these examples, it is perhaps somewhat surprising that the following theorem holds.

Theorem. *Let Y be a second-order differential equation on a manifold M with a complete flow ψ_t . Let X be an infinitesimal automorphism of Y . Then X is necessarily complete; that is, it generates a one-parameter group of diffeomorphisms of M .*

Proof. First observe that in order to prove that X is complete, it suffices to prove the uniform time τ existence for the local flow. Let $\pi : TM \rightarrow M$ be the projection, and let $\phi_t(x)$, $|t| < \tau$ be the local flow of X about $x \in M$. Let $\tilde{\phi}_t(p)$ be the lifted local flow about $p \in TM$ with $\pi(p) = x$. Then $\phi_t \circ \pi = \pi \circ \tilde{\phi}_t$. Since $[\bar{X}, Y] = 0$, $\psi_{t*}\bar{X} = \bar{X}$ for all t . This says that if $\psi_s(p) = q$ for some s , then $\tilde{\phi}_t(q) = \psi_s \circ \tilde{\phi}_t(p)$.

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Hence we may conclude existence of both a $\tau > 0$ and a local flow of $\bar{\phi}_t(q)$ for all time t with $|t| < \tau$, and thus that of the local flow of $\phi_t(\pi(q))$ for the same time $|t| < \tau$ as well.

It follows that we have time τ existence of the local flow of ϕ_t for all points $x' \in M$ such that there is an integral curve of Y that projects to a curve connecting x and x' . Continuing in this manner we may also conclude the same time τ existence for all points x'' that may be so connected to x' .

We will be done if we can show that any point $y \in M$ may be connected to x by a finite chain of such points $x, x', x'', \dots, y$, all with the same time τ existence. For this, it suffices to show that each point $z \in M$ is connected to every other point in some open neighbourhood of z . This suffices because a connected manifold is path connected, and so we can find a smooth curve $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Compactness then allows us to select a finite number of points $x, x', x'', \dots, y$ on the image of γ from the open covering provided by the following lemma to make the connection. This provides the uniform time τ estimate that we require.

Lemma. *Let Y be a second-order differential equation on a manifold M . Then there is an open neighbourhood of $x \in M$ in which every point x' is connected to x by the projection π of an integral curve of Y .*

In other words, for all points x' in some open neighbourhood of x , we may choose an initial velocity v_x at x so that the projection of the integral curve of Y through v_x contains x' . The proof of the lemma follows easily from the fact that the exponential map is onto. $\square$

This proof is patterned after the discussion of completeness for first-order G -structures in [1] and [2]. One should also note that in [3] an infinitesimal automorphism is *defined* to be complete in order to have a pretty statement about the maximal dimension of the symmetry group.

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