ON AN APPROXIMATE AUTOMORPHISM ON A C^* -ALGEBRA

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ABSTRACT. It is shown that for an approximate algebra homomorphism $f : \mathcal{B} \rightarrow \mathcal{B}$ on a Banach $*$ -algebra $\mathcal{B}$, there exists a unique algebra $*$ -homomorphism $H : \mathcal{B} \rightarrow \mathcal{B}$ near the approximate algebra homomorphism. This is applied to show that for an approximate automorphism $f : \mathcal{A} \rightarrow \mathcal{A}$ on a unital C^* -algebra $\mathcal{A}$, there exists a unique automorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ near the approximate automorphism. In fact, we show that the approximate automorphism $f : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism.

1. INTRODUCTION

Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : E_1 \rightarrow E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Rassias [7] showed that there exists a unique $\mathbb{R}$ -linear mapping $T : E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p} \|x\|^p$$

for all $x \in E_1$. Găvruta [2] generalized Rassias' result: Let G be an abelian group and X a Banach space. Denote by $\varphi : G \times G \rightarrow [0, \infty)$ a function such that

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \rightarrow X$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)$$

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for all $x \in G$. Recently, Park [6] applied Găvruta's result to linear functional equations in Banach modules over a C^* -algebra.

B. E. Johnson [3, Theorem 7.2] also investigated almost algebra $*$ -homomorphisms between Banach $*$ -algebras: Suppose that $\mathcal{U}$ and $\mathcal{B}$ are Banach $*$ -algebras that satisfy the conditions of [3, Theorem 3.1]. Then for each positive ϵ and K there is a positive δ such that if $T \in L(\mathcal{U}, \mathcal{B})$ with $\|T\| < K$, $\|T^\vee\| < \delta$ and $\|T(x^*)^* - T(x)\| \leq \delta\|x\|$ ($x \in \mathcal{U}$), then there is a $*$ -homomorphism $T' : \mathcal{U} \rightarrow \mathcal{B}$ with $\|T - T'\| < \epsilon$. Here $L(\mathcal{U}, \mathcal{B})$ is the space of bounded linear maps from $\mathcal{U}$ into $\mathcal{B}$, and $T^\vee(x, y) = T(xy) - T(x)T(y)$ ($x, y \in \mathcal{U}$). See [3] for details.

Throughout this paper, let $\mathcal{B}$ be a Banach $*$ -algebra with norm $\|\cdot\|$, and $\mathcal{B}_{sa}$ the set of selfadjoint elements in $\mathcal{B}$. Let $\mathcal{A}$ be a unital C^* -algebra with norm $\|\cdot\|$, and $\mathcal{U}(\mathcal{A})$ the set of unitary elements in $\mathcal{A}$.

In this paper, we prove that for an approximate algebra homomorphism $f : \mathcal{B} \rightarrow \mathcal{B}$ on a Banach $*$ -algebra $\mathcal{B}$, there exists a unique algebra $*$ -homomorphism $H : \mathcal{B} \rightarrow \mathcal{B}$ near the approximate algebra homomorphism. This is applied to show that for an approximate automorphism $f : \mathcal{A} \rightarrow \mathcal{A}$ on a unital C^* -algebra $\mathcal{A}$, there exists a unique automorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ near the approximate automorphism. In fact, we show that the approximate automorphism $f : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism.

2. STABILITY OF ALGEBRA $*$ -HOMOMORPHISMS ON A BANACH $*$ -ALGEBRA

We are going to show the generalized Hyers-Ulam-Rassias stability of algebra $*$ -homomorphisms on a Banach $*$ -algebra.

Theorem 2.1. *Let $f : \mathcal{B} \rightarrow \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \rightarrow [0, \infty)$ such that*

$$(2.i) \quad \tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty,$$

$$(2.ii) \quad \|D_\mu f(x, y)\| := \|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| \leq \varphi(x, y),$$

$$(2.iii) \quad \|f(x^*) - f(x)^*\| \leq \varphi(x, x),$$

$$(2.iv) \quad \|f(zw) - f(z)f(w)\| \leq \varphi(z, w)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, all $z, w \in \mathcal{B}_{sa}$, and all $x, y \in \mathcal{B}$. Then there exists a unique algebra $*$ -homomorphism $H : \mathcal{B} \rightarrow \mathcal{B}$ such that

$$(2.v) \quad \|f(x) - H(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)$$

for all $x \in \mathcal{B}$.

Proof. Put $\mu = 1 \in \mathbb{T}^1$. It follows from Găvruta's Theorem [2] that there exists a unique additive mapping $H : \mathcal{B} \rightarrow \mathcal{B}$ such that the inequality (2.v) holds. The additive mapping $H : \mathcal{B} \rightarrow \mathcal{B}$ is given by

$$H(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in \mathcal{B}$.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\|f(2^n \mu x) - 2\mu f(2^{n-1} x)\| \leq \varphi(2^{n-1} x, 2^{n-1} x)$$

for all $x \in \mathcal{B}$. Then one can show that

$$\|\mu f(2^n x) - 2\mu f(2^{n-1} x)\| \leq |\mu| \cdot \|f(2^n x) - 2f(2^{n-1} x)\| \leq \varphi(2^{n-1} x, 2^{n-1} x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. So

$$\begin{aligned} \|f(2^n \mu x) - \mu f(2^n x)\| &\leq \|f(2^n \mu x) - 2\mu f(2^{n-1} x)\| + \|2\mu f(2^{n-1} x) - \mu f(2^n x)\| \\ &\leq \varphi(2^{n-1} x, 2^{n-1} x) + \varphi(2^{n-1} x, 2^{n-1} x) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. Thus $2^{-n}\|f(2^n \mu x) - \mu f(2^n x)\| \rightarrow 0$ as $n \rightarrow \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. Hence

$$H(\mu x) = \lim_{n \rightarrow \infty} \frac{f(2^n \mu x)}{2^n} = \lim_{n \rightarrow \infty} \frac{\mu f(2^n x)}{2^n} = \mu H(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and let M be an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [4, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. Also, $H(x) = H(3 \cdot \frac{1}{3}x) = 3H(\frac{1}{3}x)$ for all $x \in \mathcal{B}$. So $H(\frac{1}{3}x) = \frac{1}{3}H(x)$ for all $x \in \mathcal{B}$. Thus

$$\begin{aligned} H(\lambda x) &= H\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x\right) = M \cdot H\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x\right) = \frac{M}{3}H\left(3\frac{\lambda}{M}x\right) \\ &= \frac{M}{3}H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $x \in \mathcal{B}$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$ ($\zeta, \eta \neq 0$) and all $x, y \in \mathcal{B}$, and $H(0x) = 0 = 0H(x)$ for all $x \in \mathcal{B}$. So the unique additive mapping $H : \mathcal{B} \rightarrow \mathcal{B}$ is a $\mathbb{C}$ -linear mapping.

By (2.i) and (2.iii), we get

$$\begin{aligned} H(x^*) &= \lim_{n \rightarrow \infty} \frac{f(2^n x^*)}{2^n} = \lim_{n \rightarrow \infty} \frac{(f(2^n x))^*}{2^n} = \left(\lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}\right)^* \\ &= H(x)^* \end{aligned}$$

for all $x \in \mathcal{B}$.

Now it follows from (2.iv) that

$$(2.1) \quad 4^{-n}\|f(2^n z \cdot 2^n w) - f(2^n z)f(2^n w)\| \leq \frac{1}{4^n}\varphi(2^n z, 2^n w)$$

for all $z, w \in \mathcal{B}_{sa}$; but $\sum_{j=0}^{\infty} 4^{-j}\varphi(2^j z, 2^j w) \leq \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j z, 2^j w)$ for all $z, w \in \mathcal{B}_{sa}$. So by (2.1),

$$\begin{aligned} H(zw) &= \lim_{n \rightarrow \infty} \frac{f(4^n zw)}{4^n} = \lim_{n \rightarrow \infty} \frac{f(2^n z)f(2^n w)}{2^n \cdot 2^n} = \lim_{n \rightarrow \infty} \frac{f(2^n z)}{2^n} \cdot \lim_{n \rightarrow \infty} \frac{f(2^n w)}{2^n} \\ &= H(z)H(w) \end{aligned}$$

for all $z, w \in \mathcal{B}_{sa}$.

For elements $x, y \in \mathcal{B}$, $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$ and $y = \frac{y+y^*}{2} + i\frac{y-y^*}{2i}$, where $x_1 := \frac{x+x^*}{2}$, $x_2 := \frac{x-x^*}{2i}$, $y_1 := \frac{y+y^*}{2}$, and $y_2 := \frac{y-y^*}{2i}$ are selfadjoint. Since H is $\mathbb{C}$ -linear,

$$\begin{aligned} H(xy) &= H(x_1y_1 - x_2y_2 + ix_1y_2 + ix_2y_1) \\ &= H(x_1y_1) - H(x_2y_2) + iH(x_1y_2) + iH(x_2y_1) \\ &= H(x_1)H(y_1) - H(x_2)H(y_2) + iH(x_1)H(y_2) + iH(x_2)H(y_1) \\ &= (H(x_1) + iH(x_2))(H(y_1) + iH(y_2)) \\ &= H(x_1 + ix_2)H(y_1 + iy_2) \\ &= H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{B}$. Hence the additive mapping $H : \mathcal{B} \rightarrow \mathcal{B}$ is an algebra $*$ -homomorphism satisfying the inequality (2.v), as desired. $\square$

Corollary 2.2. *Let $f : \mathcal{B} \rightarrow \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| &\leq \theta(\|x\|^p + \|y\|^p), \\ \|f(x^*) - f(x)^*\| &\leq 2\theta\|x\|^p, \\ \|f(zw) - f(z)f(w)\| &\leq \theta(\|z\|^p + \|w\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $z, w \in \mathcal{B}_{sa}$, and all $x, y \in \mathcal{B}$. Then there exists a unique algebra $*$ -homomorphism $H : \mathcal{B} \rightarrow \mathcal{B}$ such that

$$\|f(x) - H(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$$

for all $x \in \mathcal{B}$.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 2.1. $\square$

Theorem 2.3. *Let $f : \mathcal{B} \rightarrow \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \rightarrow [0, \infty)$ satisfying (2.i), (2.iii), and (2.iv) such that*

$$(2.vi) \quad \|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| \leq \varphi(x, y)$$

for $\mu = 1, i$, and all $x, y \in \mathcal{B}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$, then there exists a unique algebra $*$ -homomorphism $H : \mathcal{B} \rightarrow \mathcal{B}$ satisfying the inequality (2.v).

Proof. Put $\mu = 1$ in (2.vi). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $H : \mathcal{B} \rightarrow \mathcal{B}$ satisfying the inequality (2.v). By the same reasoning as in the proof of [7, Theorem], the additive mapping $H : \mathcal{B} \rightarrow \mathcal{B}$ is $\mathbb{R}$ -linear.

Put $\mu = i$ in (2.vi). By the same method as in the proof of Theorem 2.1, one can obtain that

$$H(ix) = \lim_{n \rightarrow \infty} \frac{f(2^n ix)}{2^n} = \lim_{n \rightarrow \infty} \frac{if(2^n x)}{2^n} = iH(x)$$

for all $x \in \mathcal{B}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{B}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{B}$. Hence the additive mapping $H : \mathcal{B} \rightarrow \mathcal{B}$ is $\mathbb{C}$ -linear.

The rest of the proof is the same as the proof of Theorem 2.1. $\square$

3. STABILITY OF AUTOMORPHISMS ON A UNITAL C^* -ALGEBRA

We are going to show the generalized Hyers-Ulam-Rassias stability of automorphisms on a unital C^* -algebra.

Theorem 3.1. *Let $f : \mathcal{A} \rightarrow \mathcal{A}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ satisfying (2.i) and (2.ii) such that*

$$(3.i) \quad \|f(u^*) - f(u)^*\| \leq \varphi(u, u)$$

for all $u \in \mathcal{U}(\mathcal{A})$. Assume that $\lim_{n \rightarrow \infty} \frac{f(2^n e)}{2^n}$ is invertible, where e is the identity of $\mathcal{A}$. Then the bijective mapping $f : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$ -linear mapping $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$(3.ii) \quad \|f(x) - \alpha(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)$$

for all $x \in \mathcal{A}$. The $\mathbb{C}$ -linear mapping $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ is given by

$$\alpha(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

for all $x \in \mathcal{A}$.

By (2.i) and (3.i), we get

$$\begin{aligned} \alpha(u^*) &= \lim_{n \rightarrow \infty} \frac{f(2^n u^*)}{2^n} = \lim_{n \rightarrow \infty} \frac{(f(2^n u))^*}{2^n} = \left(\lim_{n \rightarrow \infty} \frac{f(2^n u)}{2^n} \right)^* \\ &= \alpha(u)^* \end{aligned}$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since α is $\mathbb{C}$ -linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [5, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$,

$$\begin{aligned} \alpha(x^*) &= \alpha\left(\sum_{j=1}^m \overline{\lambda_j} u_j^*\right) = \sum_{j=1}^m \overline{\lambda_j} \alpha(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} \alpha(u_j)^* = \left(\sum_{j=1}^m \lambda_j \alpha(u_j)\right)^* \\ &= \alpha\left(\sum_{j=1}^m \lambda_j u_j\right)^* = \alpha(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$.

Since $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{A}$,

$$(3.1) \quad \alpha(xy) = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n xy) = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x) f(2^n y) = \alpha(x) f(2^n y)$$

for all $x, y \in \mathcal{A}$. By the additivity of α and (3.1),

$$2^n \alpha(xy) = \alpha(2^n xy) = \alpha(x(2^n y)) = \alpha(x) f(2^n y)$$

for all $x \in \mathcal{A}$. Hence

$$(3.2) \quad \alpha(xy) = \frac{1}{2^n} \alpha(x) f(2^n y) = \alpha(x) \frac{1}{2^n} f(2^n y)$$

for all $x, y \in \mathcal{A}$. Taking the limit in (3.2) as $n \rightarrow \infty$, we obtain

$$\alpha(xy) = \alpha(x)\alpha(y)$$

for all $x, y \in \mathcal{A}$. By (3.1),

$$(3.3) \quad \alpha(x) = \alpha(ex) = \alpha(e)f(x)$$

for all $x \in \mathcal{A}$. Since $\alpha(e) = \lim_{n \rightarrow \infty} \frac{f(2^n e)}{2^n}$ is invertible and the mapping $f : \mathcal{A} \rightarrow \mathcal{A}$ is bijective, the $\mathbb{C}$ -linear mapping $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ is bijective. So the $\mathbb{C}$ -linear mapping $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism.

Now by (3.3),

$$\alpha(e)\alpha(x) = \alpha(ex) = \alpha(x) = \alpha(e)f(x)$$

for all $x \in \mathcal{A}$. Since $\alpha(e)$ is invertible, $\alpha(x) = f(x)$ for all $x \in \mathcal{A}$. Hence the bijective mapping $f : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism, as desired. $\square$

Corollary 3.2. *Let $f : \mathcal{A} \rightarrow \mathcal{A}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| &\leq \theta(\|x\|^p + \|y\|^p), \\ \|f(u^*) - f(u)^*\| &\leq 2\theta \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, and all $x, y \in \mathcal{A}$. Assume that $\lim_{n \rightarrow \infty} \frac{f(2^n e)}{2^n}$ is invertible. Then the bijective mapping $f : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 3.1. $\square$

Theorem 3.3. *Let $f : \mathcal{A} \rightarrow \mathcal{A}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ satisfying (2.i), (2.vi), and (3.i). Assume that $\lim_{n \rightarrow \infty} \frac{f(2^n e)}{2^n}$ is invertible. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the bijective mapping $f : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism.*

Proof. By the same reasoning as in the proof of Theorem 2.3, there exists a unique $\mathbb{C}$ -linear mapping $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ satisfying the inequality (3.ii).

The rest of the proof is the same as the proof of Theorem 3.1. $\square$

Remark. In each statement of this section, one can replace the condition that f satisfies $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{A}$ by the condition that f satisfies $f(zw) = f(z)f(w)$ for all positive elements $z, w \in \mathcal{A}$.

By the same method as in the proof of Theorem 3.1, one can show that

$$\alpha(zw) = \alpha(z)\alpha(w)$$

for all positive elements $z, w \in \mathcal{A}$. For elements $x, y \in \mathcal{A}$, $x = x_1 - x_2 + ix_3 - ix_4$ and $y = y_1 - y_2 + iy_3 - iy_4$, where $x_1, x_2, x_3, x_4, y_1, y_2, y_3$, and y_4 are positive elements (see [1, Lemma 38.8]); but α is $\mathbb{C}$ -linear. By a similar method to the proof of Theorem 2.1, one can easily show that

$$\alpha(xy) = \alpha(x)\alpha(y)$$

for all $x, y \in \mathcal{A}$.

REFERENCES

1. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York, Heidelberg and Berlin, 1973. MR **54**:11013
2. P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436. MR **95e**:47089
3. B. E. Johnson, *Approximately multiplicative maps between Banach algebras*, J. London Math. Soc. (2) **37** (1988), 294–316. MR **89h**:46072
4. R. V. Kadison and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266. MR **87g**:47078
5. R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Vol. I: Elementary Theory, Pure and Applied Mathematics, Vol. 100, Academic Press, New York, 1983. MR **85j**:46099
6. C. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720. MR **2003i**:46045
7. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300. MR **80d**:47094

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