BACKWARD UNIQUENESS FOR SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

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ABSTRACT. We address the backward uniqueness property for the equation $u_t - \Delta u = w_j \partial_j u + vu$ in $\mathbb{R}^n \times (T_0, 0]$. We show that under rather general conditions on v and w , $u|_{t=0} = 0$ implies that u vanishes to infinite order for all points $(x, 0)$. It follows that the backward uniqueness holds if $w = 0$ and $v \in L^\infty([0, T_0], L^p(\mathbb{R}^n))$ when $p > n/2$. The borderline case $p = n/2$ is also covered with an additional continuity and smallness assumption.

1. INTRODUCTION

A question of a solution semigroup of an equation on $\mathbb{R}^n$,

$$u_t - \Delta u + f(x, u, \nabla u) = 0,$$

being one-to-one is clearly reduced to the following problem: If a solution u of

$$(1.1) \quad u_t - \Delta u + w_j \partial_j u + vu = 0$$

on $\mathbb{R}^n \times (-T_0, 0]$ vanishes for $t = 0$, does it necessarily vanish for $t < 0$? This question of backward uniqueness was raised by Lax [L] and addressed by various authors ([L], [P], [LP], [G], [AN1], [AN2]). The approach by weighted inequalities, resembling the Carleman inequality method, has been employed by Lees and Protter [P], [LP]. In particular, the backward uniqueness property was proved for the differential inequality

$$(u_t - \Delta u)^2 \leq \frac{C}{t^2} u^2 + C |\nabla u|^2.$$

When comparing with the equation (1.1), this covers the case $v \in L^\infty((T_0, 0), L^{p_1}(\mathbb{R}^n))$ and $w \in L^\infty((T_0, 0), L^{p_2}(\mathbb{R}^n))$ when $p_1 \geq n$ and $p_2 = \infty$. The logarithmic convexity approach, introduced by Agmon and Nirenberg [AN1], [AN2] and used in [A], [BT], [G], is a lot simpler. However, it leads to the same range of exponents p_1, p_2 (cf. [G]). Based on forward uniqueness results, it seems that backward uniqueness should hold when $p_1 \geq n/2$ and $p_2 \geq n$ with sufficiently small norms when $p_1 = n/2$ and $p_2 = n$.

Recently, Escauriaza and Vega (cf. [E], [EV]) showed that if u , a solution of (1.1), vanishes of infinite order in space-time at $(x, t) = (0, 0)$, and if $w = 0$ and $v \in L^\infty((T_0, 0), L^{p_1}(\mathbb{R}^n))$ where $p_1 \geq n/2$ (with sufficiently small norm when $p_1 =$

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$n/2$), then it vanishes identically for $t \in (T_0, 0)$. If one tries to apply this result to backward uniqueness, then one runs into a problem of showing that $u(\cdot, 0) \equiv 0$ implies that u vanishes of infinite order at $(x, 0)$ for every $x \in \mathbb{R}^n$. This is the main result of our note. In Theorem 2.1, we namely prove that if u vanishes for $t = 0$, and if $p_1 > n/2$ and $p_2 > n$, then u vanishes of infinite order at every point $(x, 0)$ ($x \in \mathbb{R}^n$). This shows that the backward uniqueness property holds for all $p_1 > n/2$ and $w = 0$. As for the borderline case, it holds if the corresponding norm of v is sufficiently small and $v(\cdot, t)$ is continuous as an $L^{n/2}$ -valued function.

2. THE MAIN RESULTS

Fix any $T_0 < 0$ and $n \in \mathbb{N}$. Let

$$(2.1) \quad u \in C^1([T_0, 0], H^1(\mathbb{R}^n)) \cap C([T_0, 0], H^2(\mathbb{R}^n))$$

be a solution of

$$(2.2) \quad u_t - \Delta u = f.$$

Note that $f \in C([T_0, 0], L^2(\mathbb{R}^n))$. On f , we impose the following assumptions:

- (A1) $\|f\|_{L^2} \leq K_1 \|\Delta u\|_{L^2} + K_2 \|u\|_{L^2}$ for every $t \in [T_0, 0]$, where $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(\mathbb{R}^n)}$,
- (A2) $-(f, u) \leq K_3 \|\nabla u\|_{L^2}^2 + K_4 \|u\|_{L^2}^2$ for every $t \in [T_0, 0]$,
- (A3) $(f, u) \leq K_5 \|\nabla u\|_{L^2}^2 + K_6 \|u\|_{L^2}^2$ for every $t \in [T_0, 0]$

where K_1, K_2, K_3, K_4, K_5 , and $K_6 \geq 0$. In the logarithmic convexity approach to uniqueness ([A], [AN1], [AN2], [G]), the Dirichlet quotient

$$Q(t) = \frac{\|\nabla u(t)\|_{L^2}^2}{\|u(t)\|_{L^2}^2}$$

plays a distinguished role; here and in the sequel, we write $u(t) = u(\cdot, t)$. Also, as usual, the symbol C denotes a sufficiently large universal constant depending only on n ; it has a different value at each occurrence.

The following is our main result.

Theorem 2.1. *Let u be as above. Assume $u(0) \equiv 0$ and $u(t) \not\equiv 0$ for $t \in [T_0, 0)$. If $K_1 < 1$ and $K_5 < 1$, then*

$$\|u(t)\|_{L^2} \leq \|u(T_1)\|_{L^2} \left| \frac{t}{T_1} \right|^M \exp(K_6(t - T_1))$$

for $t \in [T_1, 0)$, where

$$T_1 = \max \left\{ T_0, -\frac{1 - K_1}{CK_1K_2} \right\}$$

and

$$M = \frac{(1 - K_5)(1 - K_1)}{CK_1^2}.$$

Of particular interest is the equation of the form

$$(2.3) \quad u_t - \Delta u = w_j \partial_j u + vu$$

where u is as in (2.1).

Corollary 2.2. *Assume u , which solves (2.3), satisfies $u(0) \equiv 0$ and $u(t) \not\equiv 0$ for $t \in [T_0, 0)$. Assume $w_j \in C([T_0, 0], L^n(\mathbb{R}^n))$ and $v \in C([T_0, 0], L^{n/2}(\mathbb{R}^n))$ (mere continuity at 0 and $v \in L^\infty([T_0, 0], L^{n/2}(\mathbb{R}^n))$ suffice). Then*

$$\|u(t)\|_{L^2} = o(|t|^m), \quad \text{as } t \rightarrow 0-,$$

i.e.,

$$\lim_{t \rightarrow 0-} \frac{\|u(t)\|_{L^2}}{|t|^m} = 0$$

for every $m \in \mathbb{N}$.

Proof of Corollary 2.2. Let $\epsilon > 0$ be arbitrary. Then there exist $T_1 < 0$ and decompositions $w_j = w_{j1} + w_{j2}$ and $v = v_1 + v_2$ so that $w_{j1}, w_{j2} \in C([T_1, 0], L^n(\mathbb{R}^n))$ and $v_1, v_2 \in C([T_1, 0], L^{n/2}(\mathbb{R}^n))$ with

$$\|w_{j1}\|_{L^\infty([T_1, 0], L^n(\mathbb{R}^n))}, \|v_1\|_{L^\infty([T_1, 0], L^{n/2}(\mathbb{R}^n))} \leq \epsilon$$

and

$$w_{j2}, v_2 \in L^\infty([T_1, 0], L^\infty(\mathbb{R}^n)).$$

Then, by Sobolev imbedding, (A1), (A2), and (A3) hold with

$$K_1 = K_3 = K_5 = C\epsilon$$

where $f = w_j \partial_j u + vu$, where C depends only on n , and where K_2, K_4 , and K_6 are some positive constants. If $K_1, K_5 < 1$ (which holds if $\epsilon < 1/C$), then Theorem 2.1 implies

$$\|u(t)\|_{L^2} = \mathcal{O}(|t|^{(1-C\epsilon)/C^2\epsilon^2}), \quad \text{as } t \rightarrow 0-$$

and Corollary 2.2 indeed follows from Theorem 2.1. $\square$

Corollary 2.3. *Assume that u , which solves (2.3) on $t \in [T_0, 0)$, satisfies $u(0) \equiv 0$. Assume $w_j = 0$ and $v \in C([T_0, 0], L^{n/2}(\mathbb{R}^n))$ (mere continuity at 0 suffices) with $\|v\|_{L^\infty([T_0, 0], L^{n/2}(\mathbb{R}^n))} < \epsilon$ where $\epsilon = \epsilon(n) > 0$. Then $u(t) \equiv 0$ for $t \in [T_0, 0]$.*

Proof of Corollary 2.3. The previous corollary implies that

$$\|u(t)\|_{L^2} = \mathcal{O}(|t|^m), \quad \text{as } t \rightarrow 0-$$

for all $m \in \mathbb{N}$. Under this assumption, a result of Escauriaza and Vega [EV] implies that $u(t) = 0$ for all $t \in [T_0, 0]$. $\square$

The same conclusion holds if $w_j = 0$ and $v \in L^\infty([T_0, 0], L^p(\mathbb{R}^n))$ when $p \in (n/2, \infty]$; in this case we use [E].

3. PROOF OF THE MAIN RESULT

In this section, we prove Theorem 2.1. We start with a simple lemma.

Lemma 3.1. *Let $x \in C^1([T_0, 0), [0, \infty))$ be such that*

$$(3.1) \quad \dot{x} \leq Ax^\alpha + B$$

on $[T_0, 0)$ where $T_0 < 0$, $A > 0$, $B \geq 0$, and $\alpha > 1$. Assume that x is unbounded on $[T_0, 0)$. Then for

$$(3.2) \quad t \in \left[-\frac{1}{4B^{(\alpha-1)/\alpha} A^{1/\alpha}}, 0 \right) \cap [T_0, 0),$$

we have

$$(3.3) \quad x(t) > \frac{1}{2^{(\alpha+1)/(\alpha-1)} |t|^{1/(\alpha-1)} A^{1/(\alpha-1)}}.$$

Proof of Lemma 3.1. Denote by $\gamma(t)$ the right-hand side of (3.3). Assume, contrary to the assertion, that there exists t satisfying (3.2) such that

$$(3.4) \quad x(t) \leq \gamma(t).$$

Using the Gronwall lemma, we shall show that (3.4) implies that

$$(3.5) \quad x(\tau) \leq 2\gamma(t), \quad \tau \in (t, 0)$$

and this contradicts the fact that x is unbounded. It remains to be shown that (3.4), together with $x \in C^1([T_0, 0), [0, \infty))$ and (3.1) on $[t, 0)$, implies (3.5). Assume this is not true. Then we can find

$$\tau_0 = \min\{\tau \in (t, 0) : x(\tau) = 2\gamma(t)\}.$$

Note that $t < \tau_0 < 0$. Then

$$\dot{x}(\tau) \leq Ax(\tau)^\alpha + B \leq 2^\alpha A\gamma(t)^\alpha + B$$

for $\tau \in [t, \tau_0]$, which implies that

$$x(\tau_0) \leq \gamma(t) + (2^\alpha A\gamma(t)^\alpha + B)(\tau_0 - t) < \gamma(t) + (2^\alpha A\gamma(t)^\alpha + B)|t|.$$

Using the definition of $\gamma(t)$ and (3.2), one can check that $2^\alpha A\gamma(t)^\alpha \leq \gamma(t)/2|t|$ and $B \leq \gamma(t)/2|t|$. We get $x(\tau_0) < 2\gamma(t)$, which is a contradiction. $\square$

Proof of Theorem 2.1. Assume that $u(0) \equiv 0$ and $u(t) \not\equiv 0$ for $t \in [T_0, 0)$. First, we claim that $Q(t)$ is unbounded on $[T_0, 0)$. From (2.2) and (A2), we deduce

$$(3.6) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + Q(t) \|u\|_{L^2}^2 &= (f, u) \\ &\geq -K_3 \|\nabla u\|_{L^2}^2 - K_4 \|u\|_{L^2}^2 = -\|u\|_{L^2}^2 (K_3 Q(t) + K_4). \end{aligned}$$

Integrating this inequality, we get

$$\|u(t)\|_{L^2} \geq \|u(T_0)\|_{L^2} \exp \left(-(1 + K_3) \int_{T_0}^t Q(\tau) d\tau - K_4(t - T_0) \right)$$

provided $t \in [T_0, 0)$. If $Q(t)$ were bounded, we would have $u(0) \not\equiv 0$, which contradicts $u(0) \equiv 0$.

Next, we claim that

$$Q(t) \geq \frac{1 - K_1}{8K_1^2|t|}, \quad t \in [T_1, 0)$$

where

$$(3.7) \quad T_1 = \max \left\{ T_0, -\frac{1 - K_1}{4K_1K_2} \right\}.$$

By [AN1] and [AN2], we have an identity

$$\frac{1}{2} Q'(t) + \|(A - Q(t)I)v\|_{L^2}^2 = \frac{1}{\|u\|_{L^2}^2} (f, (A - Q(t)I))$$

where $A = -\Delta$ and $v = u/\|u\|_{L^2}$. Therefore, by assumption (A1),

$$\begin{aligned} & \frac{1}{2}Q'(t) + \|(A - Q(t)I)v\|_{L^2}^2 \\ & \leq \frac{\|f\|_{L^2}}{\|u\|_{L^2}} \|(A - Q(t)I)v\|_{L^2} \leq (K_1\|Av\|_{L^2} + K_2) \|(A - Q(t)I)v\|_{L^2} \end{aligned}$$

since $\|v\|_{L^2} = 1$. Now, using

$$\|Av\|_{L^2} \leq \|(A - Q(t)I)v\|_{L^2} + Q(t)$$

we obtain

$$\begin{aligned} & \frac{1}{2}Q'(t) + \|(A - Q(t)I)v\|_{L^2}^2 \\ & \leq K_1 \|(A - Q(t)I)v\|_{L^2}^2 + K_1 Q(t) \|(A - Q(t)I)v\|_{L^2} + K_2 \|(A - Q(t)I)v\|_{L^2}. \end{aligned}$$

Since $K_1 < 1$, we get by the Cauchy-Schwarz inequality,

$$Q'(t) \leq \frac{K_1^2}{1 - K_1} Q(t)^2 + \frac{K_2^2}{1 - K_1}.$$

By Lemma 3.1,

$$Q(t) \geq \frac{1 - K_1}{8K_1^2|t|} = -\frac{1 - K_1}{8K_1^2t}$$

for $t \in [T_1, 0)$, where T_1 is as in (3.7), and the claim is established. In order to conclude the proof, we use the first equality in (3.6) and assumption (A3) to obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + Q(t) \|u\|_{L^2}^2 \leq (K_5 Q(t) + K_6) \|u\|_{L^2}^2$$

from which

$$\frac{d}{dt} \|u\|_{L^2} + ((1 - K_5)Q(t) - K_6) \|u\|_{L^2} \leq 0.$$

Since $K_5 < 1$, we get

$$\frac{d}{dt} \|u\|_{L^2} + \left(-\frac{(1 - K_5)(1 - K_1)}{8K_1^2t} - K_6 \right) \|u\|_{L^2} \leq 0$$

for $t \in [T_1, 0)$. Integrating this inequality, we obtain

$$\|u(t)\|_{L^2} \leq \|u(T_1)\|_{L^2} \exp \left(\frac{(1 - K_5)(1 - K_1)}{8K_1^2} \log \left| \frac{t}{T_1} \right| + K_6(t - T_1) \right)$$

for all $t \in [T_1, 0)$. □

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