WEAK SELECTIONS AND PSEUDOCOMPACTNESS

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ABSTRACT. We prove that every pseudocompact space that admits a weak selection is sequentially compact.

1. INTRODUCTION

Throughout this paper all the spaces will be Tychonoff. If X is a space, $\mathcal{F}(X)$ denotes the set of all nonempty closed subsets of X , and $\mathcal{F}_2(X) = \{A \in \mathcal{F}(X) : |A| \leq 2\}$. The *Vietoris Topology* on $\mathcal{F}_2(X)$ is generated by the sets of the form $\langle V, U \rangle = \{A \in \mathcal{F}_2(X) : A \subseteq V \cup U \text{ and } A \cap V \neq \emptyset \neq A \cap U\}$, where V and U are nonempty open subsets of X . A weak selection on X is a continuous function $f : \mathcal{F}_2(X) \rightarrow X$ such that $f(A) \in A$ for every $A \in \mathcal{F}_2(X)$.

In [3], the authors proved that Ψ -spaces do not admit a weak selection. Years before, E. K. van Douwen [2] proved that every countably compact space with a weak selection is sequentially compact (see also [3]). In this paper, we show that countable compactness can be replaced by pseudocompactness. Our main result solves Problems 5.1 and 5.2 from [1]. J. van Mill and E. Wattel [4] proved that a compact space X has a weak selection iff it is orderable. On the other hand, in [1, Th. 1.16] and [5, Th. 2.1], it is shown that if $X \times X$ is pseudocompact, then X admits a weak selection iff $\beta(X)$ is orderable. This result and Theorem 2.3 below generalize van Mill and Wattel's theorem in the following way: A pseudocompact space X admits a weak selection iff $\beta(X)$ is orderable.

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2. THE RESULT

Lemma 2.1. *Let X be a pseudocompact space with a weak selection $f : \mathcal{F}_2(X) \rightarrow X$. If $\{V_n : n < \omega\}$ is a family of pairwise disjoint nonempty open subsets of X such that*

$$f[\langle V_n, V_m \rangle] \subseteq V_{\min\{n,m\}}$$

for every $n, m < \omega$, then the sequence $(V_n)_{n < \omega}$ converges to some point of X .

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Proof. Since X is pseudocompact, the family $\{V_n : n < \omega\}$ has at least an accumulation point in X . Suppose that $x, y \in X$ are two accumulation points of $\{V_n : n < \omega\}$. Without loss of generality we may assume that $f(\{x, y\}) = x$. Let $V \in \mathcal{N}(x)$ and $U \in \mathcal{N}(y)$ such that

i. $V \cap U = \emptyset$; and

ii. $f[\langle V, U \rangle] \subseteq V$.

Choose $n, m < \omega$ so that $n < m$ and there are $a \in V_n \cap U$ and $b \in V_m \cap V$. Then, we have that $f(\{a, b\}) = a$ by the hypothesis, and $f(\{a, b\}) = b$ by clause ii, but this is impossible. Therefore, $\{V_n : n < \omega\}$ has only one accumulation point $z \in X$. Now, assume that there is $W \in \mathcal{N}(z)$ such that $A = \{n < \omega : V_n \setminus \overline{W} \neq \emptyset\}$ is infinite. Then, by the pseudocompactness of X , the family $\{V_n \setminus \overline{W} : n \in A\}$ has an accumulation point in X which is different from z , but this is a contradiction. Thus, $V_n \rightarrow z$, as we desire. $\square$

Lemma 2.2. *Let X be a pseudocompact space which admits a weak selection $f : \mathcal{F}_2(X) \rightarrow X$. If $\{V_n : n < \omega\}$ is a family of pairwise disjoint nonempty open subsets of X , then there are a convergent sequence $(W_i)_{i < \omega}$ of nonempty open subsets of X and a strictly increasing sequence $(n_i)_{i < \omega}$ of positive integers such that $W_i \subseteq V_{n_i}$ for each $i < \omega$.*

Proof. By the pseudocompactness of X , if the family $\{V_n : n < \omega\}$ has only one accumulation point in X , then it converges to some point of X . Suppose that $\{V_n : n < \omega\}$ has at least two accumulation points. Then, we can find two disjoint open subsets P_0 and Q_0 of X such that $P_0 \cap Q_0 = \emptyset$, $f[\langle P_0, Q_0 \rangle] \subseteq P_0$ and both sets $A_0 = \{k < \omega : P_0 \cap V_k \neq \emptyset\}$ and $B_0 = \{k < \omega : Q_0 \cap V_k \neq \emptyset\}$ are infinite sets. By induction and assuming at every stage that the family of open sets in turn has at least two accumulation points, for every $n < \omega$, we can define two disjoint nonempty open subsets P_n and Q_n of X such that

1. $f[\langle P_n, Q_n \rangle] \subseteq P_n$ for every $n < \omega$; and

2. the sets $A_n = \{k \in B_{n-1} : P_n \cap Q_{n-1} \cap \dots \cap Q_1 \cap Q_0 \cap V_k \neq \emptyset\}$ and $B_n = \{k \in B_{n-1} : Q_n \cap Q_{n-1} \cap \dots \cap Q_1 \cap Q_0 \cap V_k \neq \emptyset\}$ are infinite.

Now, for each $i < \omega$ we pick $n_i \in A_i$ so that $n_{i-1} < n_i$ for every $1 \leq i < \omega$, and for each $i < \omega$ we put $W_i = P_i \cap Q_{i-1} \cap \dots \cap Q_1 \cap Q_0 \cap V_{n_i}$. It is not difficult to prove that $f[\langle W_i, W_j \rangle] \subseteq W_{\min\{i, j\}}$ for distinct $i, j < \omega$. By Lemma 2.1, $(W_i)_{i < \omega}$ converges to some point $x \in X$. $\square$

Theorem 2.3. *If X is a pseudocompact space with a weak selection $f : \mathcal{F}_2(X) \rightarrow X$, then $X \times Y$ is pseudocompact for every pseudocompact space Y .*

Proof. Let Y be a pseudocompact space, and let $\{V_n \times U_n : n < \omega\}$ be a countable family of pairwise disjoint basic nonempty open subsets of $X \times Y$. We may assume that $\{V_n : n < \omega\}$ is pairwise disjoint. By Lemma 2.2, there are a sequence $(W_i)_{i < \omega}$ of nonempty open subsets of X converging to a point $x \in X$, and a strictly increasing sequence $(n_i)_{i < \omega}$ of positive integers such that $W_i \subseteq V_{n_i}$ for each $i < \omega$. By the hypothesis, the family $\{U_{n_i} : i < \omega\}$ has an accumulation point in Y , say y . Then, (x, y) is an accumulation point of $\{V_{n_i} \times U_{n_i} : i < \omega\}$. So, $X \times Y$ is pseudocompact. $\square$

Since there are compact spaces which do not admit weak selections (for instance, $\beta(\omega)$ does not admit weak selection by Lemma 2.2), the converse of Theorem 2.3 is not true.

A direct consequence of Theorem 2.3 and the Theorems from [1] and [5] which were mentioned in the Introduction is the following.

Corollary 2.4. *Every pseudocompact space that admits a weak selection is sequentially compact.*

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