ON THE NORM OF AN IDEMPOTENT SCHUR MULTIPLIER ON THE SCHATTEN CLASS

WILLIAM D. BANKS AND ASMA HARCHARRAS

(Communicated by Andreas Seeger)

ABSTRACT. We show that if the norm of an idempotent Schur multiplier on the Schatten class S^p lies sufficiently close to 1, then it is necessarily equal to 1. We also give a simple characterization of those idempotent Schur multipliers on S^p whose norm is 1.

1. INTRODUCTION

We study norms of idempotent Schur multipliers defined on the Schatten p -class with $1 < p < \infty$, $p \neq 2$. For any idempotent Schur multiplier ϕ , we show that if the norm of ϕ lies sufficiently close to 1, then it is necessarily equal to 1. More precisely, if ϕ is an idempotent Schur multiplier on the Schatten p -class, then $\phi = 0$, $\|\phi\| = 1$, or $\|\phi\| \geq 1 + \eta_p$, where η_p is a positive constant that depends only on p . We also obtain a simple characterization of those idempotent Schur multipliers whose norm is equal to 1. When $p = 1$ or ∞ , these results have been obtained by Livshits [2], while for $p = 2$, every nonzero idempotent Schur multiplier has norm 1.

To state our results more explicitly, we need to fix some standard terminology. For every real number p in the range $1 \leq p < \infty$, denote by S^p the *Schatten p -class* over the Hilbert space ℓ_2 ; it is the Banach space of all compact operators $x : \ell_2 \rightarrow \ell_2$ with finite norm

$$\|x\|_{S^p} = \left(\operatorname{Tr} (x^* x)^{p/2} \right)^{1/p},$$

where $\operatorname{Tr}(\cdot)$ denotes the usual trace. For $p = \infty$, the space S^∞ is the Banach space of all compact operators $x : \ell_2 \rightarrow \ell_2$, equipped with the usual operator norm. The spaces S^p , $1 \leq p \leq \infty$, were considered in [4] as noncommutative analogues for the spaces ℓ_p , $1 \leq p \leq \infty$ (for a more modern reference, see [3] for example).

For $1 \leq p \leq \infty$ and a positive integer n , let S_n^p denote the Schatten p -class over the Hilbert space ℓ_2^n of dimension n .

In what follows, we make no distinction between an operator x on ℓ_2 and the corresponding matrix $(x_{ij})_{i,j \in \mathbb{N}}$ relative to the canonical basis $\{e_{ij}\}_{i,j \in \mathbb{N}}$ of S^p .

A set-theoretic map $\phi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{C}$ is said to be a *Schur multiplier* on S^p if the associated operator $T_\phi : S^p \rightarrow S^p$, defined by

$$T_\phi(x) = (\phi_{ij} x_{ij})_{i,j \in \mathbb{N}}, \quad \forall x = (x_{ij})_{i,j \in \mathbb{N}} \in S^p,$$

Received by the editors December 12, 2002 and, in revised form, April 21, 2003.

2000 *Mathematics Subject Classification*. Primary 47A30; Secondary 47B49, 47B10.

Key words and phrases. Idempotent Schur multiplier, Schatten class.

The first author was supported in part by NSF grant DMS-0070628.

is well defined and bounded on S^p . In particular, this implies that ϕ itself is a bounded map. Let $\mathcal{M}(S^p)$ denote the space of all Schur multipliers on S^p . Then $\mathcal{M}(S^p)$ is a Banach algebra when it is equipped with the pointwise product and the norm

$$\|\phi\|_{\mathcal{M}(S^p)} = \|T_\phi : S^p \rightarrow S^p\|, \quad \forall \phi \in \mathcal{M}(S^p).$$

It is well known that for pairs $1 \leq p, q \leq \infty$ with $p^{-1} + q^{-1} = 1$, the algebras $\mathcal{M}(S^p)$ and $\mathcal{M}(S^q)$ can be identified isometrically. These identifications can be done via the identity map by defining the duality between S^p and S^q with $\langle x, y \rangle = \text{Tr}(^t x y)$ for all $x \in S^p$ and $y \in S^q$.

In addition, the space $\mathcal{M}(S^2)$ can be identified isometrically with the Hilbert space $\ell_2(\mathbb{N} \times \mathbb{N})$. Consequently, when studying $\mathcal{M}(S^p)$ it suffices to reduce to the case where $2 < p \leq \infty$.

Finally, a Schur multiplier $\phi \in \mathcal{M}(S^p)$ is said to be *idempotent* provided that $T_\phi \circ T_\phi = T_\phi$; clearly, this is equivalent to the condition that ϕ maps $\mathbb{N} \times \mathbb{N}$ into the set $\{0, 1\}$. For such multipliers, one has

$$\|\phi\|_{\mathcal{M}(S^p)} = \|\phi \cdot \phi\|_{\mathcal{M}(S^p)} \leq \|\phi\|_{\mathcal{M}(S^p)}^2.$$

Hence, $\|\phi\|_{\mathcal{M}(S^p)} \geq 1$ whenever $\phi \neq 0$. Our main result is the following:

Theorem 1. *For every real number p with $1 < p < \infty$ and $p \neq 2$, there exists a constant $\eta_p > 0$ (depending only on p) such that for every nonzero idempotent Schur multiplier $\phi \in \mathcal{M}(S^p)$ with $\|\phi\|_{\mathcal{M}(S^p)} \neq 1$, the following inequality holds:*

$$\|\phi\|_{\mathcal{M}(S^p)} \geq 1 + \eta_p.$$

By the remarks above, it suffices to consider the case where $2 < p < \infty$, which we assume throughout the sequel.

2. PROOF OF THE MAIN RESULT

The proof of Theorem 1 can be split into three pieces, as follows.

Lemma 1. *Let $\Delta = (\Delta_{ij})_{1 \leq i, j \leq 2}$ with $\Delta_{11} = \Delta_{12} = \Delta_{22} = 1$ and $\Delta_{21} = 0$. Then $\|\Delta\|_{\mathcal{M}(S_2^{p'})} > \|\Delta\|_{\mathcal{M}(S_2^p)} > 1$ for $2 < p < p' \leq \infty$.*

Proof. For every $c \in \mathbb{C}$, let $x^{(c)} = (x_{ij}^{(c)})_{1 \leq i, j \leq 2}$, where $x_{11}^{(c)} = x_{12}^{(c)} = x_{22}^{(c)} = 1$ and $x_{21}^{(c)} = c$. One has

$$\|x^{(c)}\|_{S_2^p} = \left(\text{Tr} (x^{(c)*} x^{(c)})^{p/2} \right)^{1/p} = \left(\lambda_{+,c}^{p/2} + \lambda_{-,c}^{p/2} \right)^{1/p},$$

where

$$\lambda_{\pm,c} = \frac{1}{2} \left(3 + |c|^2 \pm \sqrt{5 + 8\Re(c) + 2|c|^2 + |c|^4} \right).$$

In particular, if we choose $c = (2 - p)/2$, then

$$\|\Delta\|_{\mathcal{M}(S_2^p)}^p \geq \frac{\|\Delta(x^{(c)})\|_{S_2^p}^p}{\|x^{(c)}\|_{S_2^p}^p} = \frac{\|x^{(0)}\|_{S_2^p}^p}{\|x^{(c)}\|_{S_2^p}^p} = f(p),$$

where $f(p)$ is the function

$$\frac{2^p ((3 + \sqrt{5})^{p/2} + (3 - \sqrt{5})^{p/2})}{(p^2 - 4p + 16 + (p - 4)\sqrt{p^2 + 16})^{p/2} + (p^2 - 4p + 16 - (p - 4)\sqrt{p^2 + 16})^{p/2}}.$$

Since $f(2) = 1$, $f'(2) = 0$, and

$$f''(2) = \frac{\log(3 + \sqrt{5})}{3\sqrt{5}} - \frac{\log(3 - \sqrt{5})}{3\sqrt{5}} - \frac{1}{6} > 0,$$

the Taylor expansion for $f(p)$ near $p = 2$ shows that $f(p) > 1$ if $2 < p < 2 + \varepsilon$, for some $\varepsilon > 0$. Thus,

$$\|\Delta\|_{\mathcal{M}(S_2^p)} > 1, \quad \forall 2 < p < 2 + \varepsilon.$$

Now let $p' > p > 2$ be arbitrary real numbers and let $0 < \theta < 1$ be chosen so that $1/p = (1 - \theta)/2 + \theta/p'$. By the classical results of complex interpolation, we have $S_2^p = (S_2^2, S_2^{p'})_\theta$ isometrically (for the definition and fundamental results on complex interpolation, the reader is referred to [1]); hence it follows that

$$\|\Delta\|_{\mathcal{M}(S_2^p)} \leq \|\Delta\|_{\mathcal{M}(S_2^2)}^{1-\theta} \|\Delta\|_{\mathcal{M}(S_2^{p'})}^\theta.$$

Taking $2 < p < 2 + \varepsilon$ with ε sufficiently small, and using the obvious fact that $\|\Delta\|_{\mathcal{M}(S_2^2)} = 1$, the preceding relation and our results above imply that $\|\Delta\|_{\mathcal{M}(S_2^{p'})} > 1$ for all $2 < p' \leq \infty$. Since $0 < \theta < 1$, the above relation further implies that $\|\Delta\|_{\mathcal{M}(S_2^{p'})} > \|\Delta\|_{\mathcal{M}(S_2^p)}$ for $2 < p < p' \leq \infty$. This completes the proof. $\square$

It has been shown in [2] that $\|\Delta\|_{\mathcal{M}(S_2^\infty)} = 2/\sqrt{3}$, which provides an upper bound for $\|\Delta\|_{\mathcal{M}(S_2^p)}$ for any $p > 2$. On the other hand, in the notation of Lemma 1 and taking $c = -1$, we have for $p > 2$,

$$\|\Delta\|_{\mathcal{M}(S_2^p)} \geq \frac{\|x^{(0)}\|_{S_2^p}}{\|x^{(-1)}\|_{S_2^p}} = \left(\frac{(3 + \sqrt{5})^{p/2} + (3 - \sqrt{5})^{p/2}}{2^{p+1}} \right)^{1/p} > \frac{\sqrt{3 + \sqrt{5}}}{2^{1+1/p}}.$$

It remains an interesting question to determine the precise value of $\|\Delta\|_{\mathcal{M}(S_2^p)}$ for any p in the range $2 < p < \infty$; this will not be needed, however, in what follows.

We now define, for each p in the range $2 < p < \infty$,

$$\eta_p = -1 + \|\Delta\|_{\mathcal{M}(S_2^p)}.$$

In view of Lemma 1, η_p is strictly positive.

Definition. A map ϕ defined on $\mathbb{N} \times \mathbb{N}$ (or any of its subsets) is said to be *triangle-free* if there are no integers i, j, k, l such that $\phi_{ij} = \phi_{il} = \phi_{kj} = 1$ and $\phi_{kl} = 0$.

The following lemma is an easy consequence of Lemma 1; the proof is omitted.

Lemma 2. Fix $p > 2$, and suppose that $\phi \in \mathcal{M}(S^p)$ is idempotent. If

$$\|\phi\|_{\mathcal{M}(S^p)} < 1 + \eta_p,$$

then ϕ is triangle-free.

Finally, we have

Lemma 3. If a map $\phi : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ is nonzero and triangle-free, then $\|\phi\|_{\mathcal{M}(S^p)} = 1$ for every real number $p > 2$.

Proof. For any positive integer n , denote by $\phi^{(n)}$ the restriction of ϕ to the subset $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ of $\mathbb{N} \times \mathbb{N}$. Recalling the well-known fact

$$\|\phi\|_{\mathcal{M}(S^p)} = \sup_{n \geq 1} \|\phi^{(n)}\|_{\mathcal{M}(S_n^p)},$$

we see that it suffices to show that $\|\phi^{(n)}\|_{\mathcal{M}(S_n^p)} = 1$ whenever $\phi^{(n)} \neq 0$.

To this end, let $n \geq 1$ be fixed with $\phi^{(n)} \neq 0$. For every integer $1 \leq i \leq n$, define the row sum

$$c_i = \#\{1 \leq j \leq n \mid \phi_{ij}^{(n)} = 1\}.$$

To show $\|\phi^{(n)}\|_{\mathcal{M}(S_n^p)} = 1$, we may freely permute the rows and/or the columns of $\phi^{(n)}$ in any way that we want; in particular, without loss of generality, we may assume that

$$c_1 \geq c_2 \geq c_3 \geq \dots \geq c_n,$$

and that

$$\phi_{11}^{(n)} = \phi_{12}^{(n)} = \phi_{13}^{(n)} = \dots = \phi_{1c_1}^{(n)} = 1.$$

Since ϕ is triangle-free, for every $1 \leq i \leq n$ there are only two possibilities:

- (α) $\phi_{ij} = 1$ for all $1 \leq j \leq c_1$, and $\phi_{ij} = 0$ for all $j > c_1$;
- (β) $\phi_{ij} = 0$ for all $1 \leq j \leq c_1$.

After permuting the rows if necessary, we may assume that (α) occurs for $1 \leq i \leq r_1$, and that (β) occurs for $i > r_1$. Then

$$\phi^{(n)} = \phi_1 \oplus \phi'_1,$$

where ϕ_1 is an $r_1 \times c_1$ rectangular matrix with every entry equal to 1, and ϕ'_1 is an $(n - r_1) \times (n - c_1)$ rectangular matrix whose entries are equal to 0 or 1 and which is triangle-free. If $\phi'_1 = 0$, we stop; otherwise, we repeat the same argument with $\phi^{(n)}$ replaced by ϕ'_1 , obtaining

$$\phi^{(n)} = \phi_1 \oplus \phi_2 \oplus \phi'_2.$$

We continue in this way until the process stops, at which point we have

$$\phi^{(n)} = \phi_1 \oplus \phi_2 \oplus \dots \oplus \phi_s,$$

where every ϕ_k , $1 \leq k \leq s$, is an $r_k \times c_k$ rectangular matrix, all of the entries of $\phi_1, \dots, \phi_{s-1}$ are equal to 1, and the entries of ϕ_s are all equal to 1 or all equal to 0. By adding some additional zero rows and/or zero columns to $\phi^{(n)}$ if necessary, we may also assume that $r_k = c_k$ for $1 \leq k \leq s$. Then

$$\|\phi^{(n)}\|_{\mathcal{M}(S_n^p)} = \sup_{1 \leq k \leq s} \|\phi_k\|_{\mathcal{M}(S_{r_k}^p)} = 1,$$

and the result follows. $\square$

Theorem 1 is an immediate consequence of Lemmas 1–3, as the reader can easily verify.

Examining the proof of Theorem 1, we see that for a nonzero idempotent Schur multiplier ϕ , the following assertions are equivalent:

- (a) for some $p > 2$, $\phi : S^p \rightarrow S^p$ has norm 1;
- (b) ϕ is triangle-free;
- (c) ϕ is equivalent to a multiplier of the form $\phi_1 \oplus \phi_2 \oplus \phi_3 \oplus \dots$, where each ϕ_j has all of its entries equal to 1 or all of its entries equal to 0;
- (d) $\phi : S^\infty \rightarrow S^\infty$ has norm 1;
- (e) for every p , $\phi : S^p \rightarrow S^p$ has norm 1.

ACKNOWLEDGMENTS

We thank G. Pisier for bringing this question to our attention; the question was inspired by a talk given by V. Paulsen at the AMS Sectional Meeting held in Irvine, California, in November, 2001.

REFERENCES

- [1] J. Bergh and J. Löfström, *Interpolation spaces: An introduction*, Grundlehren der Mathematischen Wissenschaften, Band 223, Springer-Verlag, Berlin-New York, 1976. MR **58**:2349
- [2] L. Livshits, *A note on 0-1 Schur multipliers*, Linear Algebra Appl. **222** (1995), 15–22. MR **96d**:15040
- [3] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, R.I., 1969. MR **39**:7447
- [4] R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 27, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960. MR **22**:9878

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65211
E-mail address: `bbanks@math.missouri.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65211
E-mail address: `harchars@math.missouri.edu`