ON CLEAVABILITY OF CONTINUA OVER LOTS

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ABSTRACT. It is shown that any continuum cleavable over a LOTS L is embeddable into L .

1. INTRODUCTION

This work is devoted to cleavability over LOTS (linearly ordered topological spaces). The general concept of cleavability was introduced by A. V. Arhangel'skii (see [Ar2]). Comprehensive surveys on this subject can be found in [Ar5] and [Ar6].

A space X is *cleavable* over a space Y *along* $A \subseteq X$ if there exists a continuous mapping f from X to Y such that $f(A) \cap f(X \setminus A) = \emptyset$. A space X is *cleavable* over a space Y if X is cleavable over Y along each $A \subseteq X$. A space X is said to *condense* into a space Y if there exists a one-to-one continuous mapping of X into Y .

The following are the main problems in this field.

Problem 1.1. *When does cleavability of X over Y imply that X embeds (condenses) into Y ?*

Problem 1.2. *Which properties of Y are possessed by every space cleavable over Y ?*

Interesting results related to these problems are obtained in [Ar2], [Ar3], [Ar4], [Bal], [Bu], [Tka], and [Yas]. In this article it will be shown that any continuum cleavable over a LOTS L is embeddable into L . (Throughout the paper, all spaces are assumed to be Hausdorff, and by a continuum we mean a connected compact topological space, not necessarily linearly ordered.) Connectedness of a compactum is needed in our theorem, as shown in the following example.

Example 1.3. Let X, Y be three-point and two-point discrete compacta, respectively. It is easy to see that Y is a LOTS and X is cleavable over Y . (To cleave X over Y along A , just collapse two points that are both in A or its complement.) Obviously, X is not embeddable in Y , since Y lacks one extra point. However, X is a LOTS!

Thus, the following question remains of interest.

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Question 1.4 (A. V. Arhangel'skii). *Let X be a compactum cleavable over a LOTS. Is X a LOTS?*

In [Ar3], Arhangel'skii answered this question for cleavability over metric spaces. He proved that any compactum cleavable over a metric LOTS is a metric LOTS itself.

For a space X cleavable over a LOTS, a set $A \subset X$ is called *absolute* for X if any continuous mapping that cleaves X over a LOTS along A is one-to-one. Let X be a LOTS. The character of $x \in X$ from the right, denoted $\chi_r(x, X)$, is the smallest cardinal number τ such that there exists a set $\{x_\alpha : x_\alpha > x, \alpha < \tau\}$ whose closure contains x . Similarly, we define the character from the left, $\chi_l(x, X)$. A τ -sequence is a LOTS homeomorphic to the space of all ordinal numbers less than τ with the topology of linear order. Note that our definition of τ -sequence is non-standard. An interval of a LOTS is a set in the form $[a, b]$, (a, b) , $[a, b)$, or $(a, b]$, where $a \leq b$. If $a = b$, an interval is called *trivial*. Let f be a continuous mapping of a LOTS X . We say that elements $x, y \in X$ are *collapsed (glued up)* under f if $f(x) = f(y)$. If I is an interval in X and $f(I)$ is a single-element set, we say that I is *squeezed* into a point under f . We will use the following simple known statements.

Statement 1.5. *If X is a compact LOTS and $\chi_l(x, X) = \tau$, then there exists a τ -sequence converging to x from the left, and $cf(\tau) = \tau$.*

Statement 1.6. *If $cf(\tau) > \omega_0$ and s_1, s_2 are two τ -sequences converging to x from the left, then $s_1 \cap s_2 \neq \emptyset$.*

We will also use the following obvious fact about absolute sets.

Statement 1.7. *If X is a connected infinite LOTS and A is an absolute set for X , then A is dense in X .*

Proof. If A is not dense, then there exists a nontrivial closed interval $I \subset X \setminus A$. Squeeze I into a point. This operation defines a continuous mapping f of X onto a LOTS. Obviously, f cleaves X along A , but f is not one-to-one, since I is infinite. $\square$

Statement 1.8. *If $X = [a, b]$ is a connected infinite LOTS, then there exists a countable $Y \subset X$ such that $|\overline{Y}| = 2^{\omega_0}$.*

Proof. The argument is the same as in the classical construction of the Cantor set. To avoid repetition, let us do only the first two steps. Take arbitrary distinct $y_{1/3}, y_{2/3} \in (a, b)$. Remove $(y_{1/3}, y_{2/3})$ from $[a, b]$. What is left consists of the two components $[a, y_{1/3}]$ and $[y_{2/3}, b]$. Now take arbitrary distinct $y_{1/9}, y_{2/9} \in (a, y_{1/3})$ and $y_{7/9}, y_{8/9} \in (y_{2/3}, b)$. Again remove central thirds and repeat the above-described process. The set of all end-points of the removed intervals is countable, and its closure has cardinality 2^{ω_0} . $\square$

2. SOME GENERAL FACTS

Proposition 2.1. *Let $X = [a, b]$ be a linearly ordered continuum. Let f be a continuous mapping of X onto a LOTS. Suppose that no element in (a, b) collapses with a under f . Then, $f(a)$ is an end-point of $f([a, b])$.*

Proof. Assume the contrary. Then there exist $x, y \in (a, b]$ such that $f(x)$ and $f(y)$ are the two end-points of $f(X)$. Due to connectedness of $[x, y]$, $f([x, y]) = f(X)$.

Therefore, there exists $z \in (x, y)$ such that $f(z) = f(a)$, which contradicts the proposition's assumptions. $\square$

Proposition 2.2. *Let X be a connected LOTS. Let f be a continuous non-one-to-one mapping of X onto a LOTS. Then:*

(1) *There exist two nontrivial closed disjoint intervals in X that have the same images under f .*

(2) *Suppose S is a nowhere-dense set in X . Then there exist two nontrivial closed disjoint intervals in X that do not intersect S and have the same images under f .*

Proof. (1) Let c be a point in X that collapses with another point under f . Let C be the set of all elements in X that have the same image as c . If C contains a nontrivial open interval, then we are done. Otherwise, there exist distinct $a, b \in C$ such that no element in (a, b) collapses with c . Then, by Proposition 2.1, $f(a)$ is an end-point of $f([a, b])$ (suppose it is the left end-point). Take $d \in (a, b)$ such that $f(d)$ is the other end-point of $f([a, b])$. Due to connectedness, $f([a, b]) = f([a, d]) = f([d, b])$. Now, take an arbitrary $z \in (f(a), f(d))$. Let $a_1 \in (a, d)$ be the closest point to a such that $f(a_1) = z$ and let $b_1 \in (d, b)$ be the closest point to b such that $f(b_1) = z$. Then, by Proposition 2.1, $f([a, a_1]) = f([b_1, b])$.

(2) If $[x, y] \subseteq X \setminus S$ and f is not a homeomorphism on $[x, y]$, then (1) for $X = [x, y]$ gives the required intervals. Thus, we can assume now that f is a homeomorphism on any interval outside of S . By (1) there exist two disjoint nontrivial closed intervals I, J such that $f(I) = f(J)$. Take a nontrivial $[a, b] \subseteq I \setminus S$. Now, take a nontrivial $[c, d] \subseteq f^{-1}(f([a, b])) \cap J$ that does not intersect S . Thus, $f([c, d]) \subseteq f([a, b])$. Since f is a homeomorphism on both $[a, b]$ and $[c, d]$, there exists $[a', b'] \subseteq [a, b]$ such that $f([a', b']) = f([c, d])$. $\square$

Proposition 2.3. *Let f be a continuous mapping of a connected LOTS X onto a LOTS. Let $\chi_l(x, X) = \tau$. If no nontrivial interval is squeezed into a point under f , then either $\chi_l(f(x), f(X)) = \tau$ or $\chi_r(f(x), f(X)) = \tau$.*

Proof. Since no nontrivial interval is squeezed into a point, there exists a strictly increasing sequence $\{x_\alpha\}$ of cardinality τ that converges to x from the left such that $f(x_\alpha) < f(x)$ for each x_α (or $f(x_\alpha) > f(x)$). We will consider only the case when $f(x_\alpha) < f(x)$ for each x_α . Then, due to continuity of f , $\chi_l(f(x), f(X)) \leq \tau$. Suppose $\chi_l(f(x), f(X)) = \lambda < \tau$. Then there exists a strictly decreasing family of closed intervals $\mathcal{J} = \{J_\alpha : f(x) \text{ is the right end-point of } J_\alpha, \alpha < \lambda\}$ such that $\bigcap \mathcal{J} = \{f(x)\}$. Since $\lambda < cf(\tau)$, $\bigcap f^{-1}(\mathcal{J})$ contains a nontrivial interval of the form (y, x) . This interval, by the construction of $\mathcal{J}$, must have the same image as x , which contradicts the proposition's assumptions. $\square$

Proposition 2.4. *Let f be a continuous mapping of a connected LOTS X onto a LOTS. Suppose $x \in X$ is not an end-point of X and $\chi_l(x, X) = \tau > \chi_r(x, X) = \lambda$. If f squeezes no nontrivial interval into a point, then either $\chi_l(f(x), f(X)) = \tau > \chi_r(f(x), f(X)) = \lambda$ or $\chi_l(f(x), f(X)) = \lambda < \chi_r(f(x), f(X)) = \tau$.*

Proof. Apply Proposition 2.3 twice. $\square$

Proposition 2.5. *Let $X = [a, b]$ be a linearly ordered continuum with $\chi(a, X) = \tau$ and $\chi(b, X) = \lambda > \tau$. Let f be a continuous mapping of X onto a LOTS such*

that $f(a) = f(b)$. Let $f(a)$ be an end-point of $f(X)$. Then there exists a nontrivial interval in X whose image under f is a single-element set.

Proof. Assume that no nontrivial interval is squeezed into a point. Then, by Proposition 2.3, $\chi(f(a), f(X)) = \tau < \chi(f(b), f(X)) = \lambda$, which contradicts $f(a) = f(b)$. $\square$

Proposition 2.6. *Let f be a continuous mapping of a connected LOTS X onto a LOTS. Let $I = [a, b)$, $J = (c, d]$ be two disjoint intervals such that $f|_I, f|_J$ are embeddings, and J is to the right of I . Suppose $f(x) = f(y)$ for $x \in I$ and $y \in J$, and no element in (x, y) collapses with x under f . If f preserves (reverses) the order of I , then f reverses (preserves) the order of J .*

Proof. By Proposition 2.1, $f(x)$ is an end-point of $f([x, y])$. If the order of I is preserved, $f(x)$ is the left end-point of $f([x, y])$. Since J is open from the left, there exists $z \in J$ such that $x < z < y$. Since $f(y)$ is the left end-point of $f([x, y])$, $f(z) > f(y)$. Since $f|_J$ is an embedding and switches the order of two elements, it must reverse all of J (apply connectedness of J).

If the order of I is reversed, apply the above arguments but interchange “left” by “right”, “preserves” by “reverses”, etc. $\square$

Proposition 2.7. *Let $[a, b]$ be a connected LOTS. If (a, b) is first countable, then $\chi_r(a, [a, b]) \leq \omega_1$ and $\chi_l(b, [a, b]) \leq \omega_1$.*

Proof. If $\chi_r(a, [a, b]) = \tau > \omega_1$, then there exists a τ -sequence converging to a from the right. This sequence contains an ω_1 -subsequence converging to some element in (a, b) , which contradicts first-countability of (a, b) . $\square$

Proposition 2.8. *Let $X = [a, b]$ be a linearly ordered continuum. Let f be a continuous mapping of X onto a LOTS such that $f(a) = f(b)$. Let c, d be elements in X whose images are the two end-points of $f(X)$. Then for any $x \in X \setminus \{c, d\}$ there exists $y \in X \setminus \{x\}$ such that $f(x) = f(y)$.*

Proof. We may assume that $|f(X)| > 1$ and $c < d$. Since $f(a) = f(b)$, $f([a, c]) \cup f([d, b]) = f(X)$. But, $f([c, d]) = f(X)$ as well. Thus, any element in $[a, c) \cup (d, b]$ collapses with some element in $[c, d]$. Also, any element in (c, d) collapses with some element in $[a, c] \cup [d, b]$. Since $[a, c) \cup (c, d) \cup (d, b]$ covers all of X , except $\{c, d\}$, we are done. $\square$

Proposition 2.9. *Let f be a continuous one-to-one mapping of a connected LOTS X to a LOTS. Then f is an embedding.*

Proof. Since f is a one-to-one continuous mapping of a connected LOTS into a LOTS, f is strictly monotonic. Since $f(X)$ is a connected subspace of a LOTS, it is a LOTS itself. Thus, f is a strictly monotonic mapping of a LOTS X onto a LOTS $f(X)$, and therefore it is a homeomorphism. To see this, notice that images and inverse images of intervals are intervals. $\square$

Proposition 2.10. *Let f be a continuous mapping of a connected LOTS $X = [a, b]$ to a LOTS. If f is one-to-one on (a, b) , then f is an embedding on $[a, b]$ ($[a, b)$).*

Proof. By Proposition 2.2, f is one-to-one on $[a, b]$ ($[a, b)$), and, by Proposition 2.9, f is an embedding on $[a, b]$ ($[a, b)$). $\square$

3. LEMMAS

In our first lemma, we will partially employ a technique developed by Tkachuk [Tka].

Lemma 3.1. *Let X be a first-countable connected LOTS. Then:*

- (1) *There exists an absolute set $A \subset X$ for X .*
- (2) *For two disjoint nowhere dense sets $S_1, S_2 \subset X$, there exists an absolute set A for X that contains S_1 and does not intersect S_2 .*

Proof. It is enough to prove (2). We may assume that X is infinite (if X is a single-element set, the empty set is absolute for X). First, notice that the cardinality of X is 2^{ω_0} . Indeed, any connected LOTS is locally compact. By Proposition 2.7, X can be covered by ω_1 many first countable compacta. Now apply Arhangel'skii's theorem [Ar1].

Let $\mathcal{J}$ be the set of all nonempty open intervals that have no common elements with $S_1 \cup S_2$. Enumerate $\mathcal{J} = \{J_\alpha : \alpha < 2^{\omega_0}\}$. Let $\mathcal{P}$ be the set of all possible pairs (C, D) , where C and D are countable sets in X whose closures are disjoint and have no common elements with $S_1 \cup S_2$.

For each $(C, D) \in \mathcal{P}$, let $G_{C,D}$ be the set of all one-to-one mappings $g_{C,D}$ from C onto D that possess the following property.

Property $g_{C,D}$. *There exist 2^{ω_0} mutually disjoint sets $\{c, d\}$, where $c \in \overline{C} \setminus C$ and $d \in \overline{D} \setminus D$, such that for some sequence $\{c_n : c_m \neq c_k \text{ for } m \neq k\} \subset C$ we have $c_n \rightarrow c$ and $d \in \overline{\{g_{C,D}(c_n)\}}$.*

Enumerate $G = \bigcup_{(C,D) \in \mathcal{P}} G_{C,D} = \{g_\alpha : \alpha < 2^{\omega_0}\}$. Inductively, we will construct two disjoint sets $A = S_1 \cup \{a_\alpha : \alpha < 2^{\omega_0}\} \cup \{a'_\alpha : \alpha < 2^{\omega_0}\}$ and $B = \{b_\alpha : \alpha < 2^{\omega_0}\} \cup \{b'_\alpha : \alpha < 2^{\omega_0}\}$ with the following two properties.

(*) *For every $g_\alpha = g_{C,D} \in G$, there exists a sequence $\{c_n : c_m \neq c_k \text{ for } m \neq k\} \subset C$ such that $c_n \rightarrow a_\alpha \notin C$ and $b_\alpha \in \overline{\{g_{C,D}(c_n)\}} \setminus D$.*

(**) *For every α , both a'_α and b'_α belong to J_α .*

Construction. Assume a_β, b_β are defined for $\beta < \alpha < 2^{\omega_0}$.

Step α . Consider $g_\alpha = g_{C,D} \in G$. Let a_α and b_α be any two elements in $X \setminus (S_1 \cup S_2)$, distinct from all a_β, a'_β and b_β, b'_β for $\beta < \alpha$, that play the role of c and d in Property $g_{C,D}$. Also, take arbitrary distinct $a'_\alpha, b'_\alpha \in J_\alpha$ that are distinct from all a_β, b_β for $\beta \leq \alpha$ and from all a'_β, b'_β for $\beta < \alpha$. (This can be done because the set of already-chosen points has cardinality less than 2^{ω_0} .)

The set A contains S_1 by the definition. To see why $A \cap S_2 = \emptyset$, notice that any element of A is either in S_1 , or J_α for some α , or $\overline{C}$ for some $(C, D) \in \mathcal{P}$. Every J_α has no common elements with $S_1 \cup S_2$ by the definition. By the lemma's assumptions, S_1 and S_2 are disjoint. Also, by the definition of $\mathcal{P}$ neither $\overline{C}$ nor $\overline{D}$ has common elements with S_2 .

So now we must show that A is absolute for X . Let a continuous mapping f cleave X over a LOTS along A . Assume f is not one-to-one. By Proposition 2.2, there exist two disjoint nontrivial closed intervals I and J that do not intersect $S_1 \cup S_2$ and whose images under f coincide. We may assume that both I and J are compacta.

Notice that due to (**), both A and $X \setminus A$ are dense. Since $f(A) \cap f(X \setminus A) = \emptyset$, we have that $|f(I)| = 2^{\omega_0}$.

By Statement 1.8, there exists a countable subset $Y \subset f(I)$ with closure of cardinality 2^{ω_0} . Take countable $C \subset I$ and $D \subset J$ such that $f(C) = f(D) = Y$. We can also require that f is one-to-one on both C and D . Define a correspondence $g : C \rightarrow D$ by $g(c) = f^{-1}(f(c)) \cap D$. Since f is one-to-one on both C and D and $f(C) = f(D) = Y$, f is a well-defined one-to-one function of C onto D .

Let us show that $g \in G$. Take any $c \in \overline{C} \setminus C$ such that $f(c) \notin Y$. Since X is first countable, there exists $\{c_n : c_m \neq c_k \text{ for } m \neq k\} \subset C$ such that $c_n \rightarrow c$. Consider $\{g(c_n)\}$. Since $f(g(c_n)) = f(c_n)$ and $f(c_n) \rightarrow f(c)$, there exists $d \in \overline{\{g(c_n)\}}$ such that $f(d) = f(c)$ (here we use compactness of J). Since $f(c) \notin Y$, we have that $d \in \overline{D} \setminus D$. To finish verification of Property $g_{C,D}$, notice that there are 2^{ω_0} elements in $\overline{C}$ whose images are not in Y .

Thus, $g = g_\alpha \in G$ for some α . Therefore, there exists a sequence $\{c_n : c_m \neq c_k \text{ for } m \neq k\} \subset C$ such that $c_n \rightarrow a_\alpha \notin C$ and $b_\alpha \in \overline{\{g(c_n)\}} \setminus D$. By the definition of g , $f(b_\alpha) \in \overline{\{f(c_n)\}}$. Since f is one-to-one on D , $f(b_\alpha)$ is an accumulation point of $\{f(c_n)\}$. But the latter set has only one accumulation point, namely $f(c)$, since $\{f(c_n)\}$ is a non-constant convergent sequence. Therefore, $f(a_\alpha) = f(b_\alpha)$, which contradicts the choice of f . This concludes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Let X be a linearly ordered continuum with an open dense set of points of countable character. Then there exists an absolute set $A \subset X$ for X .*

Proof. We say that a set O is a countable character component if it is a maximal open interval containing points only of countable character. Let $\mathcal{O} = \{O_\alpha\}$ be the maximal family of mutually disjoint countable character components. Obviously, $\bigcup \mathcal{O}$ is open and dense in X . For each $O_\alpha \in \mathcal{O}$, we perform the following steps 1 and 2.

1. Let a_α, b_α be the left and right end-points of $\overline{O}_\alpha$. By Proposition 2.7, $\chi_r(a_\alpha, \overline{O}_\alpha)$ does not exceed ω_1 . If $\chi_r(a_\alpha, \overline{O}_\alpha) = \omega_1$, fix an ω_1 -sequence $r_\alpha \subset O_\alpha \setminus \{b_\alpha\}$ converging to a_α from the right. Similarly, if $\chi_l(b_\alpha, \overline{O}_\alpha) = \omega_1$, fix an ω_1 -sequence $l_\alpha \subset O_\alpha \setminus \overline{r}_\alpha$ converging to b_α from the left. If $\chi_r(a_\alpha, \overline{O}_\alpha) < \omega_1$, let $r_\alpha = \emptyset$. Similarly, if $\chi_l(b_\alpha, \overline{O}_\alpha) < \omega_1$, let $l_\alpha = \emptyset$.

2. Fix an absolute for O_α set $A_\alpha \subset O_\alpha$ that does not intersect l_α but contains r_α (Lemma 3.1).

Define a set B as follows: $b \in B$ if and only if b is the right end-point for some $O \in \mathcal{O}$, and is an accumulation point for the family $\mathcal{O}$ (i.e., a limit point of $\bigcup \mathcal{O}$ that is not a limit point of any single element of $\mathcal{O}$). Let us show that $A = \bigcup \{A_\alpha\} \cup B$ is an absolute set for X .

Let a continuous mapping f cleave X over a LOTS along A . Assume f is not one-to-one. By the choice of A , $f|_O$ is one-to-one for any $O \in \mathcal{O}$. Also, by Proposition 2.10, $f|_{\overline{O}}$ is an embedding.

By Proposition 2.2 and density of $\bigcup \mathcal{O}$, there exists $x \in A_\alpha \subset O_\alpha \in \mathcal{O}$ that collapses with some element from the right. Let $y = \inf\{z \in X : z > x, f(z) = f(x)\}$. Since O_α is open and f is one-to-one on O_α , y is strictly greater than x . By the choice of f , $y \in O_\beta \in \mathcal{O}$ or y is the right end-point of O_β for some $\beta \neq \alpha$. Assume f preserves the order of O_α . By Proposition 2.6, f reverses O_β .

Since f is one-to-one on each $O \in \mathcal{O}$, no nontrivial interval is squeezed into a point. Also, by Proposition 2.3, f preserves the character of each non-end-point of X . Since any neighborhood of an end-point of O_α or O_β contains points of uncountable character, $b_\alpha(a_\beta)$ cannot collapse with any element in $O_\beta(O_\alpha)$. Thus,

$f(b_\alpha) = f(a_\beta)$. If b_α is an accumulation point for the family $\mathcal{O}$ then, by the latter equality, $f(A)$ intersects $f(X \setminus A)$, which contradicts the choice of f (recall that a_β is the left end-point of O_β and, therefore, cannot be an accumulation point for $\mathcal{O}$ and the right end-point for some O at the same time). If b_α is not an accumulation point, then it is the boundary point of two elements in $\mathcal{O}$, and, due to maximality of each O , cannot be of countable character. Therefore, there exists an ω_1 -sequence (see Proposition 2.7) that converges to b_α from the right or left. Assume this sequence is l_α . Since f reverses O_β and preserves characters, there exists an ω_1 -sequence r_β that converges to a_β from the right. Obviously, under f , images of these sequences intersect. But l_α is not in A while r_β is in A . A similar argument works if the ω_1 -sequence (some r_γ) converges to b_α from the right. This contradiction implies that f is one-to-one. $\square$

Lemma 3.3. *Let X be a connected LOTS, and let A be a dense set that does not contain end-points and possesses the following property.*

(*) $\chi_l(x, X) < \chi_r(x, X)$ for all $x \in A$.

Then:

(1) *The set A is absolute for X .*

(2) *A set A^+ which is the union of A and an arbitrary discrete-in-itself set of elements of countable character in X is absolute for X as well.*

Proof. It is enough to prove (2) only. Let a continuous mapping f cleave X over a LOTS along A^+ . Assume f is not one-to-one. First, notice that no nontrivial open interval I collapses into a point. Indeed, there is a decreasing ω_0 -sequence converging to $x \in I$. Since x does not satisfy (*), it is not in A . The set of such x 's is dense, while $A^+ \setminus A$ is not. Therefore, each nontrivial open interval contains elements from both A^+ and its complement.

By Proposition 2.2, there are two disjoint nontrivial intervals in X whose images coincide. Since A is dense, there exists $a \in A$ that is glued up under f with an element to the right of a . Let $b = \inf\{x \in X : x > a, f(x) = f(a)\}$. Then $b \geq a$.

If $b = a$, then there exists a decreasing τ -sequence $\{x_\alpha : f(x_\alpha) = f(a), \alpha < \tau\}$ that converges to a from the right and $\tau > \omega_0$ (see (*)). Since $\tau > \omega_0$, the τ -sequence contains an ω_0 -sequence $\{y_1 = x_{\omega_0}, y_2 = x_{\omega_0 + \omega_0}, \dots\}$ that converges to $y = x_{\omega_0^2}$. Because of right-cofinality, no term y_n can satisfy (*), and, therefore, is not in A . For the same reason, $y \notin A$ either. Since $A^+ \setminus A$ is discrete in itself, it cannot contain all y_n 's and y . So, either some y_n or y does not belong to A^+ . But that is impossible, because $f(y_n) = f(y) = f(a)$ and f cleaves X along A^+ .

If $b > a$, then, due to the choice of f , $b \in A^+$. By Proposition 2.1, $f(b)$ is an end-point of $f([a, b])$. Also, by Proposition 2.5, $\chi_r(a, X) = \chi_l(b, X)$. Then, $b \in A$. By Proposition 2.4, $\chi_l(a, X) = \chi_r(b, X)$. As a result of these two equalities, $\chi_l(b, X) > \chi_r(b, X)$, which contradicts (*). $\square$

Lemma 3.4. *Let X be a connected LOTS with the following properties:*

1. *There exists an open dense $O \subseteq X$ such that $\chi_l(x, X) = \chi_r(x, X)$ for all $x \in O$.*

2. *The set of elements of uncountable character is dense in X .*

Then:

(1) *There exists a set $A \subset O$ with the following properties:*

A1. *The set of all $x \in A$ with $\chi_l(x, X) = \chi_r(x, X) = \omega_1$ is dense in X .*

A2. *All x in A are of character either ω_0 or ω_1 .*

- A3. For each $x \in A$ of character ω_1 there exist an ω_1 -sequence $r_x \subset A$ converging to x from the right and an ω_1 -sequence $l_x \subset X \setminus A$ converging to x from the left.
 (2) The set A in (1) is absolute for X .

Proof. (1) First, notice that any nonempty open interval I in X contains an x with $\chi_l(x, X) = \chi_r(x, X) = \omega_1$ (ω_0). Indeed, due to property 2 and density of O , I contains a τ -sequence ($\tau > \omega_0$) whose closure is in $O \cap I$. This sequence contains an $\omega_1(\omega_0)$ -subsequence converging to some $x \in I \cap O$ from the right (left). Then $\chi_r(x, X) = \omega_1(\omega_0)$. Due to the definition of O we have the required equality.

Let a, b be the end-points of a linear ordered compactification of X . Inductively, we will define the sets A_α and B_α such that $A = \bigcup A_\alpha$ and the set $B = \bigcup B_\alpha$ contains all the l_x 's defined in property A3.

Suppose A_β and B_β are defined for all $\beta < \alpha$.

For each pair $\{c, d\}$, where $c < d$, satisfying the following requirements 1, 2:

1. $\{c, d\} \in (\bigcup_{\beta < \alpha} (A_\beta \cup B_\beta) \cup \{a, b\})$,
2. $(c, d) \cap (\bigcup_{\beta < \alpha} (A_\beta \cup B_\beta) \cup \{a, b\}) = \emptyset$ (that is, c, d are neighbors),

pick up an arbitrary point $x \in (c, d) \cap O$ of character ω_1 . Choose two disjoint ω_1 -sequences $r_x, l_x \subset (c, d) \cap O$ converging to x from the right and left, respectively, and consisting of points of countable character.

If no such c, d exist, construction stops.

Let $A_\alpha = (\bigcup_{\beta < \alpha} A_\beta) \cup (\bigcup \{\overline{r_x}\})$ and $B_\alpha = (\bigcup_{\beta < \alpha} B_\beta) \cup (\bigcup \{l_x\})$.

Let $A = \bigcup A_\alpha$ and $B = \bigcup B_\alpha$. Properties A2, A3 as well as the property of B follow from the construction. Let us show that A possesses property A1.

Take an arbitrary nonempty open interval J . Suppose, $J \cap (A \cup B) = \emptyset$. Take $c \in \overline{A \cup B \cup \{a\}}$ the closest to J from the left and $d \in \overline{A \cup B \cup \{b\}}$ the closest to J from the right. The pair $\{c, d\}$ satisfies the construction requirements, and therefore, construction must continue. Thus, $A \cup B$ is dense. During construction, from each interval we pick up not more than a nowhere dense set of elements for A_α and B_α . Once we pick up at least two points from J , at the next step there exists a pair $\{c, d\}$ satisfying the construction requirements. At that step we pick up an $x \in (c, d)$ for A with $\chi_l(x, X) = \chi_r(x, X) = \omega_1$.

(2) Now, let us show that A is absolute for X . Let f cleave X over a LOTS along A . Assume f is not one-to-one. Due to A1 and Proposition 2.2, there exists $x \in A$ with $\chi_l(x, X) = \chi_r(x, X) = \omega_1$ that collapses with some element from its left side. Let y be the closest point from the left of x that has the same image as x .

If no such y exists, then some ω_1 -sequence converging to x from the left, and, therefore, the most part of l_x are squeezed into $f(x)$. But l_x is not in A , while x is.

If y exists, then $y \in A$ (by the choice of f). Suppose y is of countable character. By Proposition 2.1, $f(x)$ is an end-point of $f([y, x])$. Then, by Proposition 2.5, there exists an interval that is squeezed into a point under f . But any interval contains elements from both A and its complement (this follows from the definition of B and property A1). If y is of character ω_1 , then the images of r_y and l_x intersect (recall that r_y, l_x are both ω_1 -sequences). But $r_y \subset A$ while $l_x \subset B \subset X \setminus A$. Since all the possibilities bring us to contradictions, we conclude that f is one-to-one. $\square$

4. THEOREM

Theorem 4.1. *Let X be a linearly ordered continuum cleavable over a LOTS L . Then, X is embeddable to L .*

Proof. Let us define families $\mathcal{B}, \mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ as follows. *Family $\mathcal{B}$* consists of all maximal nontrivial closed intervals in X containing an open dense set of elements of countable character in X . *Family $\mathcal{C}$* consists of all maximal open intervals in X containing a dense set of elements x with $\chi_l(x, X) > \chi_r(x, X)$. *Family $\mathcal{D}$* consists of all maximal open intervals in $X \setminus \overline{\bigcup \mathcal{C}}$ containing a dense set of elements x with $\chi_l(x, X) < \chi_r(x, X)$.

Family $\mathcal{D}$ possesses the following properties:

1. *For each $D \in \mathcal{D}$, the set of all elements x with $\chi_l(x, X) > \chi_r(x, X)$ is nowhere-dense in D .*

This follows from the maximality of Family $\mathcal{C}$.

2. *Each $D \in \mathcal{D}$ contains a dense set of elements of countable character.*

Indeed, take any open interval and a decreasing countable sequence inside the interval. Let the sequence converge to x . Due to property 1, we can pick up a sequence in such a manner that $\chi_l(x, X) \leq \chi_r(x, X) = \omega_0$.

Family $\mathcal{E}$ consists of all maximal open intervals in $X \setminus \overline{\bigcup (\mathcal{B} \cup \mathcal{C} \cup \mathcal{D})}$.

Family $\mathcal{E}$ possesses the following property.

Each $E \in \mathcal{E}$ contains an open dense set of elements x with $\chi_l(x, X) = \chi_r(x, X)$, and a dense set of elements of uncountable character.

This follows from the maximality of the above families.

Let $\mathcal{S} = \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$. First, notice that elements in $\mathcal{S}$ are disjoint due to maximality of each family, and their union is dense in X . Also, the closures of two elements from the same family are disjoint.

For each element $B \in \mathcal{B}$, $C \in \mathcal{C}$, and $E \in \mathcal{E}$ fix absolute sets A_B, A_C, A_E in accordance with Lemma 3.2, Lemma 3.3 (1), and Lemma 3.4 (2).

For each $D \in \mathcal{D}$, fix an absolute set A_D from Lemma 3.3 (2), so that A_D contains a discrete-in-itself set of points of countable character accumulating to endpoints of $\overline{D}$. This can be done, thanks to property 2 of family $\mathcal{D}$.

Let $A = \bigcup \{A_S : S \in \mathcal{S}\}$. Let us show that any continuous mapping f that cleaves X over L along A is an embedding.

By the choice of each A_S , $f|_S$ is one-to-one for each $S \in \mathcal{S}$. Also, by Proposition 2.10, $f|_{\overline{S}}$ is an embedding.

Suppose f is not one-to-one. Then there exist two sets S_1, S_2 whose images intersect. This follows from the density of $\bigcup \mathcal{S}$ and Proposition 2.2.

We can assume that S_1 is not from $\mathcal{B}$. Indeed, let $x \in B \in \mathcal{B}$ collapse with some element. Take y from the right (left) of x that has the same image as x . By Proposition 2.8, any element in $[x, y]$, except maybe two points, is glued up with some other element. Since $f|_B$ is one-to-one, $[x, y] \setminus B$ is a nontrivial interval. Since no two elements of $\mathcal{B}$ are neighbors, there exists $S_1 \in \mathcal{S} \setminus \mathcal{B}$ such that $S_1 \cap (x, y) \neq \emptyset$ (recall that each $S \in \mathcal{S} \setminus \mathcal{B}$ is open). Fix this S_1 .

Without loss of generality, f preserves the order of S_1 . There exists S_2 whose image intersects the image of S_1 by a nontrivial interval and f reverses the order of S_2 . Indeed, let $x \in A_{S_1}$ collapse with some element (such an x exists, since A_{S_1} is dense in S_1). Let y be the closest element to x from the right (left) such that $f(x) = f(y)$ (such a y exists, since S_1 is open and f is one-to-one on S_1). By the

choice of f , y belongs to some $S_2 \in \mathcal{S}$. Since S_1 is open, $f(S_1) \cap f(S_2)$ contains an interval. This means that S_2 cannot be from $\mathcal{B}$, since no interval of S_1 is first countable. Thus, $S_2 \in \mathcal{S} \setminus \mathcal{B}$. By Proposition 2.6, f reverses S_2 . Fix this S_2 .

Now, we need only to show that the rest of the possible choices for S_1 and S_2 lead to contradictions.

Case 1. $S_1 \in \mathcal{E}$, $S_2 \in \mathcal{C} \cup \mathcal{D}$. As mentioned earlier, $f|_{S_1}$ and $f|_{S_2}$ are embeddings. But no open interval of S_1 is homeomorphic to any open part of S_2 . This follows from the definitions of the families.

Case 2. $S_1, S_2 \in \mathcal{C}$. Due to density of the absolute set, there exist $a_1 \in A_{S_1}$ and $a_2 \in A_{S_2}$ with $f(a_1) = f(a_2)$. Since f preserves the order of S_1 , $\chi_l(f(a_1), f(X)) > \chi_r(f(a_1), f(X))$. Since the order of S_2 is reversed, $\chi_l(f(a_2), f(X)) < \chi_r(f(a_2), f(X))$. These two inequalities contradict $f(a_1) = f(a_2)$. (For the inequalities, see Lemma 3.3, (1).)

Case 3. $S_1, S_2 \in \mathcal{D}$. The argument is similar to Case 2.

Case 4. $S_1, S_2 \in \mathcal{E}$. Due to density of the absolute set, there exist $a_1 \in A_{S_1}$ and $a_2 \in A_{S_2}$ such that $f(a_1) = f(a_2)$ and $\chi_l(f(a_i), f(X)) = \chi_r(f(a_i), f(X)) = \omega_1$ (see Lemma 3.4, (1), Property A1). Suppose $a_1 < a_2$. Since the order of S_1 is preserved while that of S_2 is reversed, $f(r_{a_2}) \cap f(l_{a_1}) \neq \emptyset$ (recall that r_{a_2}, l_{a_1} are ω_1 -sequences). But r_{a_2} is in A while l_{a_1} is in the complement of A , which contradicts the choice of f (see Lemma 3.4, (1), Property A3).

Case 5. $S_1 \in \mathcal{D}$, $S_2 \in \mathcal{C}$. Assume S_2 is to the right of S_1 . By the choice of S_1, S_2 , there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that $f(s_1) = f(s_2)$. By Proposition 2.8, any element in $[s_1, s_2]$, except maybe two points, collapses with some other element. Since the right end-point of $\overline{S_1}$ is in $[s_1, s_2]$, there exists $a_1 \in A_{S_1} \cap [s_1, s_2]$ of countable character that collapses with some other element (see the definition of A_D). Take a_2 the closest to a_1 from the right (left) such that $f(a_1) = f(a_2)$. Such an a_2 exists, since S_1 is open and f is an embedding on S_1 . By the choice of f , a_2 must belong to A_{S_3} for some $S_3 \in \mathcal{S}$. By Proposition 2.6, f reverses S_3 . Since a_1 is of countable character, a_2 must be of countable character as well. Therefore, S_3 is not in $\mathcal{C}$. The other cases have already been eliminated.

Thus, f is one-to-one, and, therefore, an embedding (recall that f is from a compactum). $\square$

Theorem 4.2. *Any continuum X cleavable over a LOTS L is embeddable into L .*

Proof. By the theorem in [Bu], any continuum cleavable over a LOTS is a LOTS itself. Now, apply Theorem 4.1. $\square$

REFERENCES

- [Ar1] A. V. Arhangel'skii, *The power of bicomacta with first axiom of countability*, Dokl. Akad. Nauk SSSR, 187 (1969), 967-970 (in Russian); English transl., Soviet Math. Dokl. 10 (1969), 951-955. MR **40**:4922
- [Ar2] A. V. Arhangel'skii, *The general concept of cleavability of a topological space*, Topology Appl., 44 (1992), 27-36. MR **93k**:54047
- [Ar3] A. V. Arhangel'skii, *Cleavability over reals*, Topology Appl., 48 (1992), 163-178. MR **94a**:54041
- [Ar4] A. V. Arhangel'skii, *On cleavability of topological spaces over R, R^n , and R^ω* , Ann. New York Acad. Sci., 659 (1992), 18-28. MR **98g**:54008
- [Ar5] A. V. Arhangel'skii, *A survey of cleavability*, Topology Appl., 54 (1993), 141-163. MR **95c**:54015

- [Ar6] A. V. Arhangel'skii, *Some recent achievements and open problems in general topology*, Russian Math. Surveys, 52:5(1997), 929-953. MR **99b**:54001
- [Bal] Z. Balogh, *There is a paracompact Q -set space in ZFC*, Proc. Amer. Math. Soc., 126:6 (1998), 1827-1833. MR **98g**:54084
- [Bu] R. Z. Buzyakova, *Splittability of compacta over linearly ordered spaces and over the class of all linearly ordered spaces*, Moscow Univ. Math. Bull., 51:3 (1996), 44-45. MR **98b**:54044
- [Tka] V. V. Tkachuk, *A note on splittable spaces*, Comment. Math. Univ. Carolinae, 33(1992), 551-555. MR **94a**:54071
- [Yas] I. V. Yashchenko, *Embeddings into R^n and R^ω and cleavability*, Moscow Univ. Math. Bull., 47:2(1992).
- [Eng] R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann, Berlin, revised ed., 1989. MR **91c**:54001

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