FUNCTIONAL ANALYSIS PROOFS OF ABEL'S THEOREMS

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ABSTRACT. We give alternative proofs to the classical theorems of Abel, using the concept of Berezin symbol.

1.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. The sequence $\{a_n\}_{n=0}^{\infty}$ is Abel convergent (written (A) convergent) to a if the limit

$$\lim_{t \rightarrow 1^-} (1-t) \sum_{n=0}^{\infty} a_n t^n = a$$

exists. The series $\sum_{n=0}^{\infty} a_n$ is (A) convergent to L if the sequence of partial sums $\{s_n\}_{n=0}^{\infty}$ (where $s_n \stackrel{\text{def}}{=} \sum_{k=0}^n a_k$) is (A) convergent to L .

The following famous results are due to Abel (e.g., see [1], [2]).

Theorem 1 (Theorem of Abel). *If $\{a_n\}_{n=0}^{\infty}$ converges to a , then $\{a_n\}_{n=0}^{\infty}$ (A) converges to a .*

Theorem 2 (Theorem of Abel). *If the series $\sum_{n=0}^{\infty} a_n$ converges to L , then $\sum_{n=0}^{\infty} a_n$ is (A) convergent to L .*

This paper presents functional analysis proofs of these results. To give our proofs, we first define what is meant by a Berezin symbol.

2.

Let H^2 denote the Hardy space of functions analytic on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a general bounded operator A on the Hardy space, the Berezin symbol of A is (see [3]) the function $\tilde{A}$ defined by

$$\tilde{A}(\lambda) = \left(A \hat{k}_{\lambda}, \hat{k}_{\lambda} \right), \quad \lambda \in \mathbb{D},$$

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where $\hat{k}_\lambda$ is the normalized reproducing kernel of H^2 . The function $\hat{k}_\lambda$ is defined by

$$\hat{k}_\lambda(z) = \frac{\sqrt{1-|\lambda|^2}}{1-\bar{\lambda}z}$$

for $z \in \mathbb{D}$ and has the property that $(f, \hat{k}_\lambda) = \sqrt{1-|\lambda|^2} f(\lambda)$, for $f \in H^2$, and this obviously approaches 0 for $f \in H^\infty$ (the space of all bounded analytic functions on $\mathbb{D}$), and hence for all $f \in H^2$, whenever $|\lambda| \rightarrow 1^-$. Thus, the kernels $\hat{k}_\lambda$ converge weakly to zero as λ approaches the unit circle $\partial\mathbb{D}$ (i.e., the Hardy space H^2 is standard [3]). Then we have that if A is a compact operator on H^2 , then $\tilde{A}(\lambda) \rightarrow 0$ as $\lambda \rightarrow \partial\mathbb{D}$. In this sense, the Berezin symbol of a compact operator on H^2 vanishes on the boundary.

3.

Proof of Theorem 1. Let us consider the diagonal operator $\mathcal{D}_{\{a_n\}}$ on H^2 defined by

$$\mathcal{D}_{\{a_n\}} z^k = a_k z^k, \quad k = 0, 1, 2, \dots$$

Since $\{a_n\}$ is a bounded sequence, $\mathcal{D}_{\{a_n\}}$ is a bounded operator on H^2 . We now calculate the Berezin symbol of an operator $\mathcal{D}_{\{a_n\}}$. We have

$$\begin{aligned} \tilde{\mathcal{D}}_{\{a_n\}}(\lambda) &= (\mathcal{D}_{\{a_n\}}, \hat{k}_\lambda, \hat{k}_\lambda) = \sqrt{1-|\lambda|^2} \left(\mathcal{D}_{\{a_n\}} \sum_{k=0}^{\infty} \bar{\lambda}^k z^k, \hat{k}_\lambda \right) \\ &= \sqrt{1-|\lambda|^2} \left(\sum_{k=0}^{\infty} \bar{\lambda}^k \mathcal{D}_{\{a_n\}} z^k, \hat{k}_\lambda \right) \\ &= \sqrt{1-|\lambda|^2} \left(\sum_{k=0}^{\infty} \bar{\lambda}^k a_k z^k, \hat{k}_\lambda \right) \\ &= (1-|\lambda|^2) \sum_{k=0}^{\infty} a_k |\lambda|^{2k}. \end{aligned}$$

Thus,

$$\tilde{\mathcal{D}}_{\{a_n\}}(\lambda) = (1-|\lambda|^2) \sum_{k=0}^{\infty} a_k |\lambda|^{2k}, \quad \lambda \in \mathbb{D}$$

(i.e., $\tilde{\mathcal{D}}_{\{a_n\}}$ is a radial function, $\tilde{\mathcal{D}}_{\{a_n\}}(\lambda) = \tilde{\mathcal{D}}_{\{a_n\}}(|\lambda|)$), which yields

$$(1) \quad \tilde{\mathcal{D}}_{\{a_n\}}(\sqrt{t}) = (1-t) \sum_{k=0}^{\infty} a_k t^k, \quad 0 < t < 1.$$

Then from (1) we have

$$(1-t) \sum_{k=0}^{\infty} a_k t^k = (1-t) \sum_{k=0}^{\infty} (a_k - a) t^k + a(1-t) \sum_{k=0}^{\infty} t^k = \tilde{\mathcal{D}}_{\{a_n-a\}}(\sqrt{t}) + a.$$

Since by the condition of the theorem $a_n - a \rightarrow 0$ as $n \rightarrow \infty$, $\tilde{\mathcal{D}}_{\{a_n-a\}}$ is a compact operator on H^2 . Hence, its Berezin symbol vanishes on the boundary, i.e.,

$$\lim_{t \rightarrow 1^-} \tilde{\mathcal{D}}_{\{a_n-a\}}(\sqrt{t}) = 0.$$

Then the last equality yields

$$\lim_{t \rightarrow 1^-} (1-t) \sum_{k=0}^{\infty} a_k t^k = a,$$

which completes the proof. $\square$

4.

Proof of Theorem 2. The argument that was used to prove Theorem 1 can easily be modified to prove the equality

$$(2) \quad \tilde{\mathcal{D}}_{\{s_n\}}(\sqrt{t}) = \sum_{k=0}^{\infty} a_k t^k, \quad 0 < t < 1.$$

Formula (2) implies that for each $t \in (0, 1)$ the series $\sum_{k=0}^{\infty} a_k t^k$ is convergent. On the other hand,

$$\mathcal{D}_{\{s_n\}} = LI + \mathcal{D}_{\{s_n-L\}},$$

where the diagonal operator $\mathcal{D}_{\{s_n-L\}}$ is compact since by the hypothesis of the theorem $s_n - L \rightarrow 0$ as $n \rightarrow \infty$, and therefore from the formula (2) we have

$$\begin{aligned} \lim_{t \rightarrow 1^-} \sum_{k=0}^{\infty} a_k t^k &= \lim_{t \rightarrow 1^-} \tilde{\mathcal{D}}_{\{s_n\}}(\sqrt{t}) \\ &= \lim_{t \rightarrow 1^-} \left(L + \tilde{\mathcal{D}}_{\{s_n-L\}}(\sqrt{t}) \right) \\ &= L + \lim_{t \rightarrow 1^-} \tilde{\mathcal{D}}_{\{s_n-L\}}(\sqrt{t}) \\ &= L, \end{aligned}$$

which means that the series $\sum_{k=0}^{\infty} a_k (A)$ converges to L . The theorem is proved. $\square$

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