NOTE ON THE STABILITY OF PRINCIPAL BUNDLES

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ABSTRACT. We compare various notions of stability for principal bundles, and show that over a compact Riemann surface of genus greater than 2, there exist principal $\mathrm{SL}(2)$ -bundles that are Ad-stable.

1. INTRODUCTION

The moduli space of stable principal G -bundles over a compact Riemann surface was constructed by A. Ramanathan in his thesis [6]. The method was generalized recently by the first named author in [5], where he gave a construction of a quasi-projective moduli scheme for principal bundles over a projective scheme of arbitrary dimension. However, the moduli constructed in [5] may fail to include some bundles that we would like it to have. For instance, over a compact Riemann surface, the stability condition used in [5] is strictly stronger than the stability employed by Ramanathan. It has thus been of concern to the authors whether a principal bundle that is stable in the sense of [5] exists at all. The purpose of this article is to give an answer to this question, and to give a detailed account of the relation between the various stability conditions.

In section 2, we introduce the stability conditions for principal bundles used by A. Ramanathan, D. Hyeon (*Ad-stability*), and D. Mumford (for $G = \mathrm{GL}(n)$). We give an elementary proof that for $G = \mathrm{GL}(n)$, the stability condition defined by Ramanathan is equivalent to the slope stability of Mumford. We also show that the Ad-stability is strictly stronger than Ramanathan's stability, and that there are no Ad-stable principal G -bundles if G is not semisimple. (This is essentially proved in [6].)

In section 3, we briefly recall the main result in [4], and using it, construct examples of Ad-stable $\mathrm{SL}(2)$ -bundles. Furthermore, we give an example of an $\mathrm{SL}(2)$ -bundle that is stable in the sense of Ramanathan, but is not Ad-stable.

Unless otherwise stated, X will be a compact Riemann surface of genus g . The letter G will be reserved to denote a connected reductive complex holomorphic Lie group, whose Lie algebra will be denoted $\mathfrak{g}$. For any vector bundle E over X , the degree $\deg(E)$ of E is defined by the Riemann-Roch formula

$$\chi(E) = \deg(E) + \mathrm{rk}(E) \cdot (1 - g),$$

and the slope $\mu(E)$ is the quantity $\frac{\deg(E)}{\mathrm{rk}(E)}$.

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2. STABILITY NOTIONS FOR PRINCIPAL BUNDLES

Definition 1. (1) Let E be a principal G -bundle over X , and $\rho : G \rightarrow H$ a homomorphism of holomorphic complex Lie groups. Let G act diagonally on $E \times H$ by the natural right action on E , and through ρ on H . It is well known that the quotient bundle $(E \times H)/G$ exists as a principal H -bundle, and we denote it by $E_\rho(H)$, or by $E(H)$ when the representation used is clear. In a similar fashion, if F is any G -space, let $E(F)$ denote the fibre bundle $(E \times F)/G$ with fibre F , where G acts diagonally on $E \times F$.

(2) A reduction σ of the structure group of a principal G -bundle E to a subgroup P of G is a section $\sigma : X \rightarrow E/P$ of the fibre bundle $E/P = E(G/P)$. Then σ^*E is a P -bundle over X and there is a natural isomorphism of G -bundles $\sigma^*E(G) \simeq E$.

Ramanathan's definition of stability for principal bundles can be phrased in many different ways (e.g., Theorem 2.2. of [2], and [1]). The original version reads as follows.

Definition 2. A principal G -bundle E over X is said to be *semistable* if for every reduction $\sigma : X \rightarrow E/P$ of the structure group to a maximal parabolic $P \subset G$, the following inequality holds:

$$\deg \sigma^*T_{E/P} \geq 0.$$

We call E *stable* if the above inequality is always strict.

Here, $T_{E/P}$, the *tangent bundle over E/P along the fibre of E* , is defined to be the vector bundle $E(\mathfrak{g}/\mathfrak{p}) = (E \times (\mathfrak{g}/\mathfrak{p}))/P$ over E/P associated to the P -bundle $E \rightarrow E/P$ and the P action on $\mathfrak{g}/\mathfrak{p}$ ($\mathfrak{p}$ is the Lie algebra of P) induced by the adjoint representation.

This stability condition is consistent with the condition defined by Mumford, in the case $G = \mathrm{GL}(n)$. This is not hard to see, but we include an elementary proof for the sake of completeness.

Let $P \subset \mathrm{GL}(n)$ be a parabolic subgroup consisting of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in \mathrm{GL}(r)$ and $C \in \mathrm{GL}(n-r)$, with respect to a fixed basis of $\mathbb{C}^n$. Let $\sigma : X \rightarrow E/P$ be a reduction of the structure group, and let

$$\begin{pmatrix} h_{\alpha\beta} & b_{\alpha\beta} \\ 0 & q_{\alpha\beta} \end{pmatrix}$$

be the transition functions for the P -bundle σ^*E with respect to a trivializing cover $\{U_\alpha\}$. Let F be the rank r subbundle of $E(\mathbb{C}^n)$ corresponding to $(h_{\alpha\beta})$. Then the quotient bundle $E(\mathbb{C}^n)/F$ corresponds to $(q_{\alpha\beta})$, and we have the following natural isomorphism.

Proposition 1.

$$\sigma^*T_{E/P} \simeq F^* \otimes (E(\mathbb{C}^n)/F).$$

Proof. The adjoint action of P on the Lie algebra $\mathfrak{gl}(n)$ is given by the matrix multiplication

$$\begin{pmatrix} h & b \\ 0 & q \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} h^{-1} & b' \\ 0 & q^{-1} \end{pmatrix} = \begin{pmatrix} * & * \\ q \cdot C \cdot h^{-1} & * \end{pmatrix}.$$

Here, $\begin{pmatrix} h^{-1} & b' \\ 0 & q^{-1} \end{pmatrix}$ is the inverse matrix of $\begin{pmatrix} h & b \\ 0 & q \end{pmatrix}$. The places with $*$ are covered by $\mathfrak{p}$, and the lower left corner is the only part relevant to $\mathfrak{g}/\mathfrak{p}$. Hence we conclude that the adjoint action of P on $\mathfrak{g}/\mathfrak{p}$ is given by

$$\left(\begin{pmatrix} h & b \\ 0 & q \end{pmatrix}, C\right) \mapsto q \cdot C \cdot h^{-1}.$$

Therefore, the transition functions for $\sigma^*E(\mathfrak{g}/\mathfrak{p})$ are $({}^T h_{\alpha\beta}^{-1} \otimes q_{\alpha\beta})$, which correspond to $F^* \otimes (E(\mathbb{C}^n)/F)$. (See, for instance, section 0.5 of [3].) $\square$

Corollary 1. *A principal $\mathrm{GL}(n)$ -bundle E over X is semistable (resp. stable) if and only if for every subbundle F of $E(\mathbb{C}^n)$, $\mu(F) \leq \mu(E(\mathbb{C}^n))$ (resp. $\mu(F) < \mu(E(\mathbb{C}^n))$) holds, i.e., $E(\mathbb{C}^n)$ is semistable (resp. stable) in the sense of Mumford.*

Proof. We shall prove the statement for the “semistable” part only. One proves the “stable” part by simply replacing every inequality by a strict inequality.

Suppose that $E(\mathbb{C}^n)$ is semistable in the sense of Mumford, and let $\sigma : X \rightarrow E/P$ be a reduction of the structure group to a maximal parabolic subgroup P of G . Then P corresponds to a two-step flag

$$0 = V_0 \subset V_1 \subset V_2 = \mathbb{C}^n.$$

Let $F = \sigma^*(E(V_1))$. By Proposition 1, $\sigma^*T_{E/P} \simeq F^* \otimes (E(\mathbb{C}^n)/F)$. Since $E(\mathbb{C}^n)$ is Mumford semistable, we have $\mu(F) \leq \mu(E(\mathbb{C}^n))$, which is equivalent to $\mu(F) \leq \mu(E(\mathbb{C}^n)/F)$. This is in turn equivalent to $\deg F^* \otimes (E(\mathbb{C}^n)/F) \geq 0$, since

$$\begin{aligned} (*) \quad \deg F^* \otimes (E(\mathbb{C}^n)/F) &= -\deg F \cdot \mathrm{rk}(E(\mathbb{C}^n)/F) + \mathrm{rk} F \cdot \deg E(\mathbb{C}^n)/F \\ &= (\mu(E(\mathbb{C}^n)/F) - \mu(F)) \cdot \mathrm{rk}(E(\mathbb{C}^n)/F) \cdot \mathrm{rk}(F). \end{aligned}$$

Therefore, E is semistable as a principal $\mathrm{GL}(n)$ -bundle.

Conversely, let E be a semistable principal $\mathrm{GL}(n)$ -bundle. Any vector subbundle F of $E(\mathbb{C}^n)$ is of the form $\sigma^*(E(V))$ for some reduction σ of the structure group to a parabolic subgroup P corresponding to a flag $0 = V_0 \subset V \subset \mathbb{C}^n$. Since E is stable as a principal $\mathrm{GL}(n)$ -bundle, we have

$$\deg \sigma^*(T_{E/P}) = \deg(F^* \otimes (E(\mathbb{C}^n)/F)) \geq 0.$$

By (*), we know that this implies $\mu(F) \leq \mu(E(\mathbb{C}^n))$. $\square$

Now we introduce a very important lemma that rephrases the stability condition for principal bundles in terms of the well-known stability condition for vector bundles. This suggests that one may deduce the existence of a moduli space of principal bundles from the existence of a moduli space of vector bundles, which is exactly what Ramanan did in his thesis [6].

Lemma 1 ([6]). *A holomorphic principal G -bundle E over X is semistable if and only if the associated vector bundle $\mathrm{Ad}(E) = (E \times \mathfrak{g})/G$ is a semistable vector bundle.*

In fact, this is taken as the definition of semistability in [2]. Based on this lemma, we make the following definition.

Definition 3. A holomorphic principal G -bundle over X is said to be *Ad-stable* if the associated vector bundle $\mathrm{Ad}(E)$ is a stable vector bundle.

In [5], the first named author constructed a quasi-projective moduli scheme for Ad-stable principal G -bundles over a projective scheme of arbitrary dimension. (The term “Ad-stable” is not used in [5].) It is natural to ask what the relationship is between Ad-stability and stability according to Ramanathan.

Proposition 2. *If a principal G -bundle E over a compact Riemann surface X is Ad-stable, then it is stable in the sense of Ramanathan.*

Proof. Suppose that E is an Ad-stable G -bundle. Since G is reductive, there exists a nondegenerate G -invariant bilinear form on $\mathfrak{g}$. Via this form, $\mathrm{Ad}(E) = E(\mathfrak{g})$ is dual to itself and hence is of degree zero. Let P be a parabolic subgroup of G and let $\mathfrak{p}$ be its Lie algebra. Also, let $\sigma : X \rightarrow E/P$ be a reduction of the structure group to P . We have the exact sequence of P -modules (the module structures are given by the adjoint representation)

$$0 \rightarrow \mathfrak{p} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{p} \rightarrow 0$$

and the exact sequence of vector bundles

$$\begin{array}{ccccccc} 0 & \longrightarrow & \sigma^* E(\mathfrak{p}) & \longrightarrow & \sigma^* E(\mathfrak{g}) & \longrightarrow & \sigma^* E(\mathfrak{g}/\mathfrak{p}) \longrightarrow 0 \\ & & & & \parallel & & \parallel \\ & & & & \mathrm{Ad}(E) & & \sigma^* T_{E/P} \end{array}$$

Since $\mathrm{Ad}(E)$ is a stable vector bundle of degree zero, it follows that $\deg \sigma^* T_{E/P} > 0$, and we are done. $\square$

However, as remarked in [2], $\mathrm{Ad}(E)$ can be strictly semistable even if E is stable in the sense of Ramanathan. For example, for any $n > 1$, no $\mathrm{GL}(n)$ -bundle is Ad-stable, since $\mathrm{Ad}(E) = \mathrm{End}(E(\mathbb{C}^n))$ has the trivial line subbundle generated by the trivial section Id defined by $Id_x(v) = v$ for all $x \in X$ and $v \in E(\mathbb{C}^n)_x$. In fact, we know that

$$\mathrm{End}(E(\mathbb{C}^n)) = \mathrm{End}_o(E(\mathbb{C}^n)) \oplus \mathcal{O}_X,$$

where $\mathrm{End}_o(E(\mathbb{C}^n))$ is the bundle of trace-free endomorphisms. This decomposition follows from the fact that there exists a split exact sequence of $\mathrm{GL}(n)$ -modules

$$(**) \quad 0 \longrightarrow \mathfrak{sl}(n) \longrightarrow \mathfrak{gl}(n) \xrightleftharpoons[\iota]{Tr} \mathbb{C} \longrightarrow 0,$$

where the module structures are given by the adjoint representation. Here Tr is the trace map and ι is defined by $\iota(z) = \frac{z}{n}1$, 1 being the identity matrix in $\mathfrak{gl}(n)$. More generally,

Lemma 2. *There are no Ad-stable principal G -bundles if G is not semisimple.*

Proof. Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. Since we are in characteristic 0, note that G being semisimple, the connected center $Z^o(G)$ being trivial, and $\mathfrak{z} = 0$ are all equivalent. Suppose that there exists an Ad-stable G -bundle E . By Proposition 2, E is stable. So we can use Proposition 3.2 of [6] to conclude that $H^0(X, E(\mathfrak{g})) = \mathfrak{z}$. If $\mathfrak{z} \neq 0$, i.e., if G is not semisimple, then $E(\mathfrak{g})$ has a global section $\phi : \mathcal{O}_X \rightarrow E(\mathfrak{g})$ that generates $\mathcal{O}_X$ as a subsheaf of $E(\mathfrak{g})$. This contradicts the assumption that E is Ad-stable. $\square$

This shows that the construction in [5] is effective only for semisimple groups. However, even in the semisimple case, stability and Ad-stability are not equivalent. In section 4, we shall encounter an example of a stable $\mathrm{SL}(2)$ -bundle that is not Ad-stable.

3. AD-STABLE $\mathrm{SL}(2)$ -BUNDLES OVER A RIEMANN SURFACE

First we recall the main result in [4]. The fundamental group of X is generated by $2g$ elements $a_1, b_1, \dots, a_g, b_g$ with the single relation

$$\prod_{i=1}^g [a_i, b_i] = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

We denote this group by Γ_g .

Let Y denote the universal cover of X . It is a principal Γ_g -bundle over X (or, a Galois covering of X with Galois group Γ_g). Therefore, any rational representation $\rho : \Gamma_g \rightarrow \mathrm{GL}(n)$ produces a rank n vector bundle $(Y \times \mathbb{C}^n)/\Gamma_g$ over X . The main result of [4] is the following theorem.

Theorem 1. *Let X be a compact Riemann surface of genus ≥ 2 . Then a holomorphic vector bundle of degree zero on X is stable if and only if it arises from an irreducible unitary representation of the fundamental group of X .*

We need the following basic lemma, which can easily be seen when one considers the transition functions of the bundles involved.

Lemma 3. *Let E be a principal G_1 -bundle, and let $\rho_1 : G_1 \rightarrow G_2$ and $\rho_2 : G_2 \rightarrow G_3$ be homomorphisms of holomorphic complex Lie groups. Then we have a natural isomorphism*

$$(E_{\rho_1}(G_2))_{\rho_2}(G_3) \simeq E_{\rho_2 \circ \rho_1}(G_3).$$

A principal $\mathrm{SL}(n)$ -bundle can be thought of as a rank n vector bundle with trivial determinant. Therefore we can immediately deduce the following proposition from Theorem 1 and Lemma 3.

Proposition 3. *Let X be a compact Riemann surface of genus ≥ 2 . Then a holomorphic principal $\mathrm{SL}(n)$ -bundle E is Ad-stable if and only if it arises from a representation $\rho : \Gamma_g \rightarrow \mathrm{SL}(n)$ such that $\mathrm{ad}(\rho)$, the composition of the adjoint representation $\mathrm{ad} : \mathrm{SL}(n) \rightarrow \mathrm{GL}(\mathfrak{sl}(n))$ and ρ , is irreducible and unitary.*

Remark 1. Such a ρ is necessarily irreducible and unitary by Lemma 6, which will be shown later in this article. Therefore we conclude that a principal $\mathrm{SL}(n)$ -bundle is stable if it is Ad-stable. Of course, this also follows from Proposition 2.

We let Ad (resp. ad) denote the adjoint representation $\mathrm{GL}(n) \rightarrow \mathrm{GL}(\mathfrak{gl}(n))$ (resp. $\mathrm{SL}(n) \rightarrow \mathrm{GL}(\mathfrak{sl}(n))$). Hence for any $x \in \mathrm{SL}(n)$, $\mathrm{Ad}(x)$ denotes the adjoint action of x on $\mathfrak{gl}(n)$, and $\mathrm{ad}(x)$ denotes the adjoint action on $\mathfrak{sl}(n) \subset \mathfrak{gl}(n)$. Let $\{e_1, \dots, e_n\}$ denote the standard basis of $\mathbb{C}^n$. Then the standard basis for $\mathfrak{gl}(n)$ consists of the matrices $e_{ij} = e_i \otimes e_j^*$, the matrix with 1 at the ij -th place and 0 elsewhere.

Definition 4. Let G be a group and $\rho : G \rightarrow \mathrm{GL}(n)$ a representation. Then ρ is said to be *unitary* if $\rho(G) \subset \mathrm{U}(n)$.

Lemma 4. Let $x \in U(n)$. With respect to the basis $\{e_i \otimes e_j^*\}$, we have

$$(\text{Ad}(x))_{kl\,ij} = x_{ki} \bar{x}_{lj}.$$

That is,

$$\text{Ad}(x)(e_{ij}) = \sum_{k,l} x_{ki} \bar{x}_{lj} e_{kl}.$$

Proof. Let $x' = \bar{x}^T$. The adjoint action is given by

$$\begin{aligned} \text{Ad}(x)(e_{ij}) &= x \cdot e_i \otimes e_j^* \cdot x' \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 & x_{1i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_{2i} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{ni} & 0 & \cdots & 0 \end{pmatrix} \cdot x' \\ &= \begin{pmatrix} x_{1i}x'_{j1} & x_{1i}x'_{j2} & \cdots & x_{1i}x'_{jn} \\ x_{2i}x'_{j1} & x_{2i}x'_{j2} & \cdots & x_{2i}x'_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{ni}x'_{j1} & x_{ni}x'_{j2} & \cdots & x_{ni}x'_{jn} \end{pmatrix}. \end{aligned}$$

Hence the element at the kl th place of $\text{Ad}(x)(e_{ij})$ is

$$x_{ki}x'_{jl} = x_{ki}\bar{x}_{lj}.$$

□

Lemma 5. Let $x \in \text{SL}(n)$. $\text{Ad}(x)$ is unitary if and only if $\text{ad}(x) \in \text{GL}(\mathfrak{sl}(n))$ is unitary.

Proof. Recall the split exact sequence $(**)$ of $\text{GL}(n)$ -modules. From it, we see that with respect to any unitary basis containing $\frac{1}{\sqrt{n}}(e_{11} + \cdots + e_{nn})$, $\text{Ad}(x)$ is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \text{ad}(x) \end{pmatrix},$$

and it follows immediately that $\text{Ad}(x)$ is unitary if and only if $\text{ad}(x)$ is so. □

Lemma 6. Let $x \in \text{GL}(n)$. Then $\text{Ad}(x)$ is unitary if and only if x is so.

Proof. We have

$$\begin{aligned} (\text{Ad}(x) \cdot \overline{\text{Ad}(x)}^T)_{ij\,kl} &= \sum_{r,s} (\text{Ad}(x))_{ij\,rs} \cdot \left(\overline{\text{Ad}(x)}^T \right)_{rs\,kl} \\ &= \sum_{r,s} (\text{Ad}(x))_{ij\,rs} \cdot \overline{\text{Ad}(x)_{kl\,rs}} \\ &= \sum_{r,s} x_{ir} \bar{x}_{js} \cdot \overline{x_{kr} \bar{x}_{ls}} \\ &= \sum_{r,s} x_{ir} \bar{x}_{js} \cdot \bar{x}_{kr} x_{ls} \\ &= (x \cdot \bar{x}^T)_{ik} \cdot (x \cdot \bar{x}^T)_{lj}. \end{aligned}$$

Therefore, $\text{Ad}(x)$ is unitary if and only if $(xx) = x \cdot \bar{x}^T$ is diagonal and $(xx)_{ii} \cdot (xx)_{jj} = 1$ for any i, j . Thus $(xx)_{ii}^2 = 1$, but $(xx)_{ii} = \sum_s x_{is} \bar{x}_{is} \geq 0$, and it follows that (xx) is the identity matrix. □

Combining Lemma 5 and Lemma 6, we obtain

Lemma 7. Let $x \in \text{SU}(n)$. Then $\text{ad}(x)$ is unitary.

Example 1. In the rank 2 case ($x \in \mathrm{U}(2)$),

$$\mathrm{Ad}(x) = \begin{pmatrix} x_{11}\bar{x}_{11} & x_{11}\bar{x}_{12} & x_{12}\bar{x}_{11} & x_{12}\bar{x}_{12} \\ x_{11}\bar{x}_{21} & x_{11}\bar{x}_{22} & x_{12}\bar{x}_{21} & x_{12}\bar{x}_{22} \\ x_{21}\bar{x}_{11} & x_{21}\bar{x}_{12} & x_{22}\bar{x}_{11} & x_{22}\bar{x}_{12} \\ x_{21}\bar{x}_{21} & x_{21}\bar{x}_{22} & x_{22}\bar{x}_{21} & x_{22}\bar{x}_{22} \end{pmatrix},$$

and with respect to the unitary basis $\{e_{12}, e_{21}, \frac{1}{\sqrt{2}}(e_{11} - e_{22})\}$,

$$\mathrm{ad}(x) = \begin{pmatrix} x_{11}\bar{x}_{22} & x_{12}\bar{x}_{21} & \sqrt{2}x_{11}\bar{x}_{21} \\ x_{21}\bar{x}_{12} & x_{22}\bar{x}_{11} & \sqrt{2}x_{21}\bar{x}_{11} \\ \sqrt{2}x_{11}\bar{x}_{12} & \sqrt{2}x_{12}\bar{x}_{11} & |x_{11}|^2 - |x_{12}|^2 \end{pmatrix}.$$

Proposition 4. For any integer $g > 2$, there exists an irreducible unitary representation ρ of Γ_g of rank 2 such that $\mathrm{ad}(\rho)$ is also irreducible and unitary.

Proof. Let z be a complex number satisfying $z^2 \neq \bar{z}^2$, and set $r = |z|$. Let

$$A_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B_2 = \frac{1}{r\sqrt{2}} \begin{pmatrix} z & -\bar{z} \\ z & \bar{z} \end{pmatrix}, A_2 = B_2^{-1}.$$

Also, let $A_3 = A_1$, $B_3 = B_1$, and $A_i = B_i = 1$ for $i > 3$. Then $[A_1 B_1] = -1$, $[A_2 B_2] = 1$, $[A_3 B_3] = -1$ and hence $\prod_{i=1}^g [A_i B_i] = 1$.

With respect to the unitary basis $\{e'_1 = e_{12}, e'_2 = e_{21}, e'_3 = \frac{1}{\sqrt{2}}(e_{11} - e_{22})\}$, we have

$$\begin{aligned} \mathrm{ad}(A_1) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \mathrm{ad}(B_1) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathrm{ad}(B_2) &= \frac{1}{2r^2} \begin{pmatrix} z^2 & -\bar{z}^2 & \sqrt{2}r^2 \\ -z^2 & \bar{z}^2 & \sqrt{2}r^2 \\ -\sqrt{2}z^2 & -\sqrt{2}\bar{z}^2 & 0 \end{pmatrix}. \end{aligned}$$

We need to show that they do not stabilize any proper subspace. The eigenvectors of $\mathrm{ad}(A_1)$ are $\mathbb{C}(e'_1 + e'_2)$, $\mathbb{C}(e'_1 - e'_2) + \mathbb{C} \cdot e'_3$. Those of $\mathrm{ad}(B_1)$ are $\mathbb{C} \cdot e'_3$ and $\mathbb{C} \cdot e'_1 + \mathbb{C} \cdot e'_2$. Hence common eigenvectors for the two are $\mathbb{C} \cdot e'_3$, $\mathbb{C} \cdot (e'_1 - e'_2)$ and $\mathbb{C} \cdot (e'_1 + e'_2)$. But none of them is an eigenvector for $\mathrm{ad}(B_2)$.

We need to show that there is no 2-dimensional subspace stabilized by the above three transformations. Assume that W is such a subspace. Since $\mathrm{ad}(B_2)$ does not stabilize $\mathrm{Span}\{e'_1, e'_2\}$, W contains an element $w = (a, b, c) = ae'_1 + be'_2 + ce'_3$ with $c \neq 0$. Consider the following cases.

1. $a = b = 0$: Then W is spanned by e'_3 and $\lambda e'_1 + \mu e'_2$ for some $\lambda, \mu \in \mathbb{C}$. For $\mathrm{ad}(A_1)$ to stabilize $\mathrm{Span}\{e'_3, \lambda e'_1 + \mu e'_2\}$, either $\lambda = \mu$ or $\lambda = -\mu$. But $\mathrm{Span}\{e'_3, e'_1 + e'_2\}$ and $\mathrm{Span}\{e'_3, e'_1 - e'_2\}$ are not stabilized by $\mathrm{ad}(B_2)$, since

$$\begin{aligned} \mathrm{ad}(B_2)(e'_1 + e'_2) &= \frac{1}{2r^2}(z^2 + \bar{z}^2, -z^2 + \bar{z}^2, -\sqrt{2}(z^2 - \bar{z}^2)) \\ &\notin \mathrm{Span}\{e'_1 + e'_2, e'_3\}, \end{aligned}$$

and

$$\mathrm{ad}(B_2)(e'_3) = \frac{1}{2r^2}(\sqrt{2}r^2, \sqrt{2}r^2, 0) \notin \mathrm{Span}\{e'_1 - e'_2, e'_3\}.$$

2. $a \neq 0$: We can assume that w is not an eigenvector of $\text{ad}(A_1)$. (Otherwise w is of the form $(1, -1, c)$ and W is spanned by $(1, -1, c)$ and $\text{ad}(B_1)(w) = (-1, 1, c)$. That is, $W = \text{Span}\{e'_1 - e'_2, e'_3\}$; but we have already seen in case 1 that this is not stabilized by $\text{ad}(B_2)$.) Therefore, W is spanned by w and $w' = \text{ad}(A_1)(w) = (b, a, -c)$. Since $\text{ad}(B_1)(w) = (-a, -b, c)$ has to be in W , there are $x, y \in \mathbb{C}$ such that

$$(-a, -b, c) = xw + yw' = (xa + yb, xb + ya, (x - y)c).$$

From this we get $x - y = 1$ and $(x + 1)^2 - y^2 = 0$, which forces $x = 0$, $y = -1$, and $a = b$. But then W is spanned by $w = (a, a, c)$ and $\text{ad}(B_1)(w) = (-a, -a, c)$. So $W = \text{Span}\{e'_1 + e'_2, e'_3\}$, which we have already seen is not stabilized by $\text{ad}(B_2)$.

3. $a = 0$, $b \neq 0$: Then W is spanned by $w = (0, b, c)$ and $w' = \text{ad}(A_1)(w) = (b, 0, -c)$. Also,

$$\text{ad}(B_1)(w) = (0, b, -c) = xw + yw' = (yb, xb, (x - y)c)$$

implies $y = 0$ and forces $(0, -b, c) = (0, b, c)$. Since $b \neq 0$, this is impossible. $\square$

Combining this with Proposition 3, we obtain

Proposition 5. *There exist Ad-stable $\text{SL}(2)$ -bundles over any compact Riemann surface of genus > 2 .*

Example 2 (A unitary irreducible representation ρ whose adjoint $\text{Ad}(\rho)$ is not irreducible). Let

$$A_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and let all B_i and A_j , $j > 2$, be 1. Clearly $\Pi_{i=1}^g [A_i B_i] = 1$. They form an irreducible set, since they do not have any common eigenvector.

Now, let us take a look at the adjoint representation $\text{Ad}(\rho) : \Gamma_g \rightarrow GL(\mathfrak{gl}(2))$. We use the formula

$$(\text{Ad}(A))_{kl \ ij} = A_{ki} \overline{A_{lj}}$$

with respect to the standard basis, and get

$$\begin{aligned} \text{Ad}(A_1) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \text{Ad}(A_2) &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

With respect to $\{e_{11} - e_{22}, e_{12}, e_{21}\}$, one has

$$\text{ad}(A_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \text{ad}(A_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

which is certainly reducible.

Remark 2. By Theorem 1 and Proposition 2, this example provides us with a stable $\text{SL}(2)$ -bundle that is *not* Ad-stable.

REFERENCES

- [1] B. Balaji and C. S. Seshadri, *Semistable principal bundles, I*, J. Algebra **258** (2002), 321-347. MR **2003m**:14050
- [2] R. Friedman and J. W. Morgan, *Holomorphic principal bundles over elliptic curves*, Preprint (1998), math.AG/9811130.
- [3] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, Inc. (1978). MR **80b**:14001
- [4] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965), 540-567. MR **32**:1725
- [5] D. Hyeon, *Principal bundles over a projective scheme*, Trans. Amer. Math. Soc., **354** (2002), 1899-1908. MR **2003d**:14013
- [6] A. Ramanathan, *Moduli for principal bundles over algebraic curves I, II*, Proc. Indian Acad. Sci. (Math. Sci.), **106** (1996), 301-328 and 421-449. MR **98b**:14009a; MR **98b**:14009b
- [7] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. **213** (1975), 129-152. MR **51**:5979

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