DISCS IN COMPLEX MANIFOLDS WITH NO BOUNDED PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. Roughly speaking: In a complex manifold on which all bounded plurisubharmonic functions are constant, the center of a holomorphic disc and its boundary can be prescribed somewhat arbitrarily.

I. INTRODUCTION

Let $\overline{\Delta}$ be the closed unit disc in $\mathbb{C}$.

Theorem. *Let $\mathcal{M}$ be a complex manifold, equipped with some Riemannian metric. The following are equivalent.*

- (1) *Every bounded plurisubharmonic function on $\mathcal{M}$ is constant.*
- (2) *For every $p \in \mathcal{M}$ and every nonempty open subset ω of $\mathcal{M}$, and for every $\varepsilon > 0$, there exists a holomorphic map φ from a neighborhood of $\overline{\Delta}$ in $\mathbb{C}$ into $\mathcal{M}$ such that $h(0) = p$ and $h(e^{i\theta}) \in \omega$ for all $\theta \in [0, 2\pi) - E_\varepsilon$, where E_ε is a subset of $[0, 2\pi)$ of measure at most ε .*
- (3) *For every $p \in \mathcal{M}$ and every continuous map λ from the unit circle in $\mathbb{C}$ into $\mathcal{M}$, and for every $\varepsilon > 0$, there exists a holomorphic map h from a neighborhood of $\overline{\Delta}$ in $\mathbb{C}$ into $\mathcal{M}$ such that: $h(0) = p$ and $\text{dist}(h(e^{i\theta}), \lambda(e^{i\theta})) \leq \varepsilon$ for all $\theta \in [0, 2\pi) - E_\varepsilon$, where again E_ε is a set of measure at most ε .*

The above theorem seems close to being optimal. It cannot be hoped that the exceptional set E_ε can be chosen to be an interval, as is however the case in the simple situation $\mathcal{M} = \mathbb{C}^n$.

The following proposition makes the point.

Proposition. *Let $\mathcal{M}$ be a compact Riemann surface of genus ≥ 2 . Let p and $q \in \mathcal{M}$, with $p \neq q$. Let $\delta > 0$. There exists a neighborhood ω of q such that if h is a holomorphic map from a neighborhood of $\overline{\Delta}$ into $\mathcal{M}$ with $h(0) = p$ and if I is an interval of length $> \delta$ on the unit circle, then $h(I)$ cannot be included in ω .*

So we could not even take as exceptional set $(h(e^{i\theta}) \notin \omega)$ a large interval of measure $2\pi - \delta$.

In the theorem, (3) $\Rightarrow$ (2) is obvious (taking λ constant). (2) $\Rightarrow$ (1) is immediate: Assume that (2) holds, and let u be a bounded plurisubharmonic function on $\mathcal{M}$. Let p and $q \in \mathcal{M}$. Take ω to be a small neighborhood of q . (2) gives a map φ with

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$h(0) = p$ and $h(e^{i\theta}) \in \omega$ except for a set of θ s of measure at most ε . The mean value property for subharmonic functions yields

$$u(p) \leq \int_0^{2\pi} u \circ h(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Shrinking ω and letting ε tends to 0, one gets in the limit $u(p) \leq u(q)$. Similarly $u(q) \leq u(p)$; so u is constant.

(1) $\Rightarrow$ (2) was first proved in full generality in [4] (corollary to the main theorem). See also [2], and see [1], [3] for earlier results. See [5] for a simpler and very different proof.

So it only remains to prove (2) $\Rightarrow$ (3).

Preliminary comments. There is no new ingredient in the proof of the theorem. We do not give a direct proof of (1) $\Rightarrow$ (3). We have to use (2) as a first step. Then we have to repeat a construction of holomorphic discs that is used in proving (1) $\Rightarrow$ (2). This is inspired from the fundamental work of Poletsky [3]. It is however worth pointing out the following. In proving (1) $\Rightarrow$ (2) the construction of discs comes only in the proof of a very general result on the plurisubharmonicity of functions defined by the infimum of averages over the boundary of holomorphic discs (theorem in [4]). (2) is then obtained not directly but by a functional argument (not surprisingly since one has to finally use the hypothesis!). The argument is extremely simple but I have not been able to modify it in order to get a soft proof of (2) $\Rightarrow$ (3).

The proof will heavily rely on [5] (an alternate approach would rely on [4]).

II. PROOF OF THE THEOREM

II.1. As already said we have only to prove (2) $\Rightarrow$ (3). Let $\varepsilon > 0$. We select disjoint closed intervals $I_j \subset [0, 2\pi)$, $j = 1, \dots, J$, and $\theta_j \in I_j$ such that:

$$\left| [0, 2\pi) - \bigcup_{j=1}^J I_j \right| \leq \frac{\varepsilon}{2} \quad (|\cdot| \text{ denotes measure}), \text{ and if } \theta \in J_j, \text{ dist}[\lambda(\theta), \lambda(\theta_j)] \leq \frac{\varepsilon}{3}.$$

Using (2), for each $j \in \{1, \dots, J\}$ we select a holomorphic map g_j from a neighborhood of $\overline{\Delta}$ into $\mathcal{M}$ such that $g_j(0) = p$ and

$$\text{dist}(g_j(e^{i\nu}), \lambda(\theta_j)) \leq \frac{\varepsilon}{3}$$

for all $\nu \in [0, 2\pi) = E_j$, where E_j is a subset of $[0, 2\pi)$ of measure at most $\frac{\varepsilon}{2}$.

Next, we “put the g_j ’s together” and we claim that there is a $\mathcal{C}^\infty$ map Φ defined on a neighborhood of $\overline{\Delta} \times \overline{\Delta}$ in $\mathbb{C}_{z,w}^2$ into $\mathcal{M}$ such that:

$$(*) \quad \begin{cases} \Phi(z, 0) \equiv p. \\ \text{For every } z, w \mapsto \Phi(z, w) \text{ is holomorphic in } w. \\ \text{If } \theta_j \in I_j, \text{ then } \Phi(e^{i\theta_j}, w) = g_j(w). \\ \frac{\partial \Phi}{\partial \bar{z}} \text{ vanishes to infinite order along } |z| = 1. \end{cases}$$

The construction is fairly easy. For each $j \in \{1, \dots, J\}$, let S_j be the interval on the circle, $S_j = \{e^{i\theta}, \theta \in I_j\}$. Let χ_j ($j = 1, \dots, J$) be smooth functions on $\mathbb{C}$, with support respectively in small neighborhoods of S_j with disjoint closures, such that $\chi_j \equiv 1$ on S_j , $0 \leq \chi_j(e^{i\theta}) \leq 1$ and $\bar{\partial}\chi_j$ vanishes to infinite order along the unit circle. By additional cutoff we can impose $|\chi_j| < 1 + \varepsilon_j$ where ε_j is chosen so that

$g_j(\omega)$ is defined for $|w| < 1 + \varepsilon_j$. Off the unit circle, χ_j is not real valued. Then Φ can be defined in the following way:

$$\begin{cases} \text{For } z \text{ near } S_j \text{ (take disjoint neighborhoods), } \Phi(z, w) = g_j(\chi_j(z)w). \\ \text{Away from } \bigcup S_j, \Phi(z, w) = p. \end{cases}$$

Note that by construction the set of $(\theta, \nu) \in [0, 2\pi)^2$ such that $\text{dist}[\Phi(e^{i\theta}, e^{i\nu}), \lambda(\theta)] > \frac{2\varepsilon}{3}$ has measure at most $2\pi\varepsilon$. Indeed, it is included in the union of $\{(\theta, \nu), \theta \notin J_j\}$, which has measure at most $\frac{\varepsilon}{2} \times 2\pi$, and of $\bigcup_j \{(\theta, \nu), \theta \in I_j, \nu \in E_j\}$, which has measure at most $\sum |I_j| \frac{\varepsilon}{2}$. Since the map $(e^{i\theta}, e^{i\nu}) \mapsto (e^{i\theta}, e^{iN\theta}e^{i\nu})$ is area-preserving on the torus, for every integer N the set

$$\left\{ (\theta, \nu) \in [0, 2\pi)^2; \text{dist } \Phi[(e^{i\theta}, e^{iN\theta}e^{i\nu}), \lambda(\theta)] \geq \frac{2\varepsilon}{3} \right\}$$

has measure at most $2\pi\varepsilon$.

So there exists ν_N such that

$$(**) \quad \text{dist}[\Phi(e^{i\theta}, e^{iN\theta}e^{i\nu_N}), \lambda(\theta)] \leq \frac{2\varepsilon}{3} \\ \text{except for } \theta \text{ in a set } E_N \subset [0, 2\pi), \text{ with } |E_N| \leq \varepsilon.$$

The above argument is of course borrowed from [3]. For $N \in \mathbb{N}$, set $\varphi_N(z) = \Phi(z, e^{i\nu_N}z^N)$. Then the φ_N are maps from neighborhoods of $\overline{\Delta}$ into $\mathcal{M}$ that satisfy:

$$\begin{cases} \varphi_N(0) = p, \\ \text{dist}(\varphi_N(e^{i\theta}), \lambda(\theta)) \leq \frac{2\varepsilon}{3} \text{ except for } \theta \in E_N, \\ \overline{\partial}\varphi_N \text{ tends uniformly to 0 on } \overline{\Delta}. \end{cases}$$

The last property follows immediately from (*).

Note that there is no control of φ_N off Δ , and that there is no control on $\partial\varphi_N$.

Since $\overline{\partial}\varphi_N$ tends uniformly to 0 on $\overline{\Delta}$, we would like to conclude that there are holomorphic maps h_N defined on neighborhoods of $\overline{\Delta}$ into $\mathcal{M}$ such that $h_N(0) = \varphi_N(0) = p$ and

$$\sup_{|z| \leq 1} [\text{dist } \varphi_N(z), h_N(z)] \leq \frac{\varepsilon}{3}$$

for N large enough. We would then have $\text{dist}(h_N(e^{i\theta}), \lambda(\theta)) \leq \varepsilon$ except for $\theta \in E_N$; so we could take $h = h_N$.

This is a question of uniformly approximating maps with small $\overline{\partial}$ by holomorphic maps. The question has been discussed in detail in [4]. Although it has a negative general answer, there are special cases where the answer is positive. $\square$

II.2. We now describe the result proved in [4], before finishing the proof of the theorem in II.3.

We are given in a complex manifold $\widetilde{\mathcal{M}}$ open subsets $\Omega_0, \dots, \Omega_R$ with $K_1, \dots, K_R$ compact subsets of $\Omega_0, \dots, \Omega_R$ respectively. We assume that each Ω_j is biholomorphic to some open set in $\mathbb{C}^k$ ($k = \dim \widetilde{\mathcal{M}}$). Finally, we are given open sets $U_0, \dots, U_R$ in $\mathbb{C}$ such that $\overline{\Delta} \subset \bigcup_{j=0}^R U_j$ and with empty triple intersections, i.e., $U_j \cap U_k \cap U_\ell = \emptyset$ if $jk\ell$ are all distinct.

A map $\tilde{\varphi}$ from Δ into $\widetilde{\mathcal{M}}$ is then called *restricted* (relative to the above data) if

$$\tilde{\varphi}(U_j \cap \Delta) \subset K_j \quad \text{for all } j \in \{0, \dots, R\}.$$

We equip $\widetilde{\mathcal{M}}$ with some Riemannian metric. The main result of [4] is Proposition 1 in [4]:

For every $\varepsilon > 0$ there exists $\delta > 0$ such that if u is a restricted map from Δ into $\widetilde{\mathcal{M}}$ satisfying $|\bar{\partial}u| \leq \delta$, there exists a holomorphic map h from Δ into $\widetilde{\mathcal{M}}$ such that $h(0) = u(0)$ and

$$\sup_{z \in \Delta} [\text{dist}(u(z), h(z))] < \varepsilon.$$

If u is continuous on $\overline{\Delta}$, then we can find h holomorphic on a neighborhood of $\overline{\Delta}$, simply replacing $h(z)$ by $h[(1 - \eta)z]$ for $\eta > 0$ small enough.

Note: In Proposition 1 in [4] it is absolutely crucial that no shrinking of the disc be allowed and that no uniform bound on ∂u be imposed.

But, when considering only restricted maps, as above, there is an obvious bound of the kind:

$$|\partial u(z)| \leq \frac{c}{1 - |z|}.$$

Such a bound is not enough by itself in order to get a positive answer to the approximation problem for maps with small $\bar{\partial}$. Indeed, one can easily check that in Lempert's example described in [4] there is such a bound; however, approximation fails.

II.3. At the end of II.1 we had to approximate nonholomorphic maps φ_N by holomorphic maps h_N . To put ourselves in the setting of II.2 we have to replace $\mathcal{M}$ by $\widetilde{\mathcal{M}} = \mathcal{M} \times \mathbb{C}$ and the map Φ by $\widetilde{\Phi}$ defined by $\widetilde{\Phi}(z, w) = (\Phi(z, w), w)$. The advantage is that for $j = 1, \dots, J$, the map $w \longrightarrow (g_j(w), w) = \widetilde{g}_j(w)$ is now an embedding of a neighborhood of $\overline{\Delta}$ with $\widetilde{\mathcal{M}}$. By a theorem of Royden [6] (see also [1], [4] Lemma 1.1), $\widetilde{g}_j(\overline{\Delta})$ has a neighborhood Ω_j biholomorphic to an open set in $\mathbb{C}^k$. For $j = 1, \dots, J$, K_j will be a compact neighborhood of $\widetilde{g}_j(\overline{\Delta}) \subset \Omega_j$. Take $K_0 = \{p\} \times \overline{\Delta}$ and Ω_0 a small neighborhood of K_0 biholomorphic to some open set in $\mathbb{C}^k$.

Take U_0 open in $\mathbb{C}$ such that for $z \in U_0 \cap \Delta$, $\Phi(z, w) = p$; so $\widetilde{\Phi}(z, w) = (p, w)$. For $j \in \{1, \dots, J\}$, take U_j open in $\mathbb{C}$ containing I_j , such that for $z \in U_j \cap \Delta$, $\widetilde{\Phi}(z, w) \in K_j$. To do this, in order to have $\widetilde{g}_j(\chi_j(z)\overline{\Delta}) \subset K_j$ for $j = 1, \dots, J$, we may have to replace χ_j by $\tau \circ \chi_j$ where τ is a smooth function on $\mathbb{C}$ such that $\tau(z) = z$ if $|z| \leq 1$ and $\sup |\tau| \leq 1 + \eta$ for appropriate small η .

All the above can be done so that $\overline{\Delta} \subset \bigcup_{j=0}^J U_j$ and $U_j \cap U_k = \emptyset$ if $1 \leq j < k \leq J$. So there is no triple intersection.

For the map $\widetilde{\varphi}_N$ defined by

$$\widetilde{\varphi}_N(z) = (\varphi_N(z, e^{i\nu_N} z^N), z)$$

we are now in the situation described in II.2. So we can find holomorphic maps $\widetilde{h}_N$ from neighborhoods of $\overline{\Delta}$ into $\widetilde{\mathcal{M}}$ such that

$$\sup_{|z| \leq 1} \text{dist}[\widetilde{h}_N(z), \widetilde{\varphi}_N(z)] \xrightarrow{N \rightarrow \infty} 0.$$

Finally, take $h_N = \pi \circ \widetilde{h}_N$ where π is the natural projection of $\widetilde{\mathcal{M}} = \mathcal{M} \times \mathbb{C}$ onto $\mathcal{M}$. As explained at the end of II.1, this finishes the proof of the theorem.

III. PROOF OF THE PROPOSITION

Let $\chi : \Delta \rightarrow \mathcal{M}$ be a holomorphic covering map (Δ the open unit disc). We can assume $\chi(0) = p$. Let

$$d = \min_{\chi(\zeta)=q} |\zeta|.$$

Let $\delta > 0$.

Claim: *There exists $r > 0$ such that if g is a holomorphic map from Δ into Δ with $g(0) = 0$ and $|\zeta_0| \geq d$, then $|g(e^{i\theta}) - \zeta_0| \leq r$ can hold for a set of θ s $\in [0, 2\pi)$ of measure at most δ .*

This is a trivial consequence of the inequality

$$\text{Log}|g(0) - \zeta_0| \leq \int_0^{2\pi} \text{Log}|g(e^{i\theta}) - \zeta_0| \frac{d\theta}{2\pi}.$$

Take r as in the claim.

Then take a connected open neighborhood ω of q so small that each connected component of its preimage under the map χ has diameter $\leq r$.

Finally, if h is a holomorphic map from a neighborhood of $\overline{\Delta}$ into $\mathcal{M}$ with $h(0) = p$, lift this map to a holomorphic map g ($h = \chi \circ g$) with $g(0) = 0$.

If I is an interval on the unit circle of length $> \delta$, then h cannot map I in ω , since g would map I into one of the connected components of $\chi^{-1}(\omega)$. It would contradict the choice of r (taking ζ_0 in that component with $\chi(\zeta_0) = q$).

REFERENCES

- [1] F. Lárusson and R. Sigurdsson, *Plurisubharmonic functions and analytic discs on manifolds*, J. Reine Angew. Math. **501** (1998), 1–39. MR **99e**:32020
- [2] A. Edigarian, *A note on J. P. Rosay's paper: "Poletsky theory of disks on holomorphic manifolds"*, Ann. Polon. Math. **80** (2003), 125–132.
- [3] E. Poletsky, *Plurisubharmonic functions as solutions of variational problems*, Proc. Sympos. Pure Math. **52** (1991), 163–171. MR **92h**:32022
- [4] J.-P. Rosay, *Poletsky theory of disks on holomorphic manifolds*, Indiana Univ. Math. J. **52** (2003), 157–169. MR **2004a**:32053
- [5] J.-P. Rosay, *Approximation of non-holomorphic maps, and Poletsky theory of discs*, J. Korean Math. Soc. **40** (2003), no. 3, 423–434. MR **2004c**:32065
- [6] H. L. Royden, *The extension of regular holomorphic maps*, Proc. Amer. Math. Soc. **43** (1974), 306–310. MR **49**:629

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