CONSERVATIVENESS OF DIFFUSION PROCESSES WITH DRIFT

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ABSTRACT. We show the conservativeness of the Girsanov transformed diffusion process by drift $b \in L^p(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ with $p \geq 4/(2 - \sqrt{2\delta(|b|^2)/\lambda})$ or $p > 4d/(d+2)$, or $p = 2$ if $|b|^2$ is of the Hardy class with sufficiently small coefficient of energy $\delta(|b|^2) < \lambda/2$. Here $\lambda > 0$ is the lower bound of the symmetric measurable matrix-valued function $a(x) := (a_{i,j}(x))_{i,j}$ appearing in the given Dirichlet form. In particular, our result improves the conservativeness of the transformed process by $b \in L^d(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ when $d \geq 3$.

1. STATEMENT OF RESULT

In this note, we prove the conservativeness of diffusion processes on $\mathbb{R}^d$ by a Girsanov transformation. Let $\mathbb{R}^d \ni x \mapsto a(x) := (a_{i,j}(x))_{i,j=1}^d$ be a symmetric $(\mathbb{R}^d \otimes \mathbb{R}^d)$ -valued measurable function and $\mathbb{R}^d \ni x \mapsto b(x)$ an $\mathbb{R}^d$ -valued measurable function with expression $b(x) = (b^1(x), b^2(x), \dots, b^d(x))$. We assume the uniform ellipticity of a on $\mathbb{R}^d$: there exist constants $\Lambda \geq \lambda > 0$ such that

$$\lambda|\xi|^2 \leq \langle a(x)\xi, \xi \rangle_{\mathbb{R}^d} \leq \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d, x \in \mathbb{R}^d.$$

Here $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ stands for the Euclidean inner product with $|\xi| := \langle \xi, \xi \rangle_{\mathbb{R}^d}^{1/2}$. We consider the following quadratic form $(\mathcal{E}^a, H^1(\mathbb{R}^d))$:

$$\mathcal{E}^a(u, v) := \frac{1}{2} \int_{\mathbb{R}^d} \langle a(x) \nabla u(x), \nabla v(x) \rangle_{\mathbb{R}^d} dx \text{ for } u, v \in H^1(\mathbb{R}^d),$$

where $H^1(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d) \mid \text{the distributional derivatives } \partial u / \partial x^i, i = 1, 2, \dots, d \text{ are in } L^2(\mathbb{R}^d)\}$. Then $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ is a strongly local regular Dirichlet form on $L^2(\mathbb{R}^d)$ (see [8]).

Definition 1.1 (Hardy class function). A measurable function f on $\mathbb{R}^d$ is said to be of the *Hardy class* (write $f \in H_d$) with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ if there exist

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constants $\delta(|f|) \in (0, \infty)$ and $\gamma(|f|) \in [0, \infty)$ such that

$$(1.1) \quad \int_{\mathbb{R}^d} u^2 |f| dm \leq \delta(|f|) \mathcal{E}^a(u, u) + \gamma(|f|) \|u\|_{L^2}^2 \quad \text{for } u \in H^1(\mathbb{R}^d),$$

where $m(dx) := dx$ is the d -dimensional Lebesgue measure on $\mathbb{R}^d$.

Clearly, $H_d \subset L_{loc}^1(\mathbb{R}^d)$. If $f \in L^{p/2}(\mathbb{R}^d)$ with $p \geq d \geq 3$, in particular, if $f \in L^{d/2}(\mathbb{R}^d)$ with $d \geq 3$, then $f \in H_d$ and the constant $\delta(|f|)$ can be taken to be arbitrarily small (cf. [22], [24]).

Definition 1.2 (Measures of finite energy integrals, [8]). A positive Radon measure μ on $\mathbb{R}^d$ is said to be of *finite energy integral* with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ (write $\mu \in S_0$) if and only if there exists a positive constant C depending on μ such that

$$\int_{\mathbb{R}^d} |u| d\mu \leq C \sqrt{\mathcal{E}_1^a(u, u)} \quad \text{for } u \in C_0^\infty(\mathbb{R}^d).$$

Then for each $\alpha > 0$ there exists $U_\alpha^a \mu \in H^1(\mathbb{R}^d)$ such that $\mathcal{E}_\alpha^a(U_\alpha^a \mu, v) = \int_{\mathbb{R}^d} v d\mu$ for $v \in C_0^\infty(\mathbb{R}^d)$. $U_\alpha^a \mu$ is called the α -potential of $\mu \in S_0$. Furthermore, we set $S_{00} := \{\mu \in S_0 \mid \mu(\mathbb{R}^d) < \infty, U_1^a \mu \in L^\infty(\mathbb{R}^d)\}$.

Let Cap^a be the 1-capacity associated with $(\mathcal{E}^a, H^1(\mathbb{R}^d))$: $\text{Cap}^a(G) := \inf\{\mathcal{E}_1^a(u, u) \mid u \in H^1(\mathbb{R}^d), u \geq 1 \text{ a.e. on } G\}$ for any open set G , and $\text{Cap}^a(A) := \inf\{\text{Cap}^a(G) \mid G \text{ is open with } A \subset G\}$ for any subset A of $\mathbb{R}^d$. A subset N of $\mathbb{R}^d$ is said to be *exceptional* if $\text{Cap}^a(N) = 0$.

Definition 1.3 (Smooth measures, [8]). A Borel measure μ on $\mathbb{R}^d$ is said to be *smooth* with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ (write $\mu \in S$) if μ charges no exceptional set and there exists a sequence $\{F_n\}$ of closed sets such that $\mu(F_n) < \infty$ for each $n \in \mathbb{N}$ and

$$(1.2) \quad \lim_{n \rightarrow \infty} \text{Cap}^a(K \setminus F_n) = 0$$

for any compact set K of $\mathbb{R}^d$.

It is well known that $\mu \in S$ if and only if there exists a sequence $\{F_n\}$ of closed sets such that $I_{F_n} \mu \in S_{00}$ (or $I_{F_n} \mu \in S_0$) for each $n \in \mathbb{N}$ satisfying (1.2) for any compact K . For $f \in L_{loc}^1(\mathbb{R}^d)$, we see $|f|m \in S$.

Throughout this paper, we assume the following Assumptions 1.1 and 1.2.

Assumption 1.1. We assume that $|b|^2 \in H_d$ with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ and $\delta(|b|^2) < \lambda/2$.

Assumption 1.2. We assume that $|b| \in L^p(\mathbb{R}^d)$, $p \geq 2$, and when $2 < p < 4$ we further assume one of the following:

- (a) $|b|^2 m \in S_0$;
- (b) $|b|^{4-p} \in H_d$;
- (c) $p \geq 4/(2 - \sqrt{2\delta(|b|^2)/\lambda}) (> 2)$.

Remark 1.1. If $|b| \in L^p(\mathbb{R}^d)$ for $p \in (4d/(d+2), 4)$, $d \geq 1$, Assumption 1.2(a) holds, and if $|b| \in L^p(\mathbb{R}^d)$ for $p \in [4d/(d+2), 4)$, $d \geq 3$, $p \in (2, 4)$, $d = 2$, or $p \in [2, 4)$, $d = 1$, then Assumption 1.2(b) holds (see Lemma 1.1 below). Under Assumption 1.1, Assumption 1.2(b) holds for $p = 2$.

Set $\mathcal{E}^b(u, v) := \mathcal{E}^a(u, v) - \int_{\mathbb{R}^d} \langle b(x), \nabla u(x) \rangle_{\mathbb{R}^d} v(x) dx$ for $u, v \in C_0^\infty(\mathbb{R}^d)$. Then, under Assumption 1.1, there exists $M > 0$ such that for $u, v \in C_0^\infty(\mathbb{R}^d)$, $\alpha > \alpha_2(b) := \gamma(|b|^2) \sqrt{2/\lambda \delta(|b|^2)}$,

$$\begin{aligned} |\mathcal{E}^b(u, v)| &\leq M \|u\|_{H^1(\mathbb{R}^d)} \|v\|_{H^1(\mathbb{R}^d)}, \\ \mathcal{E}_\alpha^b(u, u) &\geq (1 - \sqrt{2\delta(|b|^2)/\lambda}) \mathcal{E}^a(u) + (\alpha - \alpha_2(b)) \|u\|_{L^2}^2 \geq 0. \end{aligned}$$

Hence for $\alpha > \alpha_2(b)$, $(\mathcal{E}_\alpha^b, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)$ and we denote its closure by $(\mathcal{E}_\alpha^b, H^1(\mathbb{R}^d))$. $(\mathcal{E}_\alpha^b, H^1(\mathbb{R}^d))$ is a local regular semi-Dirichlet form on $L^2(\mathbb{R}^d)$ (cf. [13], [18]).

Let $\{T_t^a\}_{t>0}$ be the $L^2(\mathbb{R}^d)$ -semigroups associated with $(\mathcal{E}^a, H^1(\mathbb{R}^d))$. Then T_t^a admits a symmetric jointly continuous heat kernel $p_t^a(x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ such that $P_t^a f(x) := \int_{\mathbb{R}^d} p_t^a(x, y) f(y) dy$ is a version of $T_t^a f$ for $f \in L^2(\mathbb{R}^d)$ and $p_t^a(x, y)$ satisfies the Aronson's estimates (see [26]): there exists an $M := M(\lambda, \Lambda, d) \in [1, \infty)$ such that for all $x, y \in \mathbb{R}^d, t > 0$,

$$(1.3) \quad \frac{1}{M t^{d/2}} e^{-M|x-y|^2/t} \leq p_t^a(x, y) \leq \frac{M}{t^{d/2}} e^{-|x-y|^2/Mt}.$$

It should be noted that the ball doubling condition and the strong Poincaré inequality hold for $(\mathcal{E}^a, H^1(\mathbb{R}^d))$, and the pseudo-distance (or intrinsic metric) derived from $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ is a complete metric compatible with the endowed topology. Hence the parabolic Harnack inequality holds for the local solution of the parabolic equation $(L^a - \frac{\partial}{\partial t})u = 0$ (cf. [19], [27]). Here L^a is the $L^2(\mathbb{R}^d)$ -generator of $(T_t^a)_{t>0}$. In the same way as [28], $\{P_t^a\}$ is a strong Feller semigroup, that is, $P_t^a f$ is bounded continuous for bounded measurable f . Furthermore, $\{P_t^a\}$ is a Feller semigroup in view of the upper Gaussian estimate and the estimation (Corollary 7.3 in [28]) of the local Hölder continuity of the local solution of the above parabolic equation (see also Example 4.5.2 in [8]). Let $\mathbf{M}^a := (\Omega, X_t, \mathcal{F}_t, P_x^a)$ be the Hunt process constructed by the Feller semigroup $\{P_t^a\}$. Then $\mathbf{M}^a$ is a doubly Feller process (cf. [4]) and is properly associated with $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ (cf. [8]). Note that $\mathbf{M}^a$ is conservative (see Example 5.7.1 in [8]).

Definition 1.4 (Dynkin and Kato class functions). A measurable function f on $\mathbb{R}^d$ is said to be of the *Dynkin class* with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ (write $f \in D_d$) (resp. *Kato class* with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ (write $f \in K_d$)) if $c_\alpha := \|R_\alpha^a |f|\|_\infty < \infty$ for some $\alpha > 0$ (resp. $\lim_{\alpha \rightarrow \infty} c_\alpha = 0$). Here $R_\alpha^a := \int_0^\infty e^{-\alpha t} P_t^a dt$ are the resolvents associated with $\mathbf{M}^a$. A measurable function f on $\mathbb{R}^d$ is said to be of the *local Dynkin class* with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ (write $f \in D_d^{loc}$) (resp. *local Kato class* with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$) (write $f \in K_d^{loc}$) if and only if $I_K f \in D_d$ (resp. $I_K f \in K_d$) for any compact set K in $\mathbb{R}^d$.

We see $K_d \subset D_d \subset H_d$ and if $f \in K_d$, $\delta(|f|)$ can be taken to be arbitrarily small (see [1], [25]). It is well known that for $2p > d \geq 2$ or $p \geq d = 1$, $L^p(\mathbb{R}^d) \subset L_{unif}^p(\mathbb{R}^d) \subset K_d \subset D_d$. Hence $L_{loc}^p(\mathbb{R}^d) \subset K_d^{loc} \subset D_d^{loc}$ (see Theorem 1.4(iii) in [1]). We can see that for $f \in D_d \cap L^1(\mathbb{R}^d)$, $|f|m \in S_{00}$ in view of Problem 4.2.1 in [8].

Definition 1.5 (Smooth measures in the strict sense, [8]). A Borel measure μ on $\mathbb{R}^d$ is said to be *smooth in the strict sense* with respect to $(\mathcal{E}^a, H^1(\mathbb{R}^d))$ (write $\mu \in S_1$) if there exists a sequence $\{E_n\}$ of Borel sets increasing to $\mathbb{R}^d$ such that

$I_{E_n}\mu \in S_{00}$ for each $n \in \mathbb{N}$ and

$$P_x^a(\lim_{n \rightarrow \infty} \tau_{E_n} = \infty) = 1, \quad x \in \mathbb{R}^d,$$

where $\tau_{E_n} := \inf\{t > 0 \mid X_t \notin E_n\}$ is the first exit time from E_n .

Since $D_d \subset H_d \subset L_{loc}^1(\mathbb{R}^d)$, $D_d^{loc} \subset L_{loc}^1(\mathbb{R}^d)$, then for $f \in D_d^{loc}$ we have $|f|m \in S_1$. It is well known that $S_{00} \subset S_1 \subset S$ and there exists a one-to-one correspondence between S (resp. S_1) and the equivalence class of $\mathbf{A}_c^+$ of PCAF admitting exceptional sets (resp. $\mathbf{A}_{c,1}^+$ of PCAF in the strict sense) under the Revuz correspondence (see Theorem 5.1.4 and 5.1.7 in [8]).

We consider a Fukushima decomposition in the strict sense for coordinate functions $e^i(x) := x^i$ ($i = 1, 2, \dots, d$), $x = (x^1, x^2, \dots, x^d)$:

$$X_t - X_0 = M_t + N_t, \quad P_x^a\text{-a.s. for all } x \in \mathbb{R}^d.$$

Here $M_t = (M_t^1, M_t^2, \dots, M_t^d)$, $N_t = (N_t^1, N_t^2, \dots, N_t^d)$, M_t^i is a local CAF in the strict sense and an MAF locally of finite energy, and N_t^i is a local CAF in the strict sense and a CAF locally of zero energy (see Theorem 2 in [7]). Under $\mathbf{M}^a$, we consider the following multiplicative functional L_t :

$$L_t(= L_t(b)) := \exp\left[\int_0^t (a^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^t (ba^{-1}b^*)(X_s) ds\right].$$

Note that $\Lambda^{-1} \int_0^t |b|^2(X_s) ds \leq \int_0^t (ba^{-1}b^*)(X_s) ds \leq \lambda^{-1} \int_0^t |b|^2(X_s) ds$ and in general the PCAF $\int_0^t (ba^{-1}b^*)(X_s) ds$ admits an exceptional N_b (see section 5.1 in [8]). Then L_t is a local P_x^a -martingale and a P_x^a -supermartingale multiplicative functional for all $x \in \mathbb{R}^d \setminus N_b$. If $|b|^2 m \in S_1$, then the PCAF $\int_0^t (ba^{-1}b^*)(X_s) ds$ admits no exceptional set, that is, $N_b = \emptyset$. If $p > d \geq 2$, then $|b|^2 \in K_d \subset D_d^{loc}$; hence $|b|^2 m \in S_1$, and L_t is an exponential P_x^a -martingale for all $x \in \mathbb{R}^d$. Precisely to say, $p_t^a f(x) := E_x^a[f(X_t)]$ satisfies the following estimate (see [17]): By Aronson's estimate (1.3) with the help of Minkovski's inequality, we have for $d \geq 1$, $t > 0$,

$$(1.4) \quad \|p_t^a\|_{L^\beta \rightarrow L^{\beta'}} \leq c(M, \beta, \beta', d) t^{-\frac{d}{2}(1/\beta - 1/\beta')}, \quad 1 \leq \beta \leq \beta' \leq \infty.$$

Hence, f^2 is a Kato class function if $f \in L^p(\mathbb{R}^d)$ with $p > d \geq 2$. Consequently, there exists a constant $c_1 := c_1(M, p, d, \|f\|_{L^p}) > 0$ such that

$$\sup_{x \in \mathbb{R}^d} E_x^a[e^{\int_0^t \{f(X_s)\}^2 ds}] \leq c_1 e^{c_1 t} \text{ for } f \in L^p(\mathbb{R}^d) \text{ with } p > d \geq 2,$$

which implies Novikov's criterion and the P_x^a -martingale property for every $x \in \mathbb{R}^d$ ([17]). The continuous supermartingale multiplicative functional L_t generates a right process in the sense of [21] (see §62 in [21]).

Lemma 1.1. *Suppose $|b| \in L^p(\mathbb{R}^d)$, $p \geq 2$. The following assertions hold:*

- (a) *Assumption 1.2(a) holds for $p \in (4d/(d+2), 4)$ with $d \geq 1$.*
- (b) *Assumption 1.2(b) holds for $p \in [4d/(d+2), 4)$ with $d \geq 3$, $p \in (2, 4)$ with $d = 2$, or $p \in [2, 4)$ with $d = 1$, respectively.*

Proof. (a) Suppose $|b| \in L^p(\mathbb{R}^d)$ for $p \in (4d/(d+2), 4)$, $d \geq 1$. By (1.4), we have $\|R_\alpha^a |b|^2\|_{L^{p/(p-2)}} \leq \int_0^\infty e^{-\alpha t} t^{-\frac{d}{2} \frac{4-p}{p}} dt \|b\|_{L^p}^2 < \infty$. Hence, for some/all $\alpha > 0$, $\int_{\mathbb{R}^d} (R_\alpha^a |b|^2) |b|^2 dm < \infty$. Equivalently, $R_\alpha^a |b|^2 \in H^1(\mathbb{R}^d)$ and $\mathcal{E}_\alpha^a(R_\alpha^a |b|^2, v) = \int_{\mathbb{R}^d} |b|^2 v dm$ for all $v \in H^1(\mathbb{R}^d)$ by Problem 4.2.1 in [8], which implies $|b|^2 m \in S_0$.

(b) If $|b| \in L^p(\mathbb{R}^d)$ for $p \in [4d/(d+2), 4)$, $d \geq 3$ (resp. $p \in (2, 4)$, $d = 2$), then $|b|^{4-p} \in L^{p/(4-p)}(\mathbb{R}^d) \subset H_d$ (resp. $|b|^{4-p} \in L^{p/(4-p)}(\mathbb{R}^d) \subset K_d$), because $p/(4-p) = 4/(4-p) - 1 \geq d/2$ for $d \geq 3$ (resp. $> d/2 = 1$ for $d = 2$). Similarly, if $|b| \in L^p(\mathbb{R}^d)$ for $p \in [2, 4)$ with $d = 1$, then $|b|^{4-p} \in L^{p/(4-p)}(\mathbb{R}^d) \subset K_d$ by $p/(4-p) = 4/(4-p) - 1 \geq 1$. $\square$

The following theorems are shown in [6] in more general context (cf. [15], [16], [23], [17], [20]).

Let $p_t^b f(x) := E_x^a[L_t f(X_t)]$, $\hat{p}_t^b f(x) := E_x^a[L_t \circ \gamma_t f(X_t)]$ for nonnegative or bounded Borel f . Here γ_t is the time reverse operator: $X_s(\gamma_t \omega) = X_{t-s}(\omega)$ for $s \in [0, t]$.

Theorem 1.1 (Contraction property, [15], [16], [23], [6]). *Suppose that Assumption 1.1 holds. If $p \geq 2/(2 - \sqrt{2\delta(|b|^2)/\lambda})$ (resp. $p \leq 2/\sqrt{2\delta(|b|^2)/\lambda}$), then $e^{-\alpha t} p_t^b$ (resp. $e^{-\alpha t} \hat{p}_t^b$) can be extended to a contraction semigroup $e^{-\alpha t} P_t^b$ (resp. $e^{-\alpha t} \hat{P}_t^b$) on $L^p(\mathbb{R}^d)$ for $\alpha \geq \alpha_p(b) := \frac{2\gamma(|b|^2)}{p} \sqrt{2/\lambda \delta(|b|^2)}$ (resp. $\alpha \geq \alpha_{p/(p-1)}(b)$).*

Note that $2/(2 - \sqrt{2\delta(|b|^2)/\lambda}) \leq 2 \leq 2/\sqrt{2\delta(|b|^2)/\lambda}$.

Theorem 1.2 (Girsanov formula, [6]). *Under Assumptions 1.1, let $e^{-\alpha t} T_t^b$ be the $L^2(\mathbb{R}^d)$ -semigroup associated with the semi-Dirichlet form $(\mathcal{E}_\alpha^b, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$, $\alpha > \alpha_2(b)$ and $\hat{T}_t^b$ the dual semigroup of T_t^b on $L^2(\mathbb{R}^d)$. Then $T_t^b f(x) = P_t^b f(x)$ (resp. $\hat{T}_t^b f(x) = \hat{P}_t^b f(x)$) a.e. $x \in \mathbb{R}^d$ for $f \in L^2(\mathbb{R}^d)$.*

We will expose the following:

Theorem 1.3 (Conservativeness of process). *Under Assumptions 1.1 and 1.2, L_t is a P_x^a -martingale for all $x \in \mathbb{R}^d \setminus N_b$. If furthermore $|b|^2 m \in S_1$, then L_t is a P_x^a -martingale for all $x \in \mathbb{R}^d$. In particular, the transformed process by L_t is conservative.*

Corollary 1.1. *For $b \in L^d(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ with $d \geq 3$, L_t is a P_x^a -martingale for all $x \in \mathbb{R}^d \setminus N_b$.*

Proof. Since $|b|^2$ is of the Hardy class and $\delta(|b|^2)$ can be chosen to be arbitrarily small for $b \in L^d(\mathbb{R}^d \rightarrow \mathbb{R}^d)$, we see $d > 4/(2 - \sqrt{2\delta(|b|^2)/\lambda})$ for sufficiently small $\delta(|b|^2)$. So Assumptions 1.1 and 1.2(c) hold. Also, Assumptions 1.1 and 1.2(a), or (b) hold for $p = d \geq 3$. $\square$

Example 1.1. Let $g \in C_0^\infty([0, \infty) \rightarrow [0, \infty])$ with $g(r) := 1/(r\sqrt{-\log r})$ on $(0, 1/e]$, $g(r) := 0$ on $[1, \infty)$ and $f(x) := g(|x|)$. Then $f \in L^d(\mathbb{R}^d)$, but $f \notin L^p(\mathbb{R}^d)$ for $p > d$. Let $\mathbf{M} := (B_t, P_x)$ be the d -dimensional Brownian motion with $d \geq 3$. Then for $b(x) := f(x)(1, 1, \dots, 1)$, $L_t := \exp[\sum_{i=1}^d \int_0^t f(B_s) dB_s^i - \frac{d}{2} \int_0^t f(B_s)^2 ds]$ is a P_x -martingale for all $x \in \mathbb{R}^d \setminus \{0\}$. In this case $N_b = \{0\}$. Note that f^2 is not of the Kato class function by Theorem 1.3(viii) in [1]. Also, Novikov's condition fails for starting point 0, because the low of the iterated logarithm of 1-dimensional Brownian motion yields $E_0[\exp[\int_0^t f(B_s)^2 ds]] \geq E_0[\int_0^t f(B_s)^2 ds] = \infty$ (cf. Example 5.1.1 in [8]).

Example 1.2 (cf. [9], [12], [3], Example (4.2) in [5]). When $d \geq 3$, the following Hardy-type inequality holds: for any $r > 0$,

$$(1.5) \quad \int_{B_r(0)} \frac{u(x)^2}{|x|^2} dx \leq \frac{4}{(d-2)^2} \int_{B_r(0)} |\nabla u(x)|^2 dx \quad \text{for } u \in H_0^1(B_r(0)).$$

Hence

$$(1.6) \quad \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2} dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \quad \text{for } u \in H^1(\mathbb{R}^d).$$

Take $\varphi \in C^\infty([0, \infty))$ with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $[0, 1/2]$, $\varphi(s) \leq 1/s^d$ for $s \in [1, \infty)$. Let $b(x) := \frac{d-2}{4+\varepsilon} \frac{\varphi(|x|)}{|x|^2} x$ for $\varepsilon > 0$. Then $|b|^2$ is of the Hardy class with respect to the Dirichlet form associated to d -dimensional Brownian motion $\mathbf{M} := (B_t, P_x)$, and $\delta(|b|^2) = 8/(4+\varepsilon)^2 < 1/2$. Note that $|b| \notin L^p(\mathbb{R}^d)$ for $p \geq d$, but $|b| \in L^p(\mathbb{R}^d)$ for $d/(d+1) < p < d$ and $|b|^2$ is not of the Kato class function when the support of φ is bounded, by Theorem 1.3(viii) in [1]. In this case $N_b = \{0\}$, $|b| \in L^2(\mathbb{R}^d)$ implies the P_x -martingale property of L_t for all starting points $x \in \mathbb{R}^d \setminus \{0\}$. Novikov's condition fails for starting point 0 as in the previous example.

2. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. First we show that L_t is a P_{gm}^a -martingale for $g \in C_0^+(\mathbb{R}^d)$. Consequently, L_t is a P_x^a -martingale for a.e. $x \in \mathbb{R}^d$. We have $P_{gm}^a(\cdot) := \int_{\mathbb{R}^d} P_x^a(\cdot) g(x) m(dx)$ and $m(dx) := dx$ is the d -dimensional Lebesgue measure. There exists an $\mathcal{E}^a$ -nest $\{F_n\}$ such that $I_{F_n}|b|^2$ is of the Kato class satisfying $I_{F_n}|b|^2 m \in S_{00}$ ([2]). Then $b_n := (I_{F_n} b^1, \dots, I_{F_n} b^d)$ gives an $L^p(\mathbb{R}^d)$ -approximation for b . Since $|b_n|^2$ is of the Kato class function, each $L_t^n := L_t(b_n)$ is an exponential P_x^a -martingale for all $x \in \mathbb{R}^d$. We put $p_t^n f(x) := E_x^a[f(X_t) L_t^n]$ for nonnegative or bounded Borel functions f . By $|b_n| \leq |b|$, $\delta(|b_n|^2)$ (resp. $\gamma(|b_n|^2)$) can be taken as a common constant $\delta(|b|^2)$ (resp. $\gamma(|b|^2)$). Hence we may set $\alpha_p(b_n) = \alpha_p(b)$ for all $p > 1$ so that $(\mathcal{E}_\alpha^{b_n}, H^1(\mathbb{R}^d))$ is coercive for $\alpha > \alpha_2(b)$. By Theorems 1.1 and 1.2, we can confirm that for $\alpha \geq \alpha_r(b)$, $e^{-\alpha t} p_t^n$ can be extended to a strongly continuous contraction semigroup $e^{-\alpha t} P_t^n$ on $L^r(\mathbb{R}^d)$ for all $r \geq 2/(2 - \sqrt{2\delta(|b|^2)/\lambda})$ and $e^{-\alpha t} P_t^n$ coincides with the $L^2(\mathbb{R}^d)$ -contractive semigroups $\{e^{-\alpha t} T_t^n\}$ associated to the semi-Dirichlet form $(\mathcal{E}_\alpha^{b_n}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ for $\alpha > \alpha_2(b)$. The association means that the resolvent $G_\alpha^n := \int_0^\infty e^{-\alpha t} T_t^n dt$ satisfies $\mathcal{E}_\alpha^{b_n}(G_\alpha^n f, v) = (f, v)_{L^2}$ for $f \in L^2(\mathbb{R}^d)$, $v \in H^1(\mathbb{R}^d)$, $\alpha > \alpha_2(b) = \alpha_2(b_n)$. Note that the contraction property of $e^{-\alpha t} P_t^n$ holds for all $r \geq 2$, because $2 > 2/(2 - \sqrt{2\delta(|b|^2)/\lambda})$.

On the other hand, for $g \in C_0^+(\mathbb{R}^d)$,

$$\begin{aligned} E_{gm}^a \left[\left(\int_0^t (a^{-1}(b_n - b))^*(X_s) dM_s \right)^2 \right] &= E_{gm}^a \left[\int_0^t ((b_n - b) a^{-1} (b_n - b)^*)(X_s) ds \right] \\ &\leq \lambda^{-1} E_{gm}^a \left[\int_0^t |b_n - b|^2(X_s) ds \right] \\ &\leq \|g\|_{L^{\frac{p}{p-2}}} t \quad \text{for } p \geq 2. \end{aligned}$$

This implies that $L_t^n \rightarrow L_t$ as $n \rightarrow \infty$ in P_{gm}^a -measure for $g \in C_0^+(\mathbb{R}^d)$.

Next we show uniform integrability of $\{L_t^n\}_{n \in \mathbb{N}}$ under P_{gm}^a . By Itô's formula, we have

$$L_t^n \log L_t^n = \int_0^t (1 + \log L_s^n) dL_s^n + \frac{1}{2} \int_0^t L_s^n (b_n) a^{-1} (b_n)^*(X_s) ds.$$

Taking an increasing sequence of stopping times $\{T_k^n\}$ such that $T_k^n \rightarrow \infty$ as $k \rightarrow \infty$ P_{gm}^a -a.e. and

$$\int_0^{t \wedge T_k^n} (1 + \log L_s^n) dL_s^n$$

is a P_{gm}^a -martingale, we have the following estimate if $p \geq 4$, or if $2 \leq p < 4$ under Assumption 1.2(a) or (b): for $\alpha > \alpha_2(b)$,

$$\begin{aligned} E_{gm}^a[L_{t \wedge T_k^n}^n \log L_{t \wedge T_k^n}^n] &= \frac{1}{2} E_{gm}^a \left[\int_0^{t \wedge T_k^n} L_s^n (b_n) a^{-1}(b_n)^*(X_s) ds \right] \\ &\leq \frac{\lambda^{-1}}{2} E_{gm}^a \left[\int_0^t L_s^n |b_n|^2 (X_s) ds \right] \\ &\leq \frac{\lambda^{-1}}{2} \int_0^t \int_{\mathbb{R}^d} g(P_s^n |b|^2) dm ds \\ &\leq \begin{cases} \frac{\lambda^{-1}}{2} \int_0^t e^{\alpha s} \|g\|_{L^{\frac{p}{p-2}}} \|e^{-\alpha s} P_s^n |b|^2\|_{L^{\frac{p}{2}}} ds & \text{if } p \in [4, \infty) \\ \frac{\lambda^{-1} e^{\alpha t}}{2} \int_{\mathbb{R}^d} (\hat{G}_\alpha^n g) |b|^2 dm & \text{if } p \in [2, 4) \end{cases} \\ &\leq \begin{cases} \frac{\lambda^{-1}}{2} \frac{e^{\alpha t} - 1}{\alpha} \|g\|_{L^{\frac{p}{p-2}}} \|b\|_{L^p}^2 & \text{if } p \in [4, \infty). \\ \frac{\lambda^{-1} e^{\alpha t}}{2} \|b\|_{L^p}^{p/2} \|(\hat{G}_\alpha^n g) |b|^{(4-p)/2}\|_{L^2} & \text{if } p \in [2, 4). \\ \frac{\lambda^{-1} e^{\alpha t}}{2} \left(\int_{\mathbb{R}^d} (R_\alpha^n |b|^2) |b|^2 dm \cdot \mathcal{E}_\alpha^n(\hat{G}_\alpha^n g) \right)^{1/2} & \text{if } p \in [2, 4). \end{cases} \end{aligned}$$

Here we use the $L^{p/2}$ -contraction property of $e^{-\alpha t} P_t^n$ for $p \geq 4$, $\alpha > \alpha_2(b)$ ($\geq \alpha_{p/2}(b)$) and $(\hat{G}_\alpha^n)_{\alpha > \alpha_2(b)}$ is the dual of $(G_\alpha^n)_{\alpha > \alpha_2(b)}$. By

$$\mathcal{E}^a(u, u) \leq (1 - \sqrt{2\delta(|b|^2)/\lambda})^{-1} \mathcal{E}_\alpha^{b_n}(u, u) \text{ for } u \in H^1(\mathbb{R}^d), \alpha > \alpha_2(b),$$

we have $\sup_{n, k \in \mathbb{N}} E_{gm}^a[L_{t \wedge T_k^n}^n \log L_{t \wedge T_k^n}^n] < \infty$. Similarly, under Assumption 1.2(c) with the help of Theorem 1.1, for $\alpha \geq \alpha_{p/2}(b)$,

$$\begin{aligned} E_{gm}^a[L_{t \wedge T_k^n}^n \log L_{t \wedge T_k^n}^n] &= \frac{1}{2} E_{gm}^a \left[\int_0^{t \wedge T_k^n} L_s^n (b_n) a^{-1}(b_n)^*(X_s) ds \right] \\ &\leq \frac{\lambda^{-1}}{2} \int_0^t e^{\alpha s} \|g\|_{L^{\frac{p}{p-2}}} \|e^{-\alpha s} P_s^n |b|^2\|_{L^{\frac{p}{2}}} ds \\ &\leq \frac{\lambda^{-1} e^{\alpha t} - 1}{2\alpha} \|g\|_{L^{\frac{p}{p-2}}} \|b\|_{L^p}^2 < \infty. \end{aligned}$$

These imply uniform integrability of $\{L_{t \wedge T_k^n}^n\}_{n, k \in \mathbb{N}}$ under P_{gm}^a . Since $I_{\{L_{t \wedge T_k^n}^n > K\}}$ is lower semicontinuous as $k \rightarrow \infty$ for each $K > 0$, we see the uniform P_{gm}^a -integrability of $\{L_t^n\}_{n \in \mathbb{N}}$. Hence L_t^n converges to L_t in $L^1(P_{gm}^a)$ as $n \rightarrow \infty$. Therefore $\{L_t\}$ is a P_{gm}^a -martingale. Finally we show the martingale property of L_t for all $x \in \mathbb{R}^d \setminus N_b$. Let $R_\alpha^b f(x) := \int_0^\infty e^{-\alpha t} p_t^b f(x) dt$, $\alpha > 0$ be the resolvent of $(p_t^b)_{t \geq 0}$. Then $R_\alpha^b 1$ is α -excessive, hence, finely continuous with respect to the transformed process by L_t . By the first argument, we see that $\alpha R_\alpha^b 1(x) = 1$ for a.e. $x \in \mathbb{R}^d$. Note that the absolute continuity condition holds with respect to the d -dimensional Lebesgue measure m for the transformed process on $\mathbb{R}^d \setminus N_b$. Indeed, suppose $f = 0$ a.e. Then for all $x \in \mathbb{R}^d \setminus N_b$, $P_x^a(f(X_t) = 0) = 1$; hence $E_x^a[L_t f(X_t)] = 0$ for all $x \in \mathbb{R}^d \setminus N_b$. The d -dimensional Lebesgue measure m has a full support with respect to the fine topology of the transformed (right) process (see Lemma 4.6 in

[14]). Therefore, $\alpha R_\alpha^b 1(x) = 1$ for all $x \in \mathbb{R}^d \setminus N_b$, and hence $p_t^b 1(x) = 1$ for all $t > 0$ and $x \in \mathbb{R}^d \setminus N_b$. This completes the proof. $\square$

Remark 2.1. (a) In [20], the conservativeness of the transformed process is proved (Proposition 3.1 in [20]) if $b \in L^d(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ with $d \geq 3$ and $\operatorname{div} b \geq 0$ in the following distributional sense:

$$\int_{\mathbb{R}^d} \langle b, \nabla u(x) \rangle_{\mathbb{R}^d} dx \leq 0 \quad \text{for } u \in C_0^\infty(\mathbb{R}^d) \text{ with } u \geq 0.$$

Our result shows that the condition $\operatorname{div} b \geq 0$ is not needed.

(b) In view of the result from uniqueness of the positive Cauchy problem (see [10], [11]), our integrability condition for $b \in L^p(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ ($p \geq 2$) (in particular, L^p -integrability at infinity) seems to be strong. However, in [10] and [11], $b \in L_{loc}^p(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ ($p > d$) is imposed when $d \geq 2$. So Examples 1.1 and 1.2 are not derived from [10], [11].

(c) Theorem 1.3 can be generalized in the framework of a regular local symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ corresponding to a conservative doubly Feller diffusion process $\mathbf{M}$ ([4]), where X is a locally compact separable metric space and m is a positive Radon measure on X with full topological support. For example, let $M \in \mathring{\mathcal{M}}_{loc}$ be a local martingale CAF locally of finite energy such that the energy measure $\mu_{\langle M \rangle}$ corresponding to the quadratic variation process $\langle M \rangle$ is of the Hardy class smooth measure with $\delta(M) := \delta(\mu_{\langle M \rangle}) < 1/2$. If $\mu_{\langle M \rangle} = \Gamma(M)m$ with $\Gamma(M) \in L^{p/2}(X; m)$ for $p \geq 4$ or $M \in \mathring{\mathcal{M}}$, or $4 > p > 4/(2 - \sqrt{2\delta(M)})$, then $L_t := \exp[M_t - \frac{1}{2}\langle M \rangle_t]$ generates a conservative diffusion process from $\mathbf{M}$ for every starting point $x \in X \setminus N_{\mu_{\langle M \rangle}}$.

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