THE LINEAR HEAT EQUATION WITH HIGHLY OSCILLATING POTENTIAL

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(Communicated by Carmen C. Chicone)

ABSTRACT. In this paper we consider the following initial value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = -Hu + V(x)u & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) \geq 0 & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

where $H = -\Delta - \frac{\beta}{|x|^2} \sin(\frac{1}{|x|^\alpha})$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$. Nonexistence of positive solutions is analyzed.

1. INTRODUCTION

Recently there has been great interest in the investigation of existence and nonexistence of positive solutions for the linear heat equation with a potential, e.g. [3], [8], [9] and [6]. The motivating work for these papers is the classical paper of Baras and Goldstein [2], in which it was shown that the initial value problem

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2} u & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \mathbb{R}^N \end{cases}$$

has no nonnegative solutions except $u \equiv 0$ if $c > C^*(N) = (\frac{N-2}{2})^2$. Moreover, all positive solutions blow up instantaneously, in the sense that if u_n is the solution of the same problem with the potential $c/|x|^2$ replaced by $V_n = \min\{c/|x|^2, n\}$, then $\lim_{n \rightarrow \infty} u_n(x, t) = \infty$ for all $x \in \mathbb{R}^N$ and $t > 0$. If $c \leq C^*(N)$, positive weak solutions do exist. Thus $C^*(N) = (\frac{N-2}{2})^2$ is the cutoff point for existence of positive solutions for the heat equation with inverse square potential $c/|x|^2$, where $C^*(N)$ is also the sharp constant in Hardy's inequality. Note that by Hardy's inequality, the infimum of the expression

$$\frac{\int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} \frac{c}{|x|^2} |\phi(x)|^2 dx}{\int_{\mathbb{R}^N} |\phi(x)|^2 dx}$$

over $0 \neq \phi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is either $-\infty$ or 0, depending on whether $c > C^*(N)$ or $c \leq C^*(N)$.

Cabr and Martel [3] extended the above result to general positive singular potentials. They discovered that existence and nonexistence of positive solutions is

Received by the editors April 21, 2003 and, in revised form, June 18, 2003.

2000 *Mathematics Subject Classification*. Primary 35K15, 35K25, 35R25.

Key words and phrases. Heat equation, instantaneous blow up, positive solutions.

largely determined by the size of the infimum of the spectrum of the symmetric operator $S = -\Delta - V$, which is

$$(1.2) \quad \sigma_{\inf} = \sigma_{\inf}(V; \Omega) = \inf_{0 \neq \phi \in Q} \frac{\int_{\Omega} |\nabla \phi|^2 dx - \int_{\Omega} V |\phi|^2 dx}{\int_{\Omega} |\phi|^2 dx}.$$

Here Q is a core of S . In dimension $N \geq 3$, one has $Q = C_0^\infty(\Omega)$; however, for $0 \in \Omega \subset \mathbb{R}^N$ and $N \leq 2$, it is convenient to take $Q = C_0^\infty(\Omega \setminus \{0\})$.

Goldstein and Zhang [8] have found an analogue of the results for (1.1) on the Heisenberg group. Namely,

$$(1.3) \quad \begin{cases} u_t(z, l, t) = \Delta_{\mathbb{H}^n} u(z, l, t) - \frac{c|z|^2}{|z|^4 + l^2} u(z, l, t), \\ u(z, l, 0) = u_0(z, l), \quad (z, l) = (x, y, l) \in \mathbb{H}^n, \quad t \in (0, T], \quad T > 0, \end{cases}$$

has no positive solution if $c > C_*(n) = n^2$. Moreover, they have shown that all positive solutions blow up instantaneously, and if $c \leq C_*(n)$, then positive weak solutions do exist.

Recently, Goldstein and Zhang [9] have extended the method of Cabré and Martel [3] to uniformly parabolic equations with positive singular potentials. Our work in this paper has been influenced by [2], [3], [8], and [9] with the following question:

- Is it possible to perturb the positive potential by a singular and sign changing potential that does not effect the nonexistence of positive solutions?

We answer this question in this paper.

In the same context there are also some related works on nonlinear parabolic problems by Garcia and Peral [5], Aguilar and Peral [1], Goldstein and Kombe [7].

2. HIGHLY SINGULAR, OSCILLATING POTENTIALS

In the present paper we consider the initial value problem

$$(2.1) \quad \begin{cases} \frac{\partial u}{\partial t} = -Hu + V(x)u & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) \geq 0 & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

where $N \geq 3$, $u_0(x)$ is not identically zero, and $0 \leq V(x) \in L_{\text{loc}}^1(\mathbb{R}^N)$. The operator H is the symmetric operator (defined on $C_c^\infty(\mathbb{R}^N \setminus \{0\})$)

$$(2.2) \quad H = -\Delta - \frac{\beta}{|x|^2} \sin\left(\frac{1}{|x|^\alpha}\right),$$

where $\alpha > 0$ and $\beta \in \mathbb{R} \setminus \{0\}$. Although the potential $V_{\text{osc}} = \frac{\beta}{|x|^2} \sin(\frac{1}{|x|^\alpha})$ is highly singular and oscillates wildly, a remarkable result has been obtained by K.-T. Sturm [12]. He proved that H satisfies the Gaussian bounds

$$(2.3) \quad \gamma_1 e^{\alpha_1 t} G_0(x, y, \beta_1 t) \leq G_{\text{osc}}(x, y, t; V_{\text{osc}}) \leq \gamma_2 e^{\alpha_2 t} G_0(x, y, \beta_2 t)$$

for all $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$, where $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}_+ \setminus \{0\}$; $\alpha_1, \alpha_2 \in \mathbb{R}$; G_{osc} is the Green's function for H , and

$$G_0(x, y, t) = (4\pi t)^{-N/2} \exp(-|x - y|^2/(4t))$$

is the Green's function for $-\Delta$ on $\mathbb{R}^N$.

Before proceeding to the main results of the paper, we present the definition of a weak solution.

Definition 2.1. We say that $u(\cdot, t)$ is a *weak solution* to (2.1) if

$$u, |\nabla u| \in L^2_{\text{loc}}(\mathbb{R}^N \times (0, T)), \quad V(\cdot)u(\cdot, t) \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$$

and

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\mathbb{R}^N} u \phi_t dx dt - \int_{\mathbb{R}^N} (u \phi)|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi dx dt \\ &= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (V + \frac{\beta}{|x|^2} \sin(\frac{1}{|x|^\alpha})) u \phi dx dt \end{aligned}$$

for all compactly supported $C^{1,1}$ -functions ϕ and $0 < t_1 < t_2 < T$.

In general we approach the problem in (2.1) via the approximate problem

$$(2.4) \quad \begin{cases} \frac{\partial u_k}{\partial t} = -Hu_k + V_k(x)u_k & \text{in } \mathbb{R}^N \times (0, \infty), \\ u_k(x, 0) = u_0(x) & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

where V_k is bounded above and $V_k(x) \uparrow V(x)$ for all $x \in \mathbb{R}^N$. Maximum principle arguments show that u_k exists (if u_0 is not too singular), is the unique positive solution to (2.4), and $u_k(x, t) \leq u(x, t)$ for all $(x, t) \in \mathbb{R}^N \times [0, \infty)$ if $u(x, t)$ exists; and the sequence $u_k(x, t)$ is nondecreasing in k .

Definition 2.2. We say that (2.1) has *instantaneous blow-up* if $\lim_{k \rightarrow \infty} u_k(x, t) = \infty$ for all $(x, t) \in \mathbb{R}^N \times (0, \infty)$.

The following is our main result.

Theorem 2.3. Let $N \geq 3$, $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \mathcal{K})$, and $Q = \{\phi \in C_c^\infty(\mathbb{R}^N \setminus \mathcal{K}) : \phi \text{ real}, \phi \not\equiv 0\}$, where $\mathcal{K}$ is a finite set. If

$$\sigma_{\inf}(V; \mathbb{R}^N \setminus \mathcal{K}) := \inf_{\phi \in Q} \frac{\int_{\mathbb{R}^N} (|\nabla \phi|^2 - (1 - \epsilon_1)V|\phi|^2) dx}{\int_{\mathbb{R}^N} |\phi|^2 dx} = -\infty$$

for some $\epsilon_1 > 0$, then the problem (2.1) has no positive solutions. Moreover, we have instantaneous blow-up.

3. PROOF OF THEOREM 2.3

The proof is by contradiction. Given any $T > 0$, let u be a solution to (2.1) in $\mathbb{R}^N \times (0, T)$ with $u_0 \geq 0$ but not identically zero. First we will proceed by approximation. We truncate V as

$$V_k = \min\{V, k\}, \quad k \in \mathbb{N},$$

and let $\{u_k\}_{k \in \mathbb{N}}$ be the unique positive solution of (2.4). We must show that $u_k(x, t) \rightarrow \infty$ for all $x \in \mathbb{R}^N$ and $t > 0$.

Let us multiply both sides of (2.4) by the test function ϕ^2/u_k with $\phi \in Q$, and integrate over $\mathbb{R}^N$:

$$(3.1) \quad \int_{\mathbb{R}^N} (V_k(x) + V_{\text{osc}}(x)) \phi^2(x) dx = \partial_t \int_{\mathbb{R}^N} (\log u_k) \phi^2 dx - \int_{\mathbb{R}^N} \Delta u_k \left(\frac{\phi^2}{u_k} \right) dx.$$

Using integration by parts, we get

$$\int_{\mathbb{R}^N} \Delta u_k \left(\frac{\phi^2}{u_k} \right) dx = - \int_{\mathbb{R}^N} \nabla u_k \cdot \nabla \left(\frac{\phi^2}{u_k} \right) dx.$$

Indeed, a direct computation shows that

$$(3.2) \quad \begin{aligned} \int_{\mathbb{R}^N} \Delta u_k \left(\frac{\phi^2}{u_k} \right) dx &= \int_{\mathbb{R}^N} |\nabla u_k|^2 \frac{\phi^2}{u_k^2} dx - 2 \int_{\mathbb{R}^N} (\nabla u_k \cdot \nabla \phi) \frac{\phi}{u_k} dx \\ &\geq - \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx. \end{aligned}$$

Substituting (3.2) into (3.1), we obtain

$$\int_{\mathbb{R}^N} V_k(x) \phi^2(x) dx + \int_{\mathbb{R}^N} V_{\text{osc}}(x) \phi^2(x) dx - \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx \leq \partial_t \int_{\mathbb{R}^N} (\log u_k) \phi^2 dx.$$

Integrating from t_1 to t_2 ($0 < t_1 < t_2$), we get

$$(3.3) \quad \begin{aligned} &\int_{\mathbb{R}^N} V_k(x) \phi^2 dx + \int_{\mathbb{R}^N} V_{\text{osc}}(x) \phi^2(x) dx - \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx \\ &\leq \frac{1}{t_2 - t_1} \int_{\mathbb{R}^N} \log \left(\frac{u_k(x, t_2)}{u_k(x, t_1)} \right) \phi^2(x) dx. \end{aligned}$$

From here on, we will use $L_w^1(\mathbb{R}^N) = \{f \in \mathbb{R}^N : \int_{\mathbb{R}^N} |f(x)| e^{-|x|} dx < \infty\}$, the L^1 -space with weight $w(x) \equiv e^{-|x|}$. Let $\rho \in C_0(\mathbb{R}^N)$ be any strictly positive function in $\mathbb{R}^N$ such that ρ converges to zero at ∞ and $\log \rho \in L_w^p(\mathbb{R}^N)$ for any $p > 1$.

Let $T > 0$. We claim that there exists at most one point $t_1 \in (0, T)$ such that $u(\cdot, t_1) \rho(\cdot) \in L_w^1(\mathbb{R}^N)$. Suppose the contrary. Then there exist $t_1, t_2 \in (0, T)$ with $t_2 > t_1$ such that

$$u(\cdot, t_i) \rho(\cdot) \in L_w^1(\mathbb{R}^N), \quad i = 1, 2.$$

Next, we will prove that

$$(3.4) \quad \log \left(u_k(\cdot, t_i) \rho(\cdot) \right) \in L_w^p(\mathbb{R}^N), \quad p \geq 1.$$

Let us write

$$\mathbb{R}^N = S_1 \cup S_2 \equiv \{x \in \mathbb{R}^N \mid u_k(x, t_1) \rho(x) \geq 1\} \cup \{x \in \mathbb{R}^N \mid u_k(x, t_1) \rho(x) < 1\}.$$

Using Jensen's inequality for the concave function $\log^p(s)$ over the set S_1 , we get

$$(3.5) \quad \int_{S_1} |\log(u_k(x, t_1) \rho(x))|^p w(x) dx \leq C \log^p \left(\int_{S_1} |u_k(x, t_1) \rho(x)| w(x) dx \right) < \infty,$$

where C is independent of k and we have used the fact that S_1 has finite Lebesgue measure.

When $x \in S_2$, we can write

$$1 > u_k(x, t_1) \rho(x) = \int_{\mathbb{R}^N} G(x, y, t_1; W_k) u_0(y) \rho(y) dy,$$

where G is the Green's function of (2.4) and $W_k = V_k + V_{\text{osc}}$. By the maximum principle,

$$\int_{\mathbb{R}^N} G(x, y, t_1; W_k) u_0(y) \rho(y) dy \geq \int_{\mathbb{R}^N} G_{\text{osc}}(x, y, t_1; V_{\text{osc}}) u_0(y) \rho(y) dy.$$

Using the Gaussian estimates (2.3), we obtain

$$\int_{\mathbb{R}^N} G(x, y, t_1; V_{\text{osc}}) u_0(y) \rho(y) dy \geq \gamma e^{\lambda t_1} \int_{\mathbb{R}^N} G_0(x, y, \beta t_1) u_0(y) \rho(y) dy,$$

where G_0 is the heat kernel of $u_t - \Delta u = 0$ and $\gamma > 0, \beta > 0, \lambda \in \mathbb{R}$.

Let us define h by

$$h(x, \beta t_1) \rho(x) \equiv \int_{\mathbb{R}^N} G_0(x, y, \beta t_1) u_0(y) \rho(x) dy.$$

Then we have

$$(3.6) \quad |\log(u_k(x, t_1) \rho(x))| \leq |\log(\gamma)| + |\lambda| t_1 + |\log(h(x, \beta t_1) \rho(x))|.$$

Clearly, $\log(\gamma)$ and $\lambda t_1 \in L_w^p(\mathbb{R}^N)$. We need to show that

$$(3.7) \quad \log h(x, \beta t_1) \in L_w^p(S_2)$$

for any $p > 1$.

If we assume $u_0 \leq c$, then $h(x, t) \leq c_2(t)$ follows immediately. But we want to minimize the number of assumptions. According to the strong maximum principle, there is, for any $t > 0$, a positive constant $c_1(t)$ such that

$$h(0, t/2) \geq c_1(t).$$

We connect $(0, \frac{t}{2})$ and (x, t) by a line l . Consider the points on l

$$x_j = \frac{2jx}{k|x|}, \quad t_j = \frac{t}{2} + \frac{jt}{k|x|}, \quad j = 0, 1, \dots, J; \quad J-1 < \frac{k|x|}{2} \leq J \in \mathbb{N},$$

where $k > 0$ is a suitable number. We can use the following Harnack inequality (see [11] or [10]):

$$\log \frac{h(x_j, t_j)}{h(x_{j+1}, t_{j+1})} \leq c \left[\frac{|x_{j+1} - x_j|^2}{t_{j+1} - t_j} + \log\left(\frac{t_{j+1}}{t_j}\right) + 1 \right],$$

where $0 < t_j < t_{j+1} < T$ and $c > 0$. Clearly we can write

$$(3.8) \quad h(x_j, t_j) \leq h(x_{j+1}, t_{j+1}) \exp \left[c \left(\frac{|x_{j+1} - x_j|^2}{t_{j+1} - t_j} + \frac{t_{j+1}}{t_j} + 1 \right) \right], \quad j = 0, \dots, J.$$

Iterating J times, we obtain

$$\begin{aligned} h(x_0, t_0) &\leq h(x_1, t_1) \exp \left[c \left(\frac{|x_1 - x_0|^2}{t_1 - t_0} + \frac{t_1}{t_0} + 1 \right) \right] \\ &\vdots \end{aligned}$$

$$c_1(t) \leq h(x_0, t_0) \leq h(x, t) \exp\left(\frac{2c|x|^2}{t}\right) \exp(kc|x|).$$

Note that

$$|x_{j+1} - x_j| = \frac{2}{k}, \quad |t_{j+1} - t_j| = \frac{t}{k|x|}, \quad j = 0, \dots, J,$$

and

$$\begin{aligned} \exp\left(c \sum \frac{t_{j+1}}{t_j}\right) &\leq \exp\left(\frac{c}{t_0} \sum t_{j+1}\right), \quad j = 0, \dots, J, \\ &\leq \exp(ck|x|). \end{aligned}$$

We can choose $k = 1/c$, and therefore

$$h(x, t) \geq c_1(t) \exp(-|x|) \exp(-2c \frac{|x|^2}{t}).$$

This is a lower bound. Next we find an upper bound for $h(x, t)$ using the same technique. By the above,

$$h(0, 2t) \geq c_2(t).$$

We connect $(0, 2t)$ and (x, t) by a line l . Consider the points on l where $x \neq 0$:

$$x_j = x(1 - \frac{j}{k|x|}), \quad t_j = t + \frac{jt}{k|x|}, \quad j = 0, 1, \dots, J; \quad J-1 < k|x| \leq J \in \mathbb{N},$$

where $k > 0$ is a suitable number. Iterating (3.8), we get

$$h(x, t) \leq c_2(t) \exp(2k|x|) \exp(c \frac{|x|^2}{t}),$$

and choosing $k = 1/c$ we obtain

$$c_1(t) \exp(-|x|) \exp(-2c \frac{|x|^2}{t}) \leq h(x, t) \leq c_2(t) \exp(2|x|) \exp(c \frac{|x|^2}{t}).$$

Clearly

$$|\log h(x, t)| \leq c_1(t) + |x| + 2c \frac{|x|^2}{t}.$$

Hence

$$\int_{S_2} |\log h(x, t)|^p w(x) dx \leq C.$$

This proves that

$$\int_{S_2} |\log(u_k(x, t_1)\rho(x))|^p w(x) dx \leq C.$$

Repeating this process for $u_k(\cdot, t_2)$, we conclude that

$$(3.9) \quad \log \frac{u_k(\cdot, t_2)}{u_k(\cdot, t_1)} = \log(u_k(\cdot, t_2)\rho(\cdot)) - \log(u_k(\cdot, t_1)\rho(\cdot)) \in L_w^p(\mathbb{R}^N)$$

for any $p > 1$.

Let us write

$$\varphi \equiv w(x) \frac{1}{t_2 - t_1} (\log \frac{u_k(x, t_2)}{u_k(x, t_1)}).$$

We know that $\varphi \in L^p(\mathbb{R}^N)$ for any $p > 1$. We will show that, for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$(3.10) \quad \int_{\mathbb{R}^N} \varphi \phi^2 dx \leq \frac{\epsilon}{(1-\epsilon)} \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx + C(\epsilon) \int_{\mathbb{R}^N} \phi^2(x) dx$$

for any $\phi \in W_0^{1,2}(\mathbb{R}^N)$.

Here is the proof. Given $\epsilon > 0$, we choose R sufficiently large so that

$$(3.11) \quad \int_{\mathbb{R}^N \setminus B(x_0, R)} \varphi^{\frac{N}{2}}(x) dx \leq \left(\frac{\epsilon}{2c_N(1-\epsilon)} \right)^{\frac{N}{2}},$$

where c_N is an appropriate constant which will be chosen later. We now use the following a priori inequality, which is an easy consequence of the Sobolev embedding theorem (see [9]). For every $\epsilon > 0$, there exists $C(\epsilon)$ such that

$$(3.12) \quad \begin{aligned} & \int_{B(x_0, R)} \varphi(x) \phi^2(x) dx \\ & \leq \frac{\epsilon}{2(1-\epsilon)} \int_{B(x_0, R)} |\nabla \phi(x)|^2 dx + C(\epsilon) \int_{B(x_0, R)} \phi^2(x) dx. \end{aligned}$$

We write

$$(3.13) \quad \int_{\mathbb{R}^N} \varphi \phi^2 dx = \int_{\mathbb{R}^N \setminus B(x_0, R)} \varphi(x) \phi^2(x) dx + \int_{B(x_0, R)} \varphi(x) \phi^2(x) dx.$$

Applying Hölder's inequality to

$$\int_{\mathbb{R}^N \setminus B(x_0, R)} \varphi(x) \phi^2(x) dx,$$

we obtain

$$\int_{\mathbb{R}^N \setminus B(x_0, R)} \varphi(x) \phi^2(x) dx \leq \left(\int_{\mathbb{R}^N \setminus B(x_0, R)} \varphi^{\frac{N}{2}}(x) dx \right)^{\frac{2}{N}} \left(\int_{\mathbb{R}^N} \phi^{\frac{2N}{N-2}}(x) dx \right)^{\frac{N-2}{N}}.$$

Applying the Gagliardo-Nirenberg-Sobolev inequality [4] leads to

$$\left(\int_{\mathbb{R}^N} \phi(x)^{\frac{2N}{N-2}}(x) dx \right)^{\frac{N-2}{N}} \leq c_N \int_{\mathbb{R}^N} |\nabla \phi|^2 dx$$

for some constant c_N . We now recall (3.11); this is the c_N appearing there.

Therefore,

$$(3.14) \quad \int_{\mathbb{R}^N \setminus B(x_0, R)} \varphi(x) \phi^2(x) dx \leq \frac{\epsilon}{2(1-\epsilon)} \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx.$$

Substituting (3.12) and (3.14) into (3.13) gives the desired inequality (3.10). Substituting (3.10) into (3.3) and passing to the limit, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} V(x) \phi^2(x) dx + \int_{\mathbb{R}^N} V_{\text{osc}}(x) \phi^2(x) dx - \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx \\ & \leq \frac{\epsilon}{1-\epsilon} \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx + C(\epsilon) \int_{\mathbb{R}^N} \phi^2(x) dx. \end{aligned}$$

Clearly

$$\begin{aligned} & \frac{1}{1-\epsilon} \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx - \int_{\mathbb{R}^N} V(x) \phi^2(x) dx - \int_{\mathbb{R}^N} V_{\text{osc}}(x) \phi^2(x) dx \\ & \geq -C(\epsilon) \int_{\mathbb{R}^N} \phi^2(x) dx. \end{aligned}$$

We can rearrange (by letting $(1-\epsilon)\beta = \epsilon\tilde{\beta}$):

$$\begin{aligned} & (1+\epsilon) \int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \int_{\mathbb{R}^N} (1-\epsilon)V \phi^2 dx - \epsilon \left(\int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \int_{\mathbb{R}^N} \tilde{V}_{\text{osc}} \phi^2 dx \right) \\ & \geq -C(\epsilon)(1-\epsilon) \int_{\mathbb{R}^N} \phi^2 dx, \end{aligned}$$

where

$$\tilde{V}_{\text{osc}} = \left(\frac{1-\epsilon}{\epsilon} \right) V_{\text{osc}}(x) = \frac{\tilde{\beta}}{|x|^2} \sin\left(\frac{1}{|x|^\alpha}\right).$$

We write

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \int_{\mathbb{R}^N} \left(\frac{1-\epsilon}{1+\epsilon} \right) V \phi^2 dx \\ & \geq \frac{\epsilon}{1+\epsilon} \left(\int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \int_{\mathbb{R}^N} V_{\text{osc}} \phi^2 dx \right) - \frac{C(\epsilon)(1-\epsilon)}{1+\epsilon} \int_{\mathbb{R}^N} \phi^2 dx. \end{aligned}$$

The quadratic form for H is bounded from below [12], i.e.,

$$\int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \int_{\mathbb{R}^N} V_{\text{osc}} \phi^2 dx \geq l \int_{\mathbb{R}^N} \phi^2 dx,$$

where $l \in \mathbb{R}$. Therefore, for some $\epsilon_1 > 0$, we have

$$(3.15) \quad \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \int_{\mathbb{R}^N} (1 - \epsilon_1) V \phi^2 dx}{\int_{\mathbb{R}^N} \phi^2 dx} \geq -C_1(\epsilon, \epsilon_1) > -\infty.$$

This contradicts our assumption in Theorem 2.3. The claim is proven, i.e., there is at most one $t_1 > 0$ such that $u(\cdot, t_1)\rho(\cdot) \in L^1_w(\mathbb{R}^N)$.

Now we will show that $u_k(x, t) \rightarrow \infty$ for all $(x, t) \in \mathbb{R}^N \times (0, \infty)$.

Given $x \in \mathbb{R}^N$ and $t \in (0, T)$, we have

$$u_k(x, t) = \int_{\mathbb{R}^N} G(x, y, t; V_k) u_k(y, t) w(y) dy,$$

where $G(x, y, t; V_k)$ is the heat kernel of $\frac{\partial u_k}{\partial t} - \Delta u_k = V_k u_k$. Using the maximum principle and the Gaussian bounds (2.3), we see that

$$u_k(x, t) \geq \gamma e^{\lambda t} \int_{\mathbb{R}^N} G_0(x, y, \frac{\beta t}{2}) u_k(y, \frac{t}{2}) \rho(y) w(y) dy$$

where $\gamma > 0, \beta > 0, \lambda \in \mathbb{R}$ and G_0 is the heat kernel of $\Delta u - u_t = 0$. Since $u(x, t)$ is the monotone limit of the $u_k(x, t)$, we can write

$$u(x, t) \geq \gamma e^{\lambda t} \int_{\mathbb{R}^N} G_0(x, y, \frac{\beta t}{2}) u(y, \frac{t}{2}) \rho(y) dy.$$

We take $\rho = \rho(y) = G_0(x, y, \frac{\beta t}{2})$. By the strong maximum principle, $\rho(y) > 0$ for any $y \in \mathbb{R}^N$.

If there is no $s \in (0, \frac{t}{2}]$ such that $\rho(\cdot)u(\cdot, s)w(\cdot) \in L^1(\mathbb{R}^N)$, then

$$u(x, t) \geq \gamma e^{\lambda t} \int_{\mathbb{R}^N} G_0(x, y, \frac{\beta t}{2}) u(y, \frac{t}{2}) w(y) dy = \gamma e^{\lambda t} \int_{\mathbb{R}^N} \rho(y) u(y, \frac{t}{2}) w(y) dy = \infty.$$

If $s \in (0, \frac{t}{2}]$ is the only point such that $\rho(\cdot)u(\cdot, s)w(\cdot) \in L^1(\mathbb{R}^N)$, then

$$\begin{aligned} u(x, \frac{t+s}{2}) &\geq \gamma e^{\lambda t} \int_{\mathbb{R}^N} G_0(x, y, \frac{\beta t}{2}) u(y, \frac{s}{2}) w(y) dy \\ &= \gamma e^{\lambda t} \int_{\mathbb{R}^N} \rho(y) u(y, \frac{s}{2}) w(y) dy \\ &= \infty. \end{aligned}$$

By the Harnack inequality [10], for any small $\delta > 0$, there exist $c, r > 0$ such that

$$G_0(z, y, \frac{\beta(t+\delta)}{2}) \geq c G_0(x, y, \frac{\beta t}{2})$$

when $z \in B(x, r) \subset \mathbb{R}^N$. Therefore, for $z \in B(x, r) \subset \mathbb{R}^N$,

$$\begin{aligned} u\left(z, \frac{t+s+\delta}{2}\right) &\geq \gamma e^{\lambda t} \int_{\mathbb{R}^N} G_0\left(z, y, \frac{\beta(t+\delta)}{2}\right) u(y, \frac{s}{2}) w(y) dy \\ &\geq c \gamma e^{\lambda t} \int_{\mathbb{R}^N} G_0\left(x, y, \frac{\beta t}{2}\right) u(y, \frac{s}{2}) w(y) dy = \infty. \end{aligned}$$

By the reproducing formula again, we know that

$$\begin{aligned} u(x, t) &\geq \int_{\mathbb{R}^N} G_0(x, z, \frac{t-s-\delta}{2}) u(z, \frac{t+s+\delta}{2}) w(z) dz \\ &\geq \int_{|x-z| \leq r} G_0(x, z, \frac{t-s-\delta}{2}) u(z, \frac{t+s+\delta}{2}) w(z) dz = \infty. \end{aligned}$$

Since (x, t) is arbitrary, this proves the blow-up part.

We now have the following corollary as an immediate consequence of Theorem 2.3.

Corollary 3.1. *Let $0 \in \Omega$, $N \geq 3$, $C^*(N) = (\frac{N-2}{2})^2$, and $V(x) = \frac{c}{|x|^2}$. Then the problem (2.1) has no positive solutions if $c > C^*(N)$.*

Remark 3.1. In the paper [6], Goldstein and Kombe conjectured that if c decreases to below $C^*(N)$ and $|\beta|$ is large enough, then there is instantaneous blow-up. We now see that in the light of Theorem 2.3, this is not so. Namely, we cannot decrease c to $c < C^*(N)$ no matter how large we choose β to get instantaneous blow-up.

ACKNOWLEDGEMENT

I am indebted to my thesis advisor Jerome A. Goldstein for helpful discussions leading to this article and for his constant support. Furthermore, I would like to thank Qi S. Zhang for helpful discussions.

REFERENCES

1. J. A. Aguilar Crespo and I. Peral Alonso, *Global behaviour of the Cauchy problem for some critical nonlinear parabolic equations*, SIAM J. Math. Anal., **31** (2000), 1270–1294. MR **2001d**:35086
2. P. Baras and J. A. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. **284** (1984), 121–139. MR **85f**:35099
3. X. Cabré and Y. Martel, *Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potentiel singulier*, C. R. Acad. Sci. Paris Sér. I. Math., **329** (1999), 973–978. MR **2000j**:35117
4. L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, Rhode Island, 1998. MR **99e**:35001
5. J. Garcia Azorero and I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differential Equations, **144** (1998), 441–476. MR **99f**:35099
6. J. A. Goldstein and Ismail Kombe, *Instantaneous blow up*, Advances in differential equations and mathematical physics (Birmingham, AL, 2002), Contemp. Math., Vol. 327, Amer. Math. Soc., Providence, RI, 2003, pp. 141–150.
7. J. A. Goldstein and I. Kombe, *Nonlinear degenerate parabolic equations with singular lower-order term*, Advances in Differential Equations **8** (2003), 1153–1192.
8. J. A. Goldstein and Q. S. Zhang, *On a degenerate heat equation with a singular potential*, J. Functional Analysis, **186** (2001), 342–359. MR **2002k**:35179
9. J. A. Goldstein and Q. S. Zhang, *Linear parabolic equations with strong singular potentials*, Trans. Amer. Math. Soc. **355** (2003), 197–211. MR **2003h**:35096
10. G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. MR **98k**:35003
11. J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134. MR **28**:2357
12. K.-T. Sturm, *Schrödinger operators with highly singular, oscillating potentials*, Manuscripta Math. **76** (1992), 367–395. MR **94m**:35077

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