NORMS ON EARTHQUAKE MEASURES AND ZYGMUND FUNCTIONS

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ABSTRACT. The infinitesimal earthquake theorem gives a one-to-one correspondence between Thurston bounded earthquake measures and normalized Zygmund bounded functions. In this paper, we provide an intrinsic proof of a theorem given in an earlier paper by the author; that is, we show that the cross-ratio norm of a Zygmund bounded function is equivalent to the Thurston norm of the earthquake measure in the correspondence.

1. INTRODUCTION

Consider the open unit disk $\mathbb{D}$ centered at the origin of the complex plane $\mathbb{C}$ as the hyperbolic plane. A *geodesic lamination* $\mathcal{L}$ in $\mathbb{D}$ is a collection of geodesics that foliate a closed subset L of $\mathbb{D}$. Here L is called the *locus* of $\mathcal{L}$, the geodesics are called the *leaves* of $\mathcal{L}$, the connected components of $\mathbb{D} \setminus L$ are called the *gaps*, and the gaps and the leaves of $\mathcal{L}$ are called the *strata* of $\mathcal{L}$. Let $\mathbb{S}^1$ denote the boundary circle of $\mathbb{D}$, and let $\mathbb{X}$ be the space $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\text{the diagonal}\}$ factorized by the equivalence relation $(a, b) \sim (b, a)$. A Borel measure σ defined on $\mathbb{X}$ is called an *earthquake measure* if there is a lamination $\mathcal{L}$ such that σ is supported on the pairs of the endpoints of the leaves in $\mathcal{L}$.

Let $\mathcal{L}$ be a lamination in $\mathbb{D}$ and β a closed geodesic segment of hyperbolic length ≤ 1 . If β is transversal to a leaf in $\mathcal{L}$, then $\mathcal{L}$ intersects β in a *parallel fashion* in the sense that there are two geodesic lines l_1 and l_2 among the lines of $\mathcal{L}$ intersecting β such that any other line in $\mathcal{L}$ intersecting β separates l_1 from l_2 . Suppose the strip S bounded by l_1 and l_2 in the unit disk is of the form $[a, b] \times [c, d]$, where a, d and b, c are the endpoints of l_1 and l_2 , respectively, and a, b, c, d are arranged on $\mathbb{S}^1$ in counter-clockwise order. We denote by

$$\sigma(\beta) = \sigma([a, b] \times [c, d]).$$

In [5], Thurston defines the *norm* of σ to be

$$(1) \quad \|\sigma\|_{Th} = \sup_{l(\beta) \leq 1} \sigma(\beta) = \sup_{l(\beta)=1} \sigma(\beta),$$

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where β is a closed geodesic segment transversal to $\mathcal{L}$ and $l(\beta)$ denotes the hyperbolic length of β . An earthquake measure is called *Thurston bounded* if it has finite Thurston norm. Let $\mathcal{M}$ be the collection of all Thurston bounded earthquake measures defined on $\mathbb{X}$. $\forall \sigma \in \mathcal{M}$, define

$$(2) \quad V_\sigma(x) = E(\sigma)(x) = \int \int E_{ab}(x) d\sigma(a, b),$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ if $x \in [a, b]$ and $E_{ab}(x) = 0$ otherwise. Here we also have the agreement that for each geodesic line $\overline{ab}$ in $\mathcal{L}$, $[a, b]$ denotes the short arc on $\mathbb{S}^1$ in counter-clockwise order. V_σ maps each point x on $\mathbb{S}^1$ to a vector tangent to $\mathbb{S}^1$ at x , and then it defines a tangent vector field on $\mathbb{S}^1$.

Simply consider the tangent vectors of $\mathbb{S}^1$ as complex numbers. A continuous tangent vector field V on $\mathbb{S}^1$ is said to be *Zygmund bounded* if

$$(3) \quad |V(e^{2\pi i(\theta+t)}) + V(e^{2\pi i(\theta-t)}) - 2V(e^{2\pi i\theta})| \leq M|t|$$

for a constant $M > 0$ and for all $0 \leq \theta < 1$ and $0 < t < \frac{1}{2}$. It is proved in [1] that for any $\sigma \in \mathcal{M}$, V_σ is Zygmund bounded; conversely, for any Zygmund bounded tangent vector field V on $\mathbb{S}^1$, there exists a Thurston bounded earthquake measure σ such that

$$(4) \quad V(x) = \pi \int \int E_{ab}(x) d\sigma(a, b) \text{ modulo a quadratic polynomial;}$$

furthermore, if two V 's differ by a quadratic polynomial, then the corresponding σ 's are the same. This is the so-called *infinitesimal earthquake theorem* (see Theorem 5.1 in [1]), which gives a one-to-one correspondence between $\mathcal{M}$ and the space of Zygmund bounded tangent vector fields on $\mathbb{S}^1$.

Motivated by the work of [3], we introduce a norm to measure the Zygmund boundedness of V and show it is equivalent to the Thurston norm of σ . Given a quadruple $Q = \{a, b, c, d\}$ consisting of four points a, b, c, d on the unit circle $\mathbb{S}^1$ arranged in counter-clockwise order, we define

$$(5) \quad cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}$$

and

$$(6) \quad V[Q] = \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d}.$$

Then the *cross-ratio norm* $\|V\|_{cr}$ of V is defined to be

$$(7) \quad \|V\|_{cr} = \sup_{cr(Q)=1} |V[Q]|.$$

In fact, $\|V\|_{cr}$ is finite if and only if V is Zygmund bounded.

It is proved in [2] that V_σ is the initial derivative to the parameter t of the earthquake curve determined by $t\sigma$, $t \geq 0$. By means of an implicit method, it is deduced in [4] that the cross-ratio norm of V_σ is equivalent to the Thurston norm of σ . On the other hand, it is a general principle that strategies in the study of earthquake theory (see [3] and [2]) transfer to parallel strategies in the infinitesimal theory. In this paper, we construct an intrinsic and direct proof for the following theorem.

Main Theorem. *There exists a universal constant $C > 0$ such that $\forall \sigma \in \mathcal{M}$,*

$$\frac{1}{C} \|\sigma\|_{Th} \leq \|V_\sigma\|_{cr} \leq C \|\sigma\|_{Th}.$$

2. PROOF

We divide the proof into two parts. We first prove that there exists $C > 0$ such that $\|\sigma\|_{Th} \leq C \|V_\sigma\|_{cr}$; we then show that there exists $C > 0$ such that $\|V_\sigma\|_{cr} \leq C \|\sigma\|_{Th}$. Before we start the proofs, let us summarize some techniques into lemmas.

Let σ denote a Thurston bounded earthquake measure, and let $V = V_\sigma$. Let B denote an orientation-preserving Möbius transformation from the upper half-plane $\mathbb{H}$ or the unit open disk $\mathbb{D}$ onto $\mathbb{D}$, and let $\tilde{\sigma} = B_*^{-1}(\sigma)$ be the pullback of σ under B (i.e., the pushforward of σ under B^{-1}). Also, define

$$(8) \quad \tilde{V}(x) = V_{\tilde{\sigma}}(x) = E(\tilde{\sigma})(x) = \int \int E_{ab}(x) d\tilde{\sigma}(a, b).$$

Lemma 1.

$$\tilde{V}(x) = \frac{V(B(x))}{B'(x)}.$$

Proof. Define $x' = B(x)$, $a' = B(a)$ and $b' = B(b)$. By using the identity

$$[B(a) - B(b)]^2 = (a - b)^2 B'(a) B'(b)$$

and the assumption that B is orientation-preserving, we have

$$\frac{[B(x) - B(a)][B(x) - B(b)]}{[B(a) - B(b)]} = \frac{B'(x)(x - a)(x - b)}{(a - b)}.$$

Then

$$\begin{aligned} V(B(x)) &= V(x') = \int \int E_{a'b'}(x') d\sigma(a', b') = \int \int \frac{(x' - a')(x' - b')}{a' - b'} d\sigma(a', b') \\ &= \int \int \frac{[B(x) - B(a)][B(x) - B(b)]}{B(a) - B(b)} d\sigma(B(a), B(b)) \\ &= \int \int \frac{B'(x)(x - a)(x - b)}{(a - b)} d\tilde{\sigma}(a, b) = B'(x) \tilde{V}(x). \end{aligned}$$

Therefore,

$$\tilde{V}(x) = \frac{V(B(x))}{B'(x)}.$$

□

Lemma 2. *For any quadruple Q of four points a, b, c, d on the real line or the unit circle in counter-clockwise order,*

$$\tilde{V}[Q] = V[B(Q)] \quad (\text{or } V[Q] = \tilde{V}[B^{-1}(Q)]).$$

Proof. Let $h_t = Id + tV$, where t is a real parameter near zero. Clearly, for each x , $h_t(x)$ is differentiable on t with $V = \frac{d}{dt} h_t|_{t=0}$. Define

$$cr(h_t(Q)) = \frac{[h_t(b) - h_t(a)][h_t(d) - h_t(c)]}{[h_t(c) - h_t(b)][h_t(d) - h_t(a)]}.$$

Observe first that

$$V[Q] = \frac{d}{dt} \ln cr(h_t(Q))|_{t=0}.$$

Let $\tilde{h}_t = B^{-1} \circ h_t \circ B$. Then $\tilde{h}_t$ is also differentiable on t , and

$$\frac{d}{dt} \tilde{h}_t(x)|_{t=0} = \frac{V(B(x))}{B'(x)} = \tilde{V}(x).$$

Therefore,

$$\tilde{V}[Q] = \frac{d}{dt} \ln cr(\tilde{h}_t(Q))|_{t=0}.$$

On the other hand,

$$cr(\tilde{h}_t(Q)) = cr(B^{-1} \circ h_t \circ B(Q)) = cr(h_t \circ B(Q)) = cr(h_t(B(Q))),$$

and then

$$\frac{d}{dt} \ln cr(\tilde{h}_t(Q))|_{t=0} = \frac{d}{dt} \ln cr(h_t(B(Q)))|_{t=0} = V[B(Q)].$$

Hence

$$\tilde{V}[Q] = V[B(Q)].$$

□

Corollary 1. $\|\tilde{V}\|_{cr} = \|V\|_{cr}$.

Lemma 3. Assume $\lambda > 0$, $-\infty \leq a < b < c < d$, and $c \leq s \leq d \leq t$. Let $V(x) = \lambda E_{s,t}(x)$ and $Q = \{a, b, c, d\}$. Consider $V[Q]$ as a function of s and t . Then $V[Q] \geq 0$, and $V[Q]$ is an increasing function on t for each fixed s and a decreasing function on s for each fixed t .

Proof. It is easy to check that $\frac{\partial}{\partial t} E_{s,t}(x) > 0$ and $\frac{\partial}{\partial s} E_{s,t}(x) < 0$ for each $s < x < t$. Clearly,

$$\begin{aligned} V[Q] &= \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d} \\ &= \frac{V(d)}{d - c} - \frac{V(d)}{d - a} = \left(\frac{1}{d - c} - \frac{1}{d - a} \right) \lambda E_{s,t}(d). \end{aligned}$$

Since $\frac{1}{d-c} - \frac{1}{d-a} > 0$,

$$\frac{\partial}{\partial t} V[Q] > 0 \quad \text{and} \quad \frac{\partial}{\partial s} V[Q] < 0.$$

Therefore $V[Q]$ is an increasing function on t for each fixed s and is a decreasing function on s for each fixed t . □

Lemma 4. Assume $\lambda > 0$, $-\infty \leq a < b < c < d \leq \infty$, and $b \leq s \leq c$ and $t \geq d$. Let $V(x) = \lambda E_{s,t}(x)$ and $Q = \{a, b, c, d\}$. Consider $V[Q]$ as a function of s and t . Then $V[Q] \geq 0$, and $V[Q]$ is increasing on s for each fixed t and also increasing on t for each fixed s .

Proof. It is straightforward to check

$$\begin{aligned} V[Q] &= \lambda \left[-\frac{d-b}{(c-b)(d-c)} E_{s,t}(c) + \frac{c-a}{(d-c)(d-a)} E_{s,t}(d) \right] \\ &= \lambda \left[-\frac{d-b}{(c-b)(d-c)} \frac{(c-s)(c-t)}{s-t} + \frac{c-a}{(d-c)(d-a)} E_{s,t}(d) \frac{(d-s)(d-t)}{s-t} \right]. \end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial s}V[Q] &= \lambda\left[\frac{(d-b)}{(c-b)(d-c)}\frac{(t-c)^2}{(s-t)^2} - \frac{(c-a)}{(d-c)(d-a)}\frac{(t-d)^2}{(s-t)^2}\right] \\ &= \frac{\lambda(t-d)^2}{(s-t)^2(d-c)}\left[\frac{d-b}{c-b}\left(\frac{t-c}{t-d}\right)^2 - \frac{c-a}{d-a}\right].\end{aligned}$$

Clearly, $\frac{d-b}{c-b}\left(\frac{t-c}{t-d}\right)^2 > 1$ and $\frac{c-a}{d-a} < 1$. Hence $\frac{\partial}{\partial s}V[Q] > 0$ when $t > d$ and $\frac{\partial}{\partial s}V[Q] \geq 0$ when $t = d$. Therefore $V[Q]$ is an increasing function on s for each fixed t .

Similarly,

$$\begin{aligned}\frac{\partial}{\partial t}V[Q] &= \lambda\left[-\frac{(d-b)}{(c-b)(d-c)}\frac{(c-s)^2}{(s-t)^2} + \frac{(c-a)}{(d-c)(d-a)}\frac{(d-s)^2}{(s-t)^2}\right] \\ &= \frac{\lambda(c-s)^2}{(s-t)^2(d-c)}\left[-\frac{d-b}{c-b} + \frac{c-a}{d-a}\left(\frac{d-s}{c-s}\right)^2\right] \\ &= \frac{\lambda(c-s)^2}{(s-t)^2(d-c)}\frac{d-b}{c-b}\left[-1 + \left(\frac{d-s}{c-s}/\frac{d-b}{c-b}\right)\left(\frac{d-s}{c-s}/\frac{d-a}{c-a}\right)\right].\end{aligned}$$

Since $a < b \leq s \leq c < d$,

$$\frac{d-s}{c-s} \geq \frac{d-b}{c-b} > 1 \quad \text{and} \quad \frac{d-s}{c-s} > \frac{d-a}{c-a} > 1.$$

Then

$$-1 + \left(\frac{d-s}{c-s}/\frac{d-b}{c-b}\right)\left(\frac{d-s}{c-s}/\frac{d-a}{c-a}\right) > 0.$$

Hence $\frac{\partial}{\partial t}V[Q] > 0$ when $b \leq s < c$ and $\frac{\partial}{\partial t}V[Q] = 0$ when $s = c$. Therefore $V[Q]$ is also an increasing function on t for each fixed s . $\square$

Theorem 1. *There exists a universal constant $C > 0$ such that*

$$(9) \quad \|\sigma\|_{Th} \leq C\|V_\sigma\|_{cr}.$$

In fact, we can take $C = \frac{(1+e)^2}{-3e^2+6e+1}$.

Proof. Let D denote a closed disk in $\mathbb{D}$ of hyperbolic diameter 1, l_1 and l_2 denote the lines in the lamination $\mathcal{L}$ of σ that bound all the lines in $\mathcal{L}$ intersecting D . Let β denote the geodesic perpendicular to both l_1 and l_2 (in the case that l_1 and l_2 share at least one endpoint, we let β be a geodesic perpendicular to l_1 such that the hyperbolic length of the segment on β between l_1 and l_2 is less than or equal to $\frac{1}{2}$). Label the endpoints of β by x and y so that the arc $[x, y]$ from x to y going in the counter-clockwise direction is not longer than the arc $[y, x]$ from y to x . Let $B : \mathbb{D} \rightarrow \mathbb{H}$ be the Möbius transformation that maps x to 0, y to ∞ , the arc $[x, y]$ to the positive half real line, and the geodesics l_1 and l_2 to the geodesics connecting -1 to 1 and $-s$ to s with $s > 1$ (in the case that l_1 and l_2 share at least one endpoint, then $B(l_1)$ or $B(l_2)$ connects $-s$ to 1 or -1 to s with $s \geq 1$). Since the hyperbolic distance between l_1 and l_2 is less than or equal to 1, $s \leq e$ (in the case that l_1 and l_2 share one endpoint, the requirement on β also implies that $s \leq e$). Let $\tilde{\sigma}$ denote the pushforward of σ under B and $\tilde{V} = E(\tilde{\sigma})$. It is clear that $\|\tilde{\sigma}\|_{Th} = \|\sigma\|_{Th}$, and Lemma 2 implies $\|\tilde{V}\|_{cr} = \|V\|_{Th}$.

Assume $a = 1$, $b = \infty$, $c = -s$ and $d = \frac{1-s}{2}$. Denote by $Q = \{a, b, c, d\}$ and $Q' = B^{-1}(Q) = \{a', b', c', d'\}$. Clearly, $cr(Q) = cr(Q') = 1$, and Lemma 2 implies $\tilde{V}[Q] = V[Q']$. Denote by u' and v' the other endpoints of l_1 and l_2 such that a', u', b', c', v', d' are arranged on $\mathbb{S}^1$ in counter-clockwise order. We divide the lines

in the lamination $\mathcal{L}$ that affect the value of $V[Q']$ into three groups. Let $\mathcal{L}_m$ denote the collection of the lines in $\mathcal{L}$ intersecting D , $\mathcal{L}_b$ denote the collection of the lines in $\mathcal{L} \setminus \mathcal{L}_m$ connecting points in $[u', b']$ to points in $(b', c']$, and $\mathcal{L}_d$ the collection of the lines in $\mathcal{L} \setminus \mathcal{L}_m$ connecting points in $[v', d']$ to points in $(d', a']$. Denote by $\sigma_i = \sigma|_{\mathcal{L}_i}$ and $V_i = E(\sigma_i)$ for $i = m, b, d$. By the linearity of the operator E , we have

$$V[Q'] = V_m[Q'] + V_b[Q'] + V_d[Q'].$$

By Lemma 3,

$$V_d[Q'] = V_d[\{a', b', c', d'\}] \geq 0 \quad \text{and} \quad V_b[\{c', d', a', b'\}] \geq 0.$$

Then

$$V_b[Q'] = V_b[\{a', b', c', d'\}] = -V_b[\{b', c', d', a'\}] = V_b[\{c', d', a', b'\}] \geq 0.$$

Therefore,

$$V[Q'] \geq V_m[Q'].$$

To complete the proof, we need to work out an explicit lower bound for $V_m[Q']$. Denote by $\tilde{\sigma}_m$ the pushforward of σ_m under B and $\tilde{V}_m = E(\tilde{\sigma}_m)$. By Lemma 2,

$$V_m[Q'] = \tilde{V}_m[B(Q')] = \tilde{V}_m[Q].$$

By Lemma 4, if we move the weights of the geodesic lines in the lamination $\tilde{\mathcal{L}}_m$ of $\tilde{\sigma}_m$ to the geodesic line connecting -1 to s , then the value of $\tilde{V}_m[\{b, c, d, a\}]$ is possibly increased, and hence the value of $\tilde{V}_m[Q] = -\tilde{V}_m[\{b, c, d, a\}]$ is possibly decreased. Therefore

$$\tilde{V}_m[Q] \geq (\lambda E_{-1,s})[Q],$$

where $\lambda = \sigma(\mathcal{L}_m)$. It is easy to check that

$$(\lambda E_{-1,s})[Q] = \lambda \frac{2E_{-1,s}(d) - E_{-1,s}(a)}{a - d}$$

and

$$\frac{2E_{-1,s}(d) - E_{-1,s}(a)}{a - d} = \frac{-3s^2 + 6s + 1}{(1 + s)^2} \geq \frac{-3e^2 + 6e + 1}{(1 + e)^2} > 0$$

for $1 \leq s \leq e$. Letting $C = \frac{(1+e)^2}{-3e^2+6e+1}$, we have

$$\|V\|_{cr} \geq V[Q'] \geq V_m[Q'] = \tilde{V}_m[Q] \geq (\lambda E_{-1,s})[Q] \geq \frac{\lambda}{C}.$$

Hence

$$\lambda \leq C\|V\|_{cr},$$

which implies

$$\|\sigma\|_{Th} \leq C\|V\|_{cr}.$$

□

Theorem 2. *There exists a universal constant $C > 0$, independent of σ , such that*

$$(10) \quad \|V_\sigma\|_{cr} \leq C\|\sigma\|_{Th}.$$

In fact, we can take $C = C_0 + 2C_1C_2$, where C_0 is the smallest positive integer greater than or equal to $\ln(3 + 2\sqrt{2})$, $C_1 = \frac{e}{e-1}$, and C_2 is the smallest positive integer greater than or equal to $\ln(e + \sqrt{e^2 - 1})$.

Remark. The inequality (10) was obtained in [1] for a different cross-ratio norm on V through a complex method which considered the holomorphic differential arising from the third derivative of $V + iH(V)$, where $H(V)$ denotes the Hilbert transformation of V . The proof of the inequality (10) in this paper is purely real. In addition, our Theorem 1 and that inequality in [1] imply that the two cross-ratio norms on V are actually equivalent (see [4]).

Lemma 5. *Let Q be a quadruple consisting of four points a, b, c, d on $\mathbb{R}^1$ or $\mathbb{S}^1$ arranged in counter-clockwise order. Take $\frac{|dz|}{y}$ as the hyperbolic metric on the upper half-plane $\mathbb{H}$. Then $cr(Q) = 1$ if and only if the geodesic $\overline{ac}$ from a to c is perpendicular to the geodesic $\overline{bd}$ from b to d , and if and only if the hyperbolic distance from $\overline{ab}$ to $\overline{cd}$ (or $\overline{bc}$ to $\overline{da}$) is equal to $\ln(3 + 2\sqrt{2})$.*

Proof. It is straightforward (see [3] for details). $\square$

Lemma 6. *Consider the upper half-plane $\mathbb{H}$. Let l_n denote the geodesic connecting $-e^{-n}$ to e^{-n} for each $n \in \{0\} \cup \mathbb{N}$ and $\mathcal{L}$ the lamination consisting of l_n 's. Suppose that σ is an earthquake measure supported on $\mathcal{L}$, and let $\lambda_n = \sigma(l_n)$. Let $Q = \{1, \infty, -1, 0\}$, and*

$$V(x) = \iint E_{a,b}(x) d\sigma(a, b).$$

There exists a constant $C_1 > 0$ such that

$$0 \leq V[Q] \leq C_1 \max_{n \geq 0} \lambda_n.$$

Proof. Define $\lambda = \max_{n \geq 0} \lambda_n$, $a_n = -e^{-n}$ and $b_n = e^{-n}$. Clearly, $V(1) = V(\infty) = V(-1) = 0$, and then $V[Q] = 2V(0)$. We need to work out $V(0)$, that is,

$$0 \leq V(0) = \iint E_{a_n, b_n}(0) d\sigma(a_n, b_n) = \sum_{n \geq 0} \lambda_n \frac{a_n b_n}{b_n - a_n} = \sum_{n \geq 0} \lambda_n \frac{e^{-n}}{2} \leq \frac{1}{2} \frac{e}{e-1} \lambda.$$

Let $C_1 = \frac{e}{e-1}$. Then $0 \leq V[Q] = 2V(0) \leq C_1 \lambda$. $\square$

We reduce the proof of Theorem 2 to Propositions 1 and 2. Let $\mathcal{L}$ be the lamination that supports σ . Given a quadruple Q of four points a, b, c, d on $\mathbb{S}^1$ in counter-clockwise order with $cr(Q) = 1$, we first assume that three points a, b and c belong to the same stratum A and estimate $V[Q]$ in this case. By a Möbius change of coordinates and Lemma 2, we may assume that $a = 1, b = \infty, c = -1$, and $d = 0$. We will show that there is a constant C such that

$$0 \leq V[Q] \leq C \|\sigma\|_{Th}.$$

Let x_n denote the point $-e^{-n}$ and y_n the point e^{-n} on the real axis for each $n \in \{0\} \cup \mathbb{N}$. Let $\mathcal{L}'$ denote the collection of the lines in $\mathcal{L}$ that connect points of the interval $[-1, 0)$ to points of $(0, 1]$, $\mathcal{L}_0^-$ the collection of the lines in $\mathcal{L}'$ that connect points of $[x_0, x_1)$ to points of $(0, y_0]$, and $\mathcal{L}_0^+$ the collection of the lines in $\mathcal{L}'$ that connect points of $[x_0, 0)$ to points of $(y_1, y_0]$. Finally, let $\mathcal{L}_0 = \mathcal{L}_0^- \cup \mathcal{L}_0^+$. Then any line in $\mathcal{L}' \setminus \mathcal{L}_0$ must connect a point in $[x_1, 0)$ to a point in $(0, y_1]$. Inductively, for each $n \in \mathbb{N}$, let $\mathcal{L}_n^-$ denote the collection of the lines in $\mathcal{L}' \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \dots \cup \mathcal{L}_{n-1})$ that connect points of $[x_n, x_{n+1})$ to points of $(0, y_n]$, and $\mathcal{L}_n^+$ the collection of the lines in $\mathcal{L}' \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \dots \cup \mathcal{L}_{n-1})$ that connect points of $[x_n, 0)$ to points of $(y_{n+1}, y_n]$, and $\mathcal{L}_n = \mathcal{L}_n^- \cup \mathcal{L}_n^+$. We have the following three lemmas.

Lemma 7. *For each $n \in \{0\} \cup \mathbb{N}$, any line in $\mathcal{L}_n$ must connect a point in $[x_n, 0)$ to a point in $(0, y_n]$.*

Proof. It can be easily proved by an induction on n . $\square$

Lemma 8. *There exists a constant $C_2 > 0$, independent of h and σ , such that $\sigma(\mathcal{L}_n) \leq C_2 \|\sigma\|_{Th}$ for any $n \in \{0\} \cup \mathbb{N}$.*

Proof. For each $n \in \{0\} \cup \mathbb{N}$, let l_n denote the geodesic line connecting the point x_n to the point y_n . Also, for any $n \in \mathbb{N}$, let l_n^- denote the geodesic connecting the point x_n to 0, and l_n^+ the geodesic connecting 0 to the point y_n . The hyperbolic distance from l_n to l_{n+1}^- (or l_{n+1}^+), $n \in \{0\} \cup \mathbb{N}$, is equal to a constant, that is equal to $\ln(e + \sqrt{e^2 - 1})$. Let C_2 denote the smallest positive integer that is greater than or equal to $\ln(e + \sqrt{e^2 - 1})$. Then

$$\sigma(\mathcal{L}_n) \leq C_2 \|\sigma\|_{Th}$$

for each $n \in \{0\} \cup \mathbb{N}$. $\square$

Let $\tilde{V}$ be the same map as defined in Lemma 6 with $\lambda_n = \sigma(\mathcal{L}_n)$.

Lemma 9. *We have the following inequality:*

$$V[Q] \leq \tilde{V}[Q].$$

Proof. Define $\sigma_n = \sigma|_{\mathcal{L}_n}$ and $V_n = E(\sigma_n)$. Let $\tilde{\sigma}_n$ denote the atomic earthquake measure with weight λ_n supported on the geodesic l_n and let $\tilde{V}_n = E(\tilde{\sigma}_n)$. By the linearity of the operator E ,

$$V[Q] = \sum_{n=0}^{\infty} V_n[Q] \quad \text{and} \quad \tilde{V}[Q] = \sum_{n=0}^{\infty} \tilde{V}_n[Q].$$

By Lemma 3 and Lemma 7, if we move the weights of the geodesic lines in $\mathcal{L}_n$ to the geodesic line l_n , we only increase $V_n[Q]$, that is, $V_n[Q] \leq \tilde{V}_n[Q]$. Therefore

$$V[Q] \leq \tilde{V}[Q].$$

$\square$

Lemmas 9, 6 and 8 imply the following proposition.

Proposition 1. *If $cr(Q) = 1$ and a, b, c belong to the same stratum of an earthquake measure $(E, \mathcal{L})$, then*

$$0 \leq V_\sigma[Q] \leq C_1 C_2 \|\sigma\|_{Th}.$$

Proposition 2. *Suppose $cr(Q) = 1$, and assume that there exists at least one geodesic line in the lamination $\mathcal{L}$ of σ that separates the vertices a and b from the vertices c and d . Then*

$$|V_\sigma[Q]| \leq (C_0 + 2C_1 C_2) \|\sigma\|_{Th}.$$

Proof. Given two points x and y on the unit circle, we use $[x, y]$ to denote the arc on $\mathbb{S}^1$ from x to y in the counter-clockwise direction. We divide the geodesic lines in $\mathcal{L}$ that affect $V[Q]$ into five groups. Let $\mathcal{L}_m$ denote the collection of the geodesic lines in $\mathcal{L}$ that connect points of the arc $[d, a]$ to points of the arc $[b, c]$. Let $\mathcal{L}_a$ denote the collection of the lines in $\mathcal{L}$ that connect points of the arc (d, a) to points of the arc (a, b) , $\mathcal{L}_b$ the collection of the lines in $\mathcal{L}$ that connect points of the arc (a, b) to points of the arc (b, c) , $\mathcal{L}_c$ the collection of the lines in $\mathcal{L}$ that connect

FIGURE 1. Five subcollections of $\mathcal{L}$ in the proof of Proposition 2

points of the arc (b, c) to points of the arc (c, d) , and finally $\mathcal{L}_d$ the collection of the lines in $\mathcal{L}$ that connect points of the arc (c, d) to points of the arc (d, a) (see Figure 1). Let σ_i denote the restriction of σ on $\mathcal{L}_i$ and $V_i = E(\sigma_i)$, where $i = m, a, b, c, d$. Clearly $\|\sigma_i\|_{Th} \leq \|\sigma\|_{Th}$ and

$$V[Q] = V_m[Q] + V_a[Q] + V_b[Q] + V_c[Q] + V_d[Q].$$

For the lamination $\mathcal{L}_d$, three points a, b, c are contained in the same stratum. By Proposition 1, we have

$$0 \leq V_d[Q] \leq C_1 C_2 \|\sigma_d\|_{Th} \leq C_1 C_2 \|\sigma\|_{Th}.$$

For the lamination $\mathcal{L}_b$, three points c, d, a are contained in the same stratum. Again by Proposition 1, we have

$$0 \leq V_b[\{c, d, a, b\}] \leq C_1 C_2 \|\sigma_b\|_{Th} \leq C_1 C_2 \|\sigma\|_{Th}.$$

Therefore,

$$0 \leq V_b[Q] = V_b[\{c, d, a, b\}] \leq C_1 C_2 \|\sigma\|_{Th}.$$

Similarly, we obtain

$$-C_1 C_2 \|\sigma\|_{Th} \leq V_a[Q] \leq 0 \text{ and } -C_1 C_2 \|\sigma\|_{Th} \leq V_c[Q] \leq 0.$$

It remains to consider $V_m[Q]$. Because of Lemma 2, by a Möbius change of coordinates, we may assume $a = -\infty$, $b = -1$, $c = 0$, $d = 1$. Then the geodesic lines in $\mathcal{L}_m$ connect the points of $[-1, 0]$ to the points of $[1, +\infty]$. By Lemma 4, if we move all the lines in $\mathcal{L}_m$ to the geodesic line from 0 to ∞ without changing the weights of the lines in $\mathcal{L}_m$ to obtain a new measure σ'_m , then $V'_m[Q] = E(\sigma'_m)[Q]$ is possibly bigger, that is, $V_m[Q] \leq V'_m[Q]$. Clearly, $V'_m(x) = 0$ for $x \in [-\infty, 0]$ and $V'_m(x) = \sigma(\mathcal{L}_m)x$ for $x \in (0, +\infty)$. Then $V'_m[Q] = V'_m(1) = \sigma(\mathcal{L}_m)$. If we let C_0

denote the smallest integer $\geq \ln(3 + 2\sqrt{2})$, then by Lemma 5, $\sigma(\mathcal{L}_m) \leq C_0\|\sigma\|_{Th}$. Therefore,

$$V_m[Q] \leq V'_m[Q] = \sigma(\mathcal{L}_m) \leq C_0\|\sigma\|_{Th}.$$

Again by Lemma 4, if we move all the lines in $\mathcal{L}_m$ to the geodesic line from -1 to 1 without changing the weights of the lines in $\mathcal{L}_m$ to obtain a new measure σ''_m , then $V''_m[Q] = E(\sigma''_m)[Q]$ is possibly smaller, that is, $V_m[Q] \geq V''_m[Q]$. Clearly, $V''_m(x) = 0$ for $x \notin (-1, 1)$ and $V''_m(x) = \sigma(\mathcal{L}_m)\frac{(x+1)(1-x)}{2}$ for $x \in (-1, 1)$. Then $V''_m[Q] = -2V''_m(0) = -\sigma(\mathcal{L}_m)$. Therefore,

$$V_m[Q] \geq V''_m[Q] = -\sigma(\mathcal{L}_m) \geq -C_0\|\sigma\|_{Th}.$$

Collecting together these estimates, we obtain

$$-(C_0 + 2C_1C_2)\|\sigma\|_{Th} \leq V_a[Q] + V_c[Q] + V_m[Q] + V_b[Q] + V_d[Q] \leq (C_0 + 2C_1C_2)\|\sigma\|_{Th};$$

that is,

$$|V[Q]| \leq (C_0 + 2C_1C_2)\|\sigma\|_{Th}.$$

□

Propositions 1 and 2 imply Theorem 2.

Proof. Consider the pattern of the geodesic lines in $\mathcal{L}$ with respect to Q , there either exists a line in $\mathcal{L}$ separating two adjacent vertices in Q from the other two or no such line exists. The former case is treated by Proposition 2 and the fact that $V[\{b, c, d, a\}] = -V[Q]$; the latter one is treated by Proposition 1 and the summability of $V[Q] = E(\sigma)[Q]$ over four subcollections of $\mathcal{L}$ that contain three vertices of Q in the same stratum. □

Finally, Theorems 1 and 2 imply our Main Theorem.

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